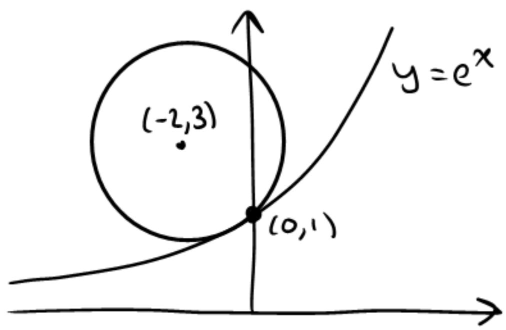
Math 352 Name: SOLUTION

Final Exam

1. [**10 points**] Let C be the plane curve $y = e^x$. Find the Cartesian equation for the osculating circle to C at the point $(0, 1)$.



$\vec{x}(t) = (t, e^t) \qquad \vec{x}(0) = (0, 1)$

$\vec{x}'(t) = (1, e^t) \qquad \vec{x}'(0) = (1, 1)$

$\vec{x}''(t) = (0, e^t) \qquad \vec{x}''(0) = (0, 1)$

$\vec{u} = \text{unit normal} = \frac{1}{\sqrt{2}}(-1, 1)$

$a_\perp = (0, 1) \cdot \vec{u} = \frac{1}{\sqrt{2}}$

$a_\perp = \kappa v^2$

$\frac{1}{\sqrt{2}} = \kappa (\sqrt{2})^2$

$\kappa = \frac{1}{2\sqrt{2}} \Rightarrow r = 2\sqrt{2}$

center $= (0, 1) + (-2, 2)$

$= (-2, 3)$

$$\boxed{(x+2)^2 + (y-3)^2 = 8}$$

2. **[14 points]** Let $\vec{X}(u,v)$ be a surface parametrization (where $0 < u < \pi/2$), and suppose that the corresponding first fundamental form is

$$g(u,v) = \begin{bmatrix} \sec u & \tan u \\ \tan u & \sec u \end{bmatrix}.$$

(a) **[8 pts]** Find a formula for the angle between the u and v coordinate lines at the point $\vec{X}(u,v)$. Your final answer should not involve any trigonometric functions.

$$\cos\theta = \frac{\vec{X}_u \cdot \vec{X}_v}{\|\vec{X}_u\| \|\vec{X}_v\|} = \frac{\tan u}{\sqrt{\sec u}\sqrt{\sec u}} = \sin u$$

$$\boxed{\theta = \frac{\pi}{2} - u}$$

(b) **[6 pts]** Is the parametrization $\vec{X}(u,v)$ conformal, equiareal, both, or neither?

<u>not</u> <u>conformal</u>

$$\sqrt{\det g} = \sqrt{\sec^2 u - \tan^2 u} = \sqrt{1} = 1$$

$$\boxed{\text{equiareal}}$$

3. **[24 points (8 pts each)]** Find the principle curvatures of the given surface at the given point.

(a) The surface $z = 4xy + 3y^2$, with upward-pointing normal vectors, at the point $(0, 0, 0)$.

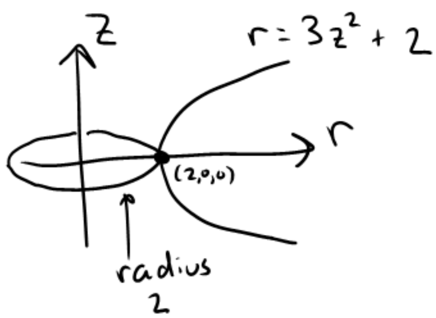
$$\frac{\partial z}{\partial x} = 4y \qquad \frac{\partial z}{\partial y} = 4x + 6y$$

$(0,0)$ is a critical point

$$\text{Hessian} = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix} \qquad \begin{aligned} \lambda_1 + \lambda_2 &= 6 \\ \lambda_1 \lambda_2 &= -16 \end{aligned}$$

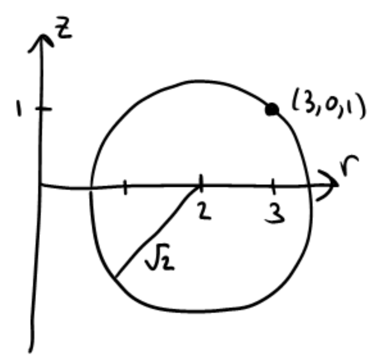
$$\boxed{-2 \text{ and } 8}$$

(b) The surface $r = 3z^2 + 2$, with outward-pointing normal vectors, at the point $(2, 0, 0)$.



$$\boxed{6, -\tfrac{1}{2}}$$

(c) The surface $(r - 2)^2 + z^2 = 2$, with outward-pointing normal vectors, at the point $(3, 0, 1)$.



$$\vec{P} = (-1,0,0)$$

$$\vec{N} = \frac{1}{\sqrt{2}}(1,0,1)$$

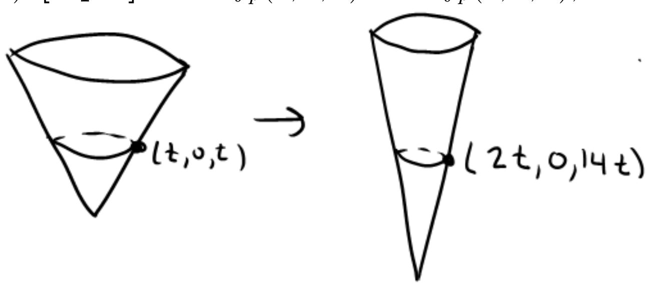
$$\vec{P} \cdot \vec{N} = -\frac{1}{\sqrt{2}}$$

$$\begin{aligned} k_n &= k(\vec{P} \cdot \vec{N}) \\ &= \frac{1}{3}\left(-\frac{1}{\sqrt{2}}\right) \\ &= -\frac{1}{3\sqrt{2}} \end{aligned}$$

$$\boxed{-\frac{1}{\sqrt{2}} \text{ and } -\frac{1}{3\sqrt{2}}}$$

4. [**12 points)**] Let S_1 be the cone $z = \sqrt{x^2 + y^2}$, let S_2 be the cone $z = 7\sqrt{x^2 + y^2}$, and let $f\colon S_1 \to S_2$ be the map $f(x, y, z) = \big(2x, 2y, 14z\big)$.

(a) [**6 pts**] Find $df_p(0, 1, 0)$ and $df_p(1, 0, 1)$, where p is the point $(t, 0, t)$.



$$\boxed{\begin{aligned} &df_p(0,1,0) \\ &= (0,2,0) \end{aligned}}$$

$$\boxed{\begin{aligned} &df_p(1,0,1) \\ &= (2,0,14) \end{aligned}}$$

(b) [**6 pts**] Let $\mathcal{R}$ be a region on S_1 with area 4. What is the area of the corresponding region $f(\mathcal{R})$ on S_2?

$$\frac{\|(0,2,0) \times (2,0,14)\|}{\|(0,1,0) \times (1,0,1)\|} = \frac{\|(0,2,0)\|\ \|(2,0,14)\|}{\|(0,1,0)\|\ \|(1,0,1)\|}$$

$$= \frac{(2)(\sqrt{200})}{(1)(\sqrt{2})} = 20$$

$$(20)(4) = \boxed{80}$$

5. [**14 points**] Let C be the helix $\vec{x}(t) = (3\cos t, 3\sin t, 4t)$.

(a) [**8 pts**] Find formulas for the Frenet vectors $\vec{T}(t)$, $\vec{P}(t)$, and $\vec{B}(t)$.

$$\vec{x}'(t) = (-3\sin t, 3\cos t, 4) \qquad s'(t) = 5$$

$$\boxed{\vec{T}(t) = \left(-\tfrac{3}{5}\sin t, \tfrac{3}{5}\cos t, \tfrac{4}{5}\right)}$$

$$\frac{d\vec{T}}{dt} = s'\kappa\vec{P}$$

$$\left(-\tfrac{3}{5}\cos t, -\tfrac{3}{5}\sin t, 0\right) = s'\kappa\vec{P}$$

$$\boxed{\vec{P}(t) = (-\cos t, -\sin t, 0)}$$

$$\vec{B} = \vec{T}\times\vec{P} = \frac{1}{5}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\sin t & 3\cos t & 4 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$\boxed{\vec{B}(t) = \left(\tfrac{4}{5}\sin t, -\tfrac{4}{5}\cos t, \tfrac{3}{5}\right)}$$

(b) [**6 pts**] Determine the curvature κ and torsion τ of this helix.

$$s'\kappa = \frac{3}{5} \qquad s' = 5, \text{ so } \qquad \boxed{\kappa = \frac{3}{25}}$$

$$\frac{d\vec{B}}{dt} = -s'\tau\vec{P}$$

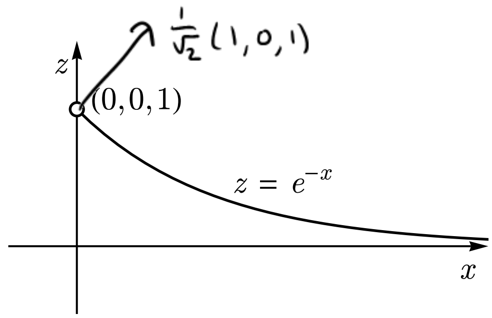
$$\uparrow$$

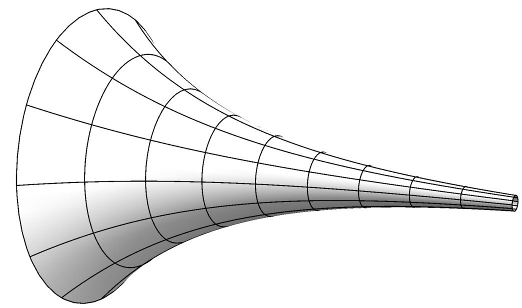
$$\left(\tfrac{4}{5}\cos t, \tfrac{4}{5}\sin t, 0\right)$$

$$-s'\tau = -\frac{4}{5}$$

$$\boxed{\tau = \frac{4}{25}}$$

6. **[12 points]** Let C be the curve $z = e^{-x}$ $(x > 0)$ in the xz-plane, and let S be the surface obtained by rotating the curve C around the x-axis, oriented with normal vectors pointing outwards.





(a) **[6 pts]** Find the image of S under the Gauss map. Express your answer as one or more inequalities defining a region on the unit sphere.

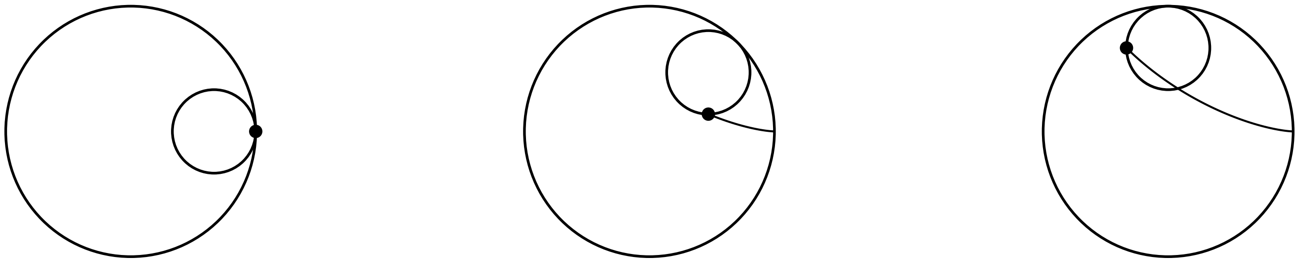
$$\boxed{0 < x < \frac{1}{\sqrt{2}}}$$

(b) **[6 pts]** Use your answer to part (a) to compute $\displaystyle\iint_S K\,dA$, where K is the Gaussian curvature of S.

area $= (2\pi)\left(\frac{1}{\sqrt{2}}\right) = \pi\sqrt{2}$

But K is negative, so $\boxed{-\pi\sqrt{2}}$

7. [**14 points**] A unit circle is rolling counterclockwise on the inside of the circle $x^2 + y^2 = 9$. A point P lies on the perimeter of the rolling circle, with initial coordinates $(3, 0)$. Find parametric equations for the curve produced by tracing the path of P.



$$\vec{x}(t) = (2\cos t + \cos 2t,\ 2\sin t - \sin 2t)$$