Math 352

Name:__________________________________

Final Exam

1. [**10 points**] Let C be the plane curve $y = e^x$. Find the Cartesian equation for the osculating circle to C at the point $(0, 1)$.

2. [**14 points**] Let $\vec{X}(u,v)$ be a surface parametrization (where $0 < u < \pi/2$), and suppose that the corresponding first fundamental form is

$$g(u,v) = \begin{bmatrix} \sec u & \tan u \\ \tan u & \sec u \end{bmatrix}.$$

(a) [**8 pts**] Find a formula for the angle between the u and v coordinate lines at the point $\vec{X}(u,v)$. Your final answer should not involve any trigonometric functions.

(b) [**6 pts**] Is the parametrization $\vec{X}(u,v)$ conformal, equiareal, both, or neither?

3. **[24 points (8 pts each)]** Find the principle curvatures of the given surface at the given point.

(a) The surface $z = 4xy + 3y^2$, with upward-pointing normal vectors, at the point $(0, 0, 0)$.

(b) The surface $r = 3z^2 + 2$, with outward-pointing normal vectors, at the point $(2, 0, 0)$.

(c) The surface $(r - 2)^2 + z^2 = 2$, with outward-pointing normal vectors, at the point $(3, 0, 1)$.

4. **[12 points)]** Let S_1 be the cone $z = \sqrt{x^2+y^2}$, let S_2 be the cone $z = 7\sqrt{x^2+y^2}$, and let $f\colon S_1 \to S_2$ be the map $f(x,y,z) = \bigl(2x, 2y, 14z\bigr)$.

(a) **[6 pts]** Find $df_p(0,1,0)$ and $df_p(1,0,1)$, where p is the point $(t,0,t)$.

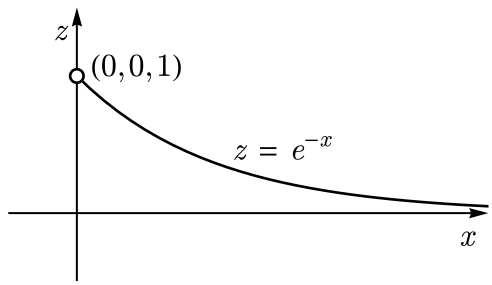
(b) **[6 pts]** Let $\mathcal{R}$ be a region on S_1 with area 4. What is the area of the corresponding region $f(\mathcal{R})$ on S_2?

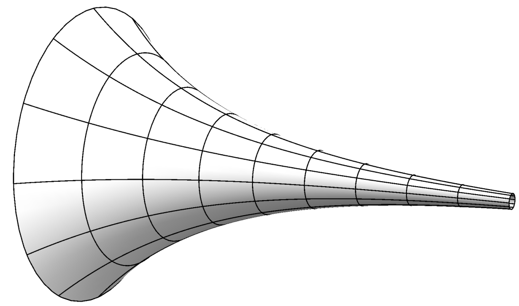
5. **[14 points]** Let C be the helix $\vec{x}(t) = (3\cos t, 3\sin t, 4t)$.

(a) **[8 pts]** Find formulas for the Frenet vectors $\vec{T}(t)$, $\vec{P}(t)$, and $\vec{B}(t)$.

(b) **[6 pts]** Determine the curvature κ and torsion τ of this helix.

6. [**12 points**] Let C be the curve $z = e^{-x}$ $(x > 0)$ in the xz-plane, and let S be the surface obtained by rotating the curve C around the x-axis, oriented with normal vectors pointing outwards.





(a) [**6 pts**] Find the image of S under the Gauss map. Express your answer as one or more inequalities defining a region on the unit sphere.

(b) [**6 pts**] Use your answer to part (a) to compute $\iint_S K\, dA$, where K is the Gaussian curvature of S.

7. [**14 points**] A unit circle is rolling counterclockwise on the inside of the circle $x^2 + y^2 = 9$. A point P lies on the perimeter of the rolling circle, with initial coordinates $(3, 0)$. Find parametric equations for the curve produced by tracing the path of P.