The Gauss Map

Outline

1. The Gauss Map

Let S be a regular surface. A **Gauss map** on S is a continuous function n that assigns to each point $p \in S$ a unit normal vector $n(p)$. If we regard unit vectors as points on the unit sphere $\mathbb{S}^2$, the we can think of n as a map $n\colon S \to \mathbb{S}^2$.

Most surfaces have two possible Gauss maps, corresponding to the two possible choices for the direction of the normal vectors. However, some surfaces such as the Möbius strip don't have a consistent way to to pick a direction for the normal vectors, meaning there isn't a Gauss map. We will use the following terminology:

- A surface is called **orientiable** if a Gauss map exists.
- A choice of a Gauss map for a surface is called an **orientation** of the surface.
- An **oriented surface** is a surface S whose orientation is specified.

2. Using a Parametrization

If $\vec{X}(u,v)$ is a parametrization of a surface S, and n is a Gauss map on S, let $\vec{N}$ denote the function

$$\vec{N}(u,v) \;=\; n\big(\vec{X}(u,v)\big).$$

That is, $\vec{N}(u,v)$ is the unit normal vector at the point $\vec{X}(u,v)$.

The function $\vec{N}$ is given by the formula

$$\vec{N} \;=\; \pm\frac{\vec{X}_u \times \vec{X}_v}{\|\vec{X}_u \times \vec{X}_v\|},$$

where the $\pm$ depends on which Gauss map n we're using. In practice, just compute the right side and then check whether or not it points in the correct direction.

3. The Differential

Let S be a surface, and let $n\colon S \to \mathbb{S}^2$ be a Gauss map for S. The differential of n at a point p is a linear transformationm

$$dn_p\colon T_pS \to T_{n(p)}\mathbb{S}^2.$$

Note that T_pS and $T_{n(p)}\mathbb{S}^2$ are actually the same vector space, since the normal vector to S^2 at $n(p)$ is equal to $n(p)$.

If $\vec{X}(u,v)$ is a parametrization of S, then

$$dn_p\big(\vec{X}_u\big) \;=\; \vec{N}_u \qquad \text{and} \qquad dn_p\big(\vec{X}_v\big) \;=\; \vec{N}_v.$$

4. Absolute Gaussian Curvature

Let S be a surface with Gauss map n. The **absolute Gaussian curvature** of S at a point p is defined by the formula

$$|K(p)| \;=\; \frac{\|\vec{N}_u \times \vec{N}_v\|}{\|\vec{X}_u \times \vec{x}_v\|},$$

where $\vec{X}(u,v)$ is a parametrization of the surface, and $\vec{N}(u,v) = n\big(\vec{X}(u,v)\big)$.

That is, the absolute Gaussian curvature $|K(p)|$ is the Jacobian of the Gauss map. The absolute Gaussian curvature $|K(p)|$ is always positive, but later we will define the **Gaussian curvature** $K(p)$, which may be positive or negative.

5. Integrating the Curvature

Let S be a surface with Gauss map n, and let R be a region on S. If n is one-to-one on R, then

$$\iint_R |K|\, dA \;=\; \text{area of } n(R),$$

where $n(R)$ denotes the image of the region R on the sphere $\mathbb{S}^2$.