Solutions to Final Exam Practice Problems

Math 352, Fall 2014

1. We have $\vec{X}_u = (\cos v, \sin v, v)$ and $\vec{X}_v = (-u\sin v, u\cos v, u)$, so

$$g(u,v) \;=\; \begin{bmatrix} \vec{X}_u\cdot\vec{X}_u & \vec{X}_u\cdot\vec{X}_v \\ \vec{X}_v\cdot\vec{X}_u & \vec{X}_v\cdot\vec{X}_v \end{bmatrix} \;=\; \boxed{\begin{bmatrix} 1+v^2 & uv \\ uv & 2u^2 \end{bmatrix}}$$

2. (a) $$\int_0^3 \sqrt{\vec{x}\,'(t)^T g\big(\vec{x}(t)\big)\,\vec{x}\,'(t)}\,dt \;=\; \int_0^3 \sqrt{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} t+1 & 1 \\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}\,dt$$

$$= \int_0^3 \sqrt{t+1}\,dt \;=\; \left[\frac{2}{3}(t+1)^{3/2}\right]_0^3 \;=\; \boxed{\frac{14}{3}}$$

(b) $$\int_0^4\int_0^4 \sqrt{\det\big(g(u,v)\big)}\,du\,dv \;=\; \int_0^4\int_0^4 \sqrt{u}\,du\,dv \;=\; 4\left[\frac{2}{3}u^{3/2}\right]_0^4 \;=\; \boxed{\frac{64}{3}}$$

3. (a) Let $\vec{x}(t) = (0,1,t)$, and note that $\vec{x}(4) = p$ and $\vec{x}\,'(4) = (0,0,1)$. The image of $\vec{x}(t)$ on the paraboloid is the curve

$$\vec{y}(t) \;=\; f(\vec{x}(t)) \;=\; \big(0, \sqrt{t}, t\big).$$

Then $df_p(0,0,1) = \vec{y}\,'(4) = \boxed{(0,1/4,1)}$.

For the other vector, let $\vec{x}(t) = (\sin t, \cos t, 4)$. Then $\vec{x}(0) = p$ and $\vec{x}\,'(0) = (1,0,0)$. The image of $\vec{x}(t)$ on the paraboloid is the curve

$$\vec{y}(t) \;=\; f(\vec{x}(t)) \;=\; \big(2\sin t, 2\cos t, 4\big).$$

Then $df_p(1,0,0) = \vec{y}\,'(0) = \boxed{(2,0,0)}$.

(b) The Jacobian is $$\frac{\|(0,1/4,1)\times(2,0,0)\|}{\|(0,0,1)\times(1,0,0)\|} \;=\; \frac{\|(0,1/4,1)\|\,\|(2,0,0)\|}{1} \;=\; \boxed{\frac{\sqrt{17}}{2}}$$

4. Observe that $\vec{X}(u,v) = (\cos u, \sin u, v)$ is a parametrization of the cylinder. Consider the function

$$\vec{F}(u,v) \;=\; f\big(\vec{X}(u,v)\big) \;=\; \big(e^{kv}\cos u, e^{kv}\sin u, e^{kv}\big).$$

We can compute $\vec{X}_u = (-\sin u, \cos u, 0)$, $\vec{X}_v = (0,0,1)$, $\vec{F}_u = e^{kv}\big(-\sin u, \cos u, 0\big)$, and $\vec{F}_v = ke^{kv}\big(\cos u, \sin u, 1\big)$. Then

$$\begin{bmatrix} \vec{X}_u\cdot\vec{X}_u & \vec{X}_u\cdot\vec{X}_v \\ \vec{X}_v\cdot\vec{X}_u & \vec{X}_v\cdot\vec{X}_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \vec{F}_u\cdot\vec{F}_u & \vec{F}_u\cdot\vec{F}_v \\ \vec{F}_v\cdot\vec{F}_u & \vec{F}_v\cdot\vec{F}_v \end{bmatrix} = \begin{bmatrix} e^{2kv} & 0 \\ 0 & 2k^2e^{2kv} \end{bmatrix}$$

The map is conformal if and only if the second matrix is a scalar multiple of the first. This occurs if and only if $2k^2 = 1$, so $\boxed{k = 1/\sqrt{2}}$ (or $k = -1/\sqrt{2}$).

5. (a) Let $f(x,y) = \cos(3x) + 6\sin(xy)$. We have

$$\frac{\partial f}{\partial x} = -3\sin(3x) + 6y\sin(xy) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 6x\cos(xy)$$

so $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$. Thus $(0,0)$ is a critical point for f, so we can compute the curvatures using the Hessian.

Observe that

$$Hf(x,y) = \begin{bmatrix} -9\cos(3x) - 6y^2\sin(xy) & 6\cos(xy) - 6xy\sin(xy) \\ 6\cos(xy) - 6xy\sin(xy) & -6x^2\sin(xy) \end{bmatrix}$$

so $Hf(0,0) = \begin{bmatrix} -9 & 6 \\ 6 & 0 \end{bmatrix}$. The eigenvalues of this are $\boxed{-12 \text{ and } 3}$.

(b) The Gaussian curvature is $\boxed{-36}$, and the mean curvature is $\boxed{-9/2}$.

6. (a) At $(2,0,0)$, both the initial arc A and the horizontal circle of radius 2 are geodesics, so the principle curvatures are $\boxed{-1/5 \text{ and } -1/2}$.

(b) At $(1,0,3)$, the initial arc A is a geodesic, so one of the principle curvatures is $-1/5$. However, the horizontal circle of radius 1 is not a geodesic. In particular, this circle has $\vec{P} = (-1,0,0)$, and the normal vector is $\vec{N} = (4/5, 0, 3/5)$ (parallel to the radius of the large circle), so $\kappa_n = \kappa\vec{P}\cdot\vec{N} = (1)(-4/5) = -4/5$. Thus, the principle curvatures are $\boxed{-1/5 \text{ and } -4/5}$.

(c) The normal vectors at the "endpoints" would be $(3/5, 0, 4/5)$ and $(3/5, 0, -4/5)$, so the image of S under the Gauss map is the region $\boxed{-4/5 < z < 4/5}$ on the unit sphere.

(d) We just need to find the area of the region $-4/5 < z < 4/5$ of the unit sphere.

SOLUTION 1: This region can parameterized by

$$\vec{X}(u,v) = \big(\cos u \cos v, \sin u \cos v, \sin v\big)$$

for $0 < u < 2\pi$ and $-\sin^{-1}(4/5) < z < \sin^{-1}(4/5)$. Then

$\vec{X}_u = (-\sin u \cos v, \cos u \cos v, 0)$ and $\vec{X}_v = (-\cos u \sin v, -\sin u \sin v, \cos v)$

so

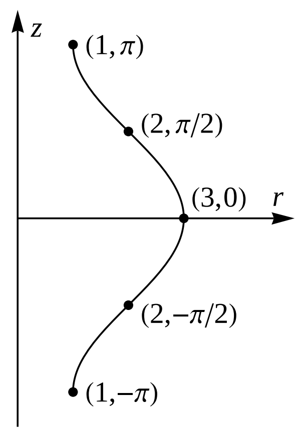
$$\|\vec{X}_u \times \vec{X}_v\| = \|\vec{X}_u\|\,\|\vec{X}_v\| = \cos v$$

so

$$\iint_S K\,dA = \int_{-\sin^{-1}(4/5)}^{\sin^{-1}(4/5)} \int_0^{2\pi} \cos v\,du\,dv = 2\pi\Big[\sin v\Big]_{-\sin^{-1}(4/5)}^{\sin^{-1}(4/5)} = \boxed{\frac{16\pi}{5}}$$

SOLUTION 2: Recall that Archimedes' map (projecting the sphere horizontally outwards to the cylinder) is area-preserving. The image of the given region is a cylinder with radius 1 and height 8/5, so the area is $(8/5)(2\pi) = \boxed{\frac{16\pi}{5}}$.

7. (a) The curve $r = 2 + \cos z$ on the rz-plane is shown below



This curve will be a geodesic on the resulting surface, and its curvature at $(3,0,0)$ is 1. The horizontal circle at $z=0$ is also a geodesic, with a curvature of 1/3. These are curving *away* from the normal vector, so the principle curvatures at $(3,0,0)$ are $\boxed{-1 \text{ and } -1/3}$.

(b) Again, the curve shown above is a geodesic, and its curvature at $(2,0,\pi/2)$ is 0. The horizontal circle at $z=\pi/2$ has a curvature of 1/2, but it is *not* a geodesic. Its principle normal vector is $\vec{P} = (-1,0,0)$, and the normal vector to the surface is $\vec{N} = \frac{1}{\sqrt{2}}(1,0,1)$, so the normal curvature is $\kappa\vec{P}\cdot\vec{N} = -\frac{1}{2\sqrt{2}}$. Thus, the principle curvatures are $\boxed{0 \text{ and } -\frac{1}{2\sqrt{2}}}$.

(c) The Gaussian curvature will be positive for $\boxed{-\pi/2 < z < \pi/2}$.

8. (a) Clearly $\boxed{\vec{T} = (0, 1, 0)}$. The parabola $y = \frac{2}{3}x^2$ has slope $4/3$ at the point $(1, 2/3)$. Therefore, $\boxed{\vec{U} = (-3/5, 0, -4/5)}$ and $\boxed{\vec{N} = (-4/5, 0, 3/5)}$.

(b) The curvature of $\vec{x}(t)$ is $\kappa = 1$, and the principle normal vector is $\vec{P} = (-1, 0, 0)$. Therefore, the normal curvature is $\kappa_n = \kappa \vec{P} \cdot \vec{N} = \boxed{4/5}$ and the geodesic curvature is $\kappa_g n = \kappa \vec{P} \cdot \vec{U} = \boxed{3/5}$.

(c) The image of this portion of the surface under the Gauss map is the region $z > 3/5$ on the unit sphere. Using techniques similar to the solution of problem 6(d) above, the area of this portion of the sphere is $\boxed{4\pi/5}$.