Final Exam Practice Problems

Math 352, Fall 2014

1. Let $\vec{X}(u,v) = (u\cos v, u\sin v, uv)$. Compute the matrix for the corresponding first fundamental form.

2. Let $\vec{X}(u,v)$ be a surface parametrization, and suppose that the corresponding first fundamental form is
$$g(u,v) = \begin{bmatrix} u+v & \sqrt{v} \\ \sqrt{v} & 1 \end{bmatrix}.$$
   (a) Let $\vec{x}(t) = (t,1)$ be a curve on the uv-plane, and let $\vec{y}(t) = \vec{X}(\vec{x}(t))$ be the corresponding curve on the surface. Compute the length of $\vec{y}(t)$ for $0 < t < 3$.

   (b) Let $\mathcal{R}$ be the region in the uv-plane defined by $0 < u < 4$ and $0 < v < 4$, and let $\vec{X}(\mathcal{R})$ be the corresponding region on the surface. Find the area of $\vec{X}(\mathcal{R})$.

3. Let C be the portion of the cylinder $x^2 + y^2 = 1$ lying above the xy-plane, let P be the paraboloid $z = x^2 + y^2$, and let $f\colon C \to P$ be the map
$$f(x,y,z) = \big(x\sqrt{z}, y\sqrt{z}, z\big).$$
   (a) Compute $df_p(0,0,1)$ and $df_p(1,0,0)$, where p is the point $(0,1,4)$.

   (b) Compute the Jacobian of f at the point $(0,1,4)$.

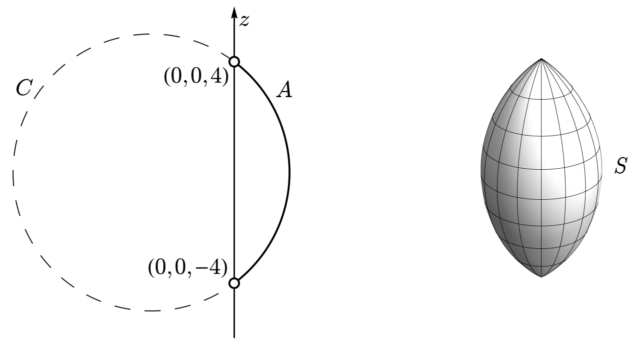
4. Let S_1 be the cylinder $x^2 + y^2 = 1$ for $z > 0$, let S_2 be the cone $z = \sqrt{x^2+y^2}$, and let $f\colon S_1 \to S_2$ be the map
$$f(x,y,z) = \big(xe^{kz}, ye^{kz}, e^{kz}\big)$$
where k is a constant. Find a value of k for which f is conformal.

5. Let S be the surface $z = \cos(3x) + 6\sin(xy)$, oriented so that normal vectors point upwards.

   (a) Compute the principle curvatures of S at the point $(0,0,1)$.

   (b) Compute the Gaussian curvature and mean curvature of S at this point.

6. Let C be a circle of radius 5 that contains the points $(0, 0, 4)$ and $(0, 0, -4)$, and let A be the (open) minor arc of C between these points. Let S be the surface obtained by rotating the arc A around the z-axis, oriented so that normal vectors point outwards.



Note that the surface S does not include the cusp points $(0, 0, 4)$ and $(0, 0, -4)$.

(a) Find the principle curvatures of S at the point $(2, 0, 0)$.

(b) Find the principle curvatures of S at the point $(1, 0, 3)$.

(c) Find the image of S under the Gauss map. Express your answer as one or more inequalities defining a region on the unit sphere.

(d) Use your answer to part (c) to evaluate $\iint_S K\, dA$, where K is the Gaussian curvature of S.

7. Let S be the surface $r = 2 + \cos z$ for $-\pi < z < \pi$, oriented with normal vectors pointing outwards.

(a) Find the principle curvatures of S at the point $(3, 0, 0)$.

(b) Find the principle curvatures of S at the point $(2, 0, \pi/2)$.

(c) For what values of z in the range $-\pi < z < \pi$ is the Gaussian curvature of S positive?

8. Let P be the paraboloid $z = \frac{2}{3}\left(x^2 + y^2\right)$, oriented so that the normal vectors point **inwards**, and consider the curve on this surface defined by $\vec{x}(t) = (\cos t, \sin t, 2/3)$.

(a) Find the vectors $\{\vec{N}, \vec{T}, \vec{U}\}$ of the Darboux frame for $\vec{x}$ at the point $(1, 0, 2/3)$.

(b) Find the normal curvature κ_n and geodesic curvature κ_g of $\vec{x}$ at the point $(1, 0, 2/3)$.

(c) Use the Gauss map to compute $\iint_{\mathcal{R}} K\, dA$, where $\mathcal{R}$ is the portion of the surface for which $z < 2/3$.