Outline: Quadratic Forms

1. **Bilinear and Quadratic Forms on $\mathbb{R}^n$**

   A **symmetric bilinear form** on $\mathbb{R}^n$ is a function $B\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

   $$B(\vec{v}, \vec{w}) \;=\; \vec{v}^{\,T} M \vec{w},$$

   where M is a symmetric $n \times n$ matrix.

   The associated **quadratic form** on $\mathbb{R}^n$ is the function $Q\colon \mathbb{R}^n \to \mathbb{R}$ defined by

   $$Q(\vec{v}) \;=\; B(\vec{v}, \vec{v}) \;=\; \vec{v}^{\,T} M \vec{v}.$$

   For example, a quadratic form on $\mathbb{R}^2$ is defined by

   $$Q(x, y) \;=\; \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; ax^2 + 2bxy + cy^2.$$

2. **Principle Axes**

   If $Q(\vec{v}) = \vec{v}^{\,T} M \vec{v}$ is a quadratic form on $\mathbb{R}^n$, the eigenspaces of M are called the **principle axes** of Q. For a quadratic form on $\mathbb{R}^2$, the associated eigenvalues are given by the following formulas:

   $$\lambda_1 \;=\; \min\{Q(\vec{u}) : \vec{u} \text{ is a unit vector}\} \qquad \text{and} \qquad \lambda_2 \;=\; \max\{Q(\vec{u}) : \vec{u} \text{ is a unit vector}\}$$

   If Q is a quadratic form on $\mathbb{R}^2$, then the level curves $Q(\vec{v}) = C$ are conic sections centered at the origin. In this case, the principle axes are the lines of symmetry of the conic section, e.g. the lines containing the major and minor axes of an ellipse. The eigenvalues determine the type of conic section, i.e. the conic section

   $$Q(\vec{v}) = C$$

   is just a rotated version of $\lambda_1 x^2 + \lambda_2 y^2 = C$.

3. **The Hessian**

   The **Hessian** of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is the matrix of second partial derivatives:

   $$Hf \;=\; \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\, \partial y} \\[2ex] \dfrac{\partial^2 f}{\partial x\, \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

   The Hessian of f at a point $\vec{p}$ is denoted $Hf(\vec{p})$.

   The Hessian should be thought of as a quadratic form. If $\vec{u}$ is a unit vector, then

   $$\vec{u}^{\,T}\, Hf\, \vec{u}$$

   is equal to the directional second derivative of f in the direction of $\vec{u}$.

4. The Second Derivative Test

The **Hessian** of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is the matrix of second partial derivatives: The Hessian can be used to classify the critical points of the function f. In particular, if $\vec{p}$ is a critical point of f, the **second derivative test** works as follows:

1. If all of the eigenvalues of $Hf(\vec{p})$ are positive, then $\vec{p}$ is a local minimum for f.
2. If all of the eigenvalues of $Hf(\vec{p})$ are negative, then $\vec{p}$ is a local maximum for f.
3. If some of the eigenvalues of $Hf(\vec{p})$ are positive and some are negative, then $\vec{p}$ is neither a local minimum nor a local maximum.

5. Bilinear and Quadratic Forms in General

Bilinear and quadratic forms can be defined on any vector space V. Specifically, a **symmetric bilinear form** on V is a function $B\colon V \times V \to \mathbb{R}$ such that

1. $B(\vec{v}, \vec{w}) = B(\vec{w}, \vec{v})$,
2. $B(\vec{v}_1 + \vec{v}_2, \vec{w}) = B(\vec{v}_1, \vec{w}) + B(\vec{v}_2, \vec{w})$, and
3. $B(\lambda\vec{v}, \vec{w}) = \lambda\, B(\vec{v}, \vec{w})$

for all $\vec{v}, \vec{v}_1, \vec{v}_2, \vec{w} \in V$ and $\lambda \in \mathbb{R}$.

If B is a symmetric bilinear form on V, the corresponding **quadratic form** on V is the function $Q\colon V \to \mathbb{R}$ defined by

$$Q(\vec{v}) = B(\vec{v}, \vec{v}).$$

If the vector space V has an inner product, it is possible to define the principle axes and eigenvalues of a quadratic form. In the case where V is two-dimensional, the eigenvalues of Q are defined by

$$\lambda_1 \;=\; \min\{Q(\vec{u}) : \vec{u} \text{ is a unit vector}\} \qquad \text{and} \qquad \lambda_2 \;=\; \max\{Q(\vec{u}) : \vec{u} \text{ is a unit vector}\}$$

and the principle axes are in the directions of the corresponding unit vectors.