Complex Dynamics

So far, we have learned about two main branches of dynamical systems:

1. **One-dimensional dynamics** is the study of dynamical systems whose state spaces are subsets of the real line. This was the subject of chapters 1 and 3 in the textbook.

2. **Multidimensional dynamics** is the study of dynamical systems whose state spaces are subset of $\mathbb{R}^n$ for some $n > 1$. This was the subject of chapters 2 and 5 in the textbook.

However, there is a third major branch of dynamical systems that the book does not discuss in much detail:

3. **Complex dynamics** is the study of dynamical systems whose state spaces are subsets of the complex plane $\mathbb{C}$ (or more generally $\mathbb{C}^n$ for $n \geq 1$).

Complex dynamics was one of the original motivations for the study of fractal geometry, and it includes some of the most famous fractals in mathematics—the Julia sets and the Mandelbrot set.

This first set of notes introduces some of the basic concepts in complex dynamics, beginning with a brief review of complex numbers. Julia sets and the Mandelbrot set will be covered in a later set of notes.

1. Complex Numbers

We assume some basic familiarity with the complex number system. Recall that a **complex number** is a number of the form

$$a + bi,$$

where a and b are real numbers, and i is a square root of -1. The set of all complex numbers is denoted $\mathbb{C}$.

If z is a complex number, we will let $\mathrm{Re}(z)$ denote the real part of z, and $\mathrm{Im}(z)$ denote the imaginary part of z. For example,

$$\mathrm{Re}(-3+4i) \;=\; -3 \qquad \text{and} \qquad \mathrm{Im}(-3+4i) \;=\; 4.$$

Note then that

$$z \;=\; \mathrm{Re}(z) \;+\; i\;\mathrm{Im}(z)$$

for any complex number z.

The four operations of arithmetic extend to the complex numbers in a natural way. For example, we can add any two complex numbers by collecting like terms:

$$(2+3i)+(6+2i) \;=\; 8+5i.$$

To multiply complex numbers, we use the distributive law, together with the rule that $i^2=-1$:

$$\begin{aligned}(2+3i)(6+2i) \;&=\; 12+4i+18i+6i^2 \\ &=\; 12+4i+18i-6 \\ &=\; 6+22i.\end{aligned}$$

Division of complex numbers is slightly more difficult. Given a fraction,

$$\frac{2+i}{3+2i},$$

we multiply the top and bottom by the **complex conjugate** of the denominator:

$$\frac{(2+i)(3-2i)}{(3+2i)(3-2i)}.$$

The denominator is now real, which allows us to simplify:

$$\frac{2+i}{3+2i} \;=\; \frac{(2+i)(3-2i)}{(3+2i)(3-2i)} \;=\; \frac{8-i}{13} \;=\; \frac{8}{13} - \frac{1}{13}\,i.$$

Euler's Formula

Using Taylor series, it is possible to extend many of the functions from calculus to the complex numbers. We will concentrate on the exponential function:

Definition: Complex Exponential
The **exponential function** $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined by the formula

$$\exp(z) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{n!} \;=\; 1 \;+\; z \;+\; \frac{1}{2!}z^2 \;+\; \frac{1}{3!}z^3 \;+\; \frac{1}{4!}z^4 \;+\; \cdots$$

If z is a complex number, then $\exp(z)$ is sometimes denoted e^z. Note, however, that complex exponentiation is not defined in general. For example, expressions like $(-i)^{1/2}$ or $(1+i)^{1+i}$ have no single, well-defined meaning.

The following theorem is one of the most famous in all of mathematics:

Theorem 1 Euler's Formula

If $\theta \in \mathbb{R}$, then

$$e^{i\theta} \;=\; \cos\theta \,+\, i\sin\theta.$$

PROOF By the definition of the complex exponential, we have

$$e^{i\theta} \;=\; 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots.$$

Expanding the powers of i gives

$$e^{i\theta} \;=\; 1 + \theta i - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}i + \frac{\theta^4}{4!} + \frac{\theta^5}{5!}i - \frac{\theta^6}{6!} - \frac{\theta^7}{7!}i + \cdots.$$

Since θ is real, we conclude that

$$\mathrm{Re}\big(e^{i\theta}\big) \;=\; 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots.$$

The series on the right is the Taylor series for $\cos\theta$, and therefore $\mathrm{Re}\big(e^{i\theta}\big) = \cos\theta$. Similarly,

$$\mathrm{Im}\big(e^{i\theta}\big) \;=\; \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots,$$

which is the Taylor series for $\sin\theta$. ■

When $\theta = \pi$, Euler's formula reduces to

$$\boxed{e^{i\pi} \;=\; -1}$$

This famous equation relating the constants e, i, and pi is known as **Euler's identity** or **Euler's magic formula**.

The Complex Plane

In the same way that $\mathbb{R}$ can the thought of geometrically as the "real line", we can think of the complex numbers $\mathbb{C}$ as forming a **complex plane**. Any complex number

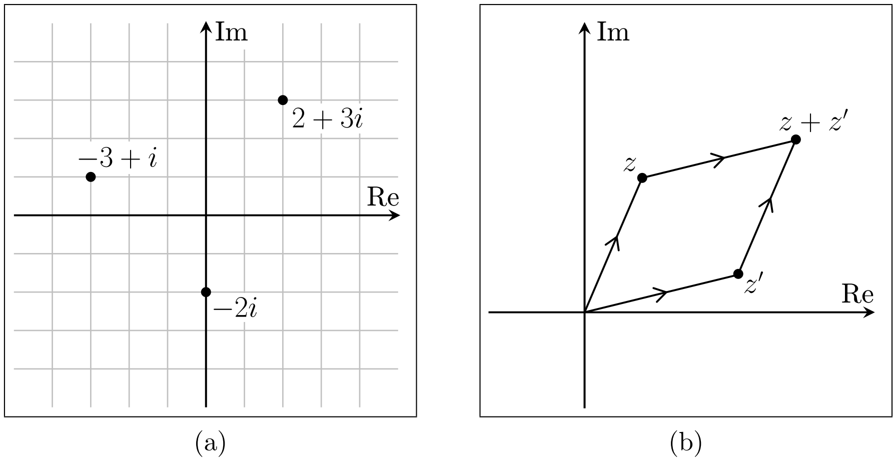

Figure 1: (a) The complex plane (b) The sum of complex numbers

$a+bi$ corresponds to a point on this plane, namely the point with horizontal coordinate a and vertical coordinate b, as shown in Figure 1a.

Using the complex plane, addition of complex numbers has the same geometric meaning as addition of vectors in $\mathbb{R}^2$. Specifically, if z and z' are complex numbers, then the points 0, z, z', and $z+z'$ form a parallelogram on the complex plane, as shown in Figure 1b.

There is also a nice geometric interpretation of multiplication, but it requires polar coordinates. As with points on the xy plane, we can represent any point on the complex plane using polar coordinates (r, θ), where r is the distance to the origin and θ is the angle from the positive real axis. For a complex number, the radius and angle have special names:

Definition: Modulus and Argument

Let z be a complex number, corresponding to a point on the complex plane. Then:

1. The distance from z to the origin is called the **modulus** or **absolute value** of z, and is denoted $|z|$.

2. The angle θ from z to the positive real axis is called the **argument** of z, and is denoted $\arg(z)$.

The modulus and argument of complex number are illustrated in Figure 2a. By the Pythagorean Theorem, the modulus of a complex number can be found by the formula

$$\boxed{|a+bi| \;=\; \sqrt{a^2+b^2}}$$

and the argument can be found using trigonometry. Conversely, if the modulus and

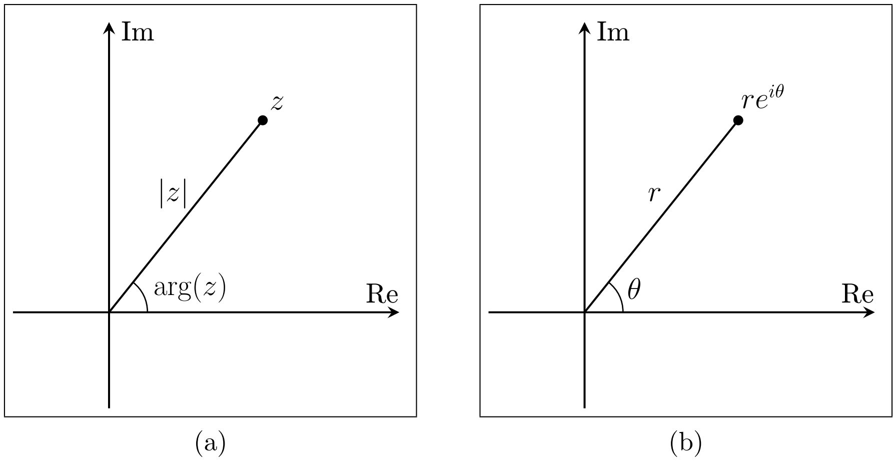

Figure 2: (a) The modulus and argument of z (b) The geometric meaning of $re^{i\theta}$

argument of a complex number are known, then the rectangular coordinates of the number are given by the formula

$$\boxed{z \;=\; (r\cos\theta) \;+\; (r\sin\theta)i \;=\; re^{i\theta}}$$

where $r = |z|$ and $\theta = \arg(z)$ (see Figure 2b). To summarize, we now have two ways of representing a complex number:

1. In rectangular coordinates, every complex number can be expressed as $a + bi$, where $a, b \in \mathbb{R}$.
2. In polar coordinates, every complex number can be expressed as $re^{i\theta}$, where $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$.

The following theorem gives a geometric interpretation of complex multiplication using polar coordinates:

Theorem 2 Multiplication in Polar Coordinates

If $r, s \in [0, \infty)$ and $\theta, \phi \in \mathbb{R}$, then

$$\left(re^{i\theta}\right)\left(se^{i\phi}\right) \;=\; (rs)e^{i(\theta+\phi)}$$

PROOF Though this may look obvious, we have not proven that $e^{x+y} = e^x e^y$ when x and y are complex. Therefore, we must prove this formula using trigonometric identities.

Recall that

$$re^{i\theta} \;=\; (r\cos\theta) \,+\, (r\sin\theta)\, i \qquad \text{and} \qquad se^{i\phi} \;=\; (s\cos\phi) \,+\, (s\sin\phi)\, i.$$

Multiplying these gives

$$\left(re^{i\theta}\right)\left(se^{i\phi}\right) \;=\; rs(\cos\theta\cos\phi - \sin\theta\sin\phi) \,+\, rs(\cos\theta\sin\phi + \sin\theta\cos\phi)\, i.$$

The terms in parentheses are the sum-of-angle formulas for cosine and sine. Thus, the above formula simplifies to

$$\left(re^{i\theta}\right)\left(se^{i\phi}\right) \;=\; rs\cos(\theta+\phi) \,+\, rs\sin(\theta+\phi)\, i \;=\; (rs)e^{i(\theta+\phi)}. \qquad \blacksquare$$

In terms of modulus and argument, the above formula theorem says that

$$\boxed{|zw| \;=\; |z|\,|w|} \qquad \text{and} \qquad \boxed{\arg(zw) \;=\; \arg(z) + \arg(w)}$$

for any two complex numbers z and w.

Using induction, Theorem 2 can be extended to larger products and powers, yielding the following formulas:

$$\left(r_1e^{i\theta_1}\right)\cdots\left(r_ne^{i\theta_n}\right) \;=\; (r_1\cdots r_n)e^{i(\theta_1+\cdots+\theta_n)} \qquad \text{and} \qquad \left(re^{i\theta}\right)^n \;=\; r^ne^{in\theta}.$$

EXAMPLE 1 Compute $(1-i)^6$.
SOLUTION Observe that $|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$ and $\arg(1-i) = 3\pi/4$. Then

$$(1-i)^6 \;=\; \left(\sqrt{2}\,e^{i(3\pi/4)}\right)^6 \;=\; \left(\sqrt{2}\right)^6 e^{i(18\pi/4)} \;=\; 2^3e^{i\pi/2} \;=\; 8i. \qquad \blacksquare$$

EXAMPLE 2 Find a complex number z for which $z^4 = -1$.
SOLUTION Let $z = re^{i\theta}$. Then

$$z^4 \;=\; \left(re^{i\theta}\right)^4 \;=\; r^4e^{i(4\theta)}.$$

Since $-1 = e^{i\pi}$, we need

$$r^4 = 1 \qquad \text{and} \qquad 4\theta \cong \pi \pmod{2\pi}.$$

One possible solution is $r = 1$ and $\theta = \pi/4$, so

$$z \;=\; e^{i\pi/4} \;=\; \cos\frac{\pi}{4} \,+\, i\sin\frac{\pi}{4} \;=\; \frac{1+i}{\sqrt{2}}. \qquad \blacksquare$$

EXAMPLE 3 Multiple Angle Formulas
Suppose we wish to find trigonometric formulas for $\cos(3\theta)$ and $\sin(3\theta)$. This is fairly easy using complex numbers. Observe that:

$$\cos(3\theta) + i\sin(3\theta) \;=\; e^{i(3\theta)} \;=\; \left(e^{i\theta}\right)^3 \;=\; (\cos\theta + i\sin\theta)^3.$$

Expanding the expression on the right yields

$$\cos(3\theta) + i\sin(3\theta) \;=\; \cos^3\theta \,+\, i\cos^2\theta\sin\theta \,-\, \cos\theta\sin^2\theta \,-\, i\sin^3\theta$$

so

$$\cos(3\theta) \;=\; \cos^3\theta - \cos\theta\sin^2\theta \qquad \text{and} \qquad \sin(3\theta) \;=\; \cos^2\theta\sin\theta - \sin^3\theta.$$

For higher multiples of θ, we can combine this technique with the Binomial Theorem to derive explicit formulas for $\cos(n\theta)$ and $\sin(n\theta)$. In general, the formulas are finite sums of the following forms:

$$\cos(n\theta) \;=\; \cos^n\theta \,-\, \binom{n}{2}\cos^{n-2}\theta\sin^2\theta \,+\, \binom{n}{4}\cos^{n-4}\theta\sin^4\theta \,-\, \cdots$$

$$\sin(n\theta) \;=\; \binom{n}{1}\cos^{n-1}\theta\sin\theta \,-\, \binom{n}{3}\cos^{n-3}\theta\sin^3\theta \,+\, \binom{n}{5}\cos^{n-5}\theta\sin^5\theta \,-\, \cdots.$$

For example,

$$\begin{aligned}\cos(5\theta) \;&=\; \cos^5\theta \,-\, \binom{5}{2}\cos^3\theta\sin^2\theta \,+\, \binom{5}{4}\cos\theta\sin^4\theta \\ &=\; \cos^5\theta \,-\, 10\cos^3\theta\sin^2\theta \,+\, 5\cos\theta\sin^4\theta\end{aligned}$$

■

Complex Polynomials

A **complex polynomial** is any polynomial whose coefficients are complex numbers. For example, the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) \;=\; (1+i)z^3 + 5z^2 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z + (4i - 1)$$

is a complex polynomial of degree three.

For the following theorem, recall that a **root** of a complex polynomial f is a complex number p for which $f(p) = 0$.

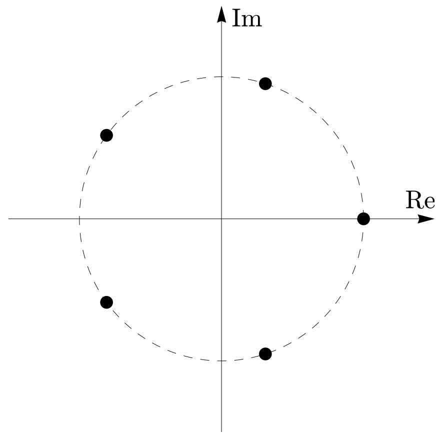

Figure 3: The five 5th roots of unity.

Theorem 3 Fundamental Theorem of Algebra

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d > 0$. Then f can be written in the form

$$f(z) \;=\; a(z-r_1)(z-r_2)\cdots(z-r_d),$$

where $a \neq 0$ and $r_1, \ldots, r_d$ are complex constants. In particular, f has at least one complex root.

EXAMPLE 4 Roots Of Unity

Consider the polynomial

$$f(z) \;=\; z^n - 1$$

where $n \in \{2, 3, 4, \ldots\}$. Any root of this polynomial must satisfy the equation

$$z^n \;=\; 1.$$

That is, it must be an nth root of the number 1.

According to the Fundamental Theorem of Algebra, we can factor this polynomial into linear factors

$$z^n - 1 \;=\; (z-\omega_1)(z-\omega_2)\cdots(z-\omega_n).$$

The corresponding roots $\omega_1, \ldots, \omega_n$ are called the ***n*th roots of unity**. In polar coordinates, these roots can be written

$$\omega_1 = e^{i(2\pi/n)}, \quad \omega_2 = e^{i(4\pi/n)}, \quad \ldots, \quad \omega_k = e^{i(2k\pi/n)}, \quad \ldots, \quad \omega_n = 1.$$

Note that

$$\left(e^{i(2k\pi/n)}\right)^n \;=\; e^i(2k\pi) \;=\; 1,$$

so these are in fact nth roots of 1. Geometrically, the roots $\omega_1, \ldots, \omega_n$ correspond to n equally spaced points around the unit circle, as shown in Figure 3. ■

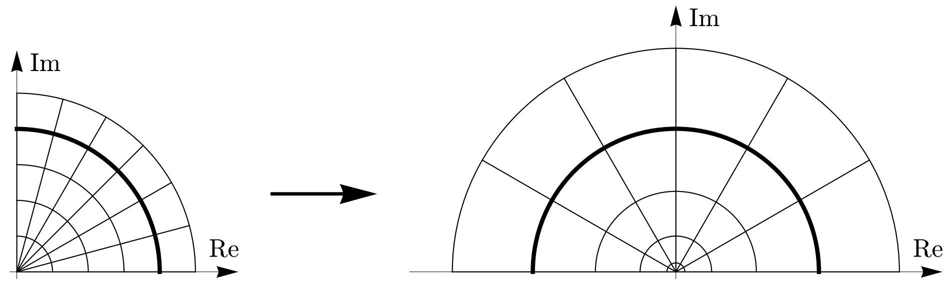

Figure 4: The image of a quarter-circle in the first quadrant under the map $f(z) = z^2$. The unit circle has been highlighted.

2. Complex Maps

A **complex map** is any continuous function from the complex plane to itself. If f is a complex map, then we can iterate f to obtain a **complex dynamical system**.

EXAMPLE 5 Multiplication by a Constant

Let $f\colon \mathbb{C} \to \mathbb{C}$ be the map $f(z) = cz$, where c is a complex constant. If we let $c = ae^{i\phi}$ and $z = re^{\theta}$, this map can be written

$$f\big(re^{i\theta}\big) \;=\; (ar)e^{i(\theta+\phi)}.$$

That is, f stretches the complex plane by a factor of a, and rotates the plane counterclockwise through an angle of ϕ.

The map f always has a fixed point at 0. If $a < 1$, then this fixed point is attracting, and every orbit spirals in towards the fixed point. If $a > 1$, then 0 is repelling, and every orbit spirals out toward infinity. When $a = 1$, the map is simply a rotation, with each orbit traveling along a circle centered at the origin. ■

EXAMPLE 6 Complex Squaring

Let $f\colon \mathbb{C} \to \mathbb{C}$ be the map $f(z) = z^2$. In polar coordinates, this map can be written

$$f\big(re^{i\theta}\big) \;=\; r^2e^{i(2\theta)}.$$

That is, f squares the radius of any complex number, and doubles the angle.

The effect of f on the first quadrant is shown in Figure 4. Geometrically, f maps the first quadrant onto the upper half plane, doubling the angles between radial lines. The second quadrant would map onto the lower half plane, the third onto the upper half plane again, and the fourth onto the lower half plane. ■

Complex Derivatives

We now define the derivative of a complex map:

Definition: Complex Derivative/Holomorphic Map
Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex map. The **derivative** of f is the function $f'\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f'(z) \;=\; \lim_{h\to 0} \frac{f(z+h)-f(z)}{h},$$

where h is a complex number. If the derivative $f'(z)$ exists for all $z \in \mathbb{C}$, then f is said to be **holomorphic**.

This definition is algebraically identical to the standard definition of the derivative for a function of a real variable. As a result, all of the usual differentiation rules—including the power rule, the product rule, and the chain rule—can be used for holomorphic maps.

EXAMPLE 7 Let $f\colon \mathbb{C} \to \mathbb{C}$ be the map

$$f(z) = iz^2 + (3+i)z + 5.$$

Then f is holomorphic, with

$$f'(z) \;=\; 2iz + (3+i).$$

More generally, any polynomial function $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic, with f' being a polynomial of smaller degree. ∎

Recall that for a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, the derivative $f'(x)$ can be interpreted as the factor by which f stretches the line at the point x. This follows from the formula

$$\Delta f \;\approx\; f'(x)\cdot \Delta x.$$

where Δx is the difference between two points in the domain, and Δf is the difference between corresponding points in the range.

A similar formula holds for the derivative of a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$. Specifically, if Δz is the difference between two points in the domain and Δf is the difference between the corresponding points in the range, then

$$\Delta f \;\approx\; f'(z)\cdot \Delta z.$$

In this case, however, multiplication by $f'(z)$ can both stretch and rotate the complex plane. In particular, the modulus $|f'(z)|$ will be the "stretch factor" near z, while the argument $\arg\bigl(f'(z)\bigr)$ will be the "rotation factor".

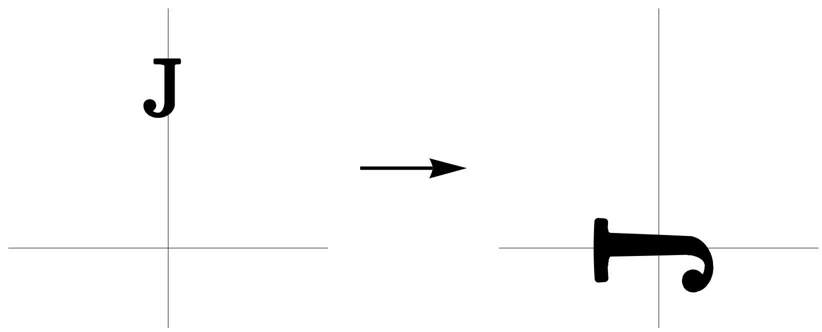

Figure 5: The image of a letter J under the map $f(z) = z^2 + 1$. Since the derivative is $2i$, the map stretches the J by a factor of 2 and rotates it 90° counterclockwise.

EXAMPLE 8 Geometric Meaning of the Derivative

Let $f(z) = z^2 + 1$, and consider the behavior of this map near $z = i$. It is easy to check that

$$f(i) \;=\; 0 \qquad \text{and} \qquad f'(i) \;=\; 2i.$$

Since $|f'(i)| = 2i$, the map f should have a "stretch factor" of 2 near the point i. Moreover, since $\arg\bigl(f'(i)\bigr) = \pi/2 = 90°$, the map f should have a "rotation factor" of 90° near this point.

Figure 5 shows the effect of this map on a letter J drawn in the complex plane. The first part of the figure shows the domain, with the letter J centered at the point i, while the second part shows the image of the J under the map f. Since $f(i) = 0$, the image of the letter J is centered at the origin. The size and orientation of the J is determined by the derivative: the image has been stretched by a factor of two, and has been rotated 90° counterclockwise. ■

Stability and Chaos

Because $|f'(p)|$ represents the stretch factor of a holomorphic map, we get the following test for the stability of a fixed point:

Theorem 4 Stability of Fixed Points

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map, and let $p \in \mathbb{C}$ be a fixed point for f.

1. *If $|f'(p)| < 1$, then p is a sink.*

2. *If $|f'(p)| > 1$, then p is a source.*

EXAMPLE 9 Complex Fixed Points
Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function $f(z) = iz^2 + z + 2$. We can find the fixed points for f by solving the equation

$$iz^2 + z + 2 = z.$$

This equation has two solutions, namely $z = 1 + i$ and $z = -1 - i$, so each of these is a fixed point for f.

To determine whether these are sources or sinks, we must find the derivative $f'(z)$. Since the complex derivative is defined in essentially the same way as the real derivative, all of the usual differentiation rules apply, and in particular

$$f'(z) \;=\; 2iz + 1.$$

Then

$$|f'(1+i)| \;=\; |-1+2i| \;=\; \sqrt{5} \;>\; 1,$$

so $1 + i$ is a source. Similarly,

$$|f'(-1-i)| \;=\; |3-2i| \;=\; \sqrt{13} \;>\; 1,$$

so $-1 - i$ is a source as well. ■

Because the Chain Rule works the same way for holomorphic maps as it does for one-dimensional maps, we get the following test for the stability of periodic cycles:

Theorem 5 Stability of Cycles

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic map, and let $p_1, \ldots, p_n$ be a periodic cycle for f.

1. *If $|f'(p_1) \cdots f'(p_n)| < 1$, then $\{p_1, \ldots, p_n\}$ is a periodic sink.*

2. *If $|f'(p_1) \cdots f'(p_n)| > 1$, then $\{p_1, \ldots, p_n\}$ is a periodic source.*

More generally, we can define Lyapunov numbers for orbits in the complex plane:

Definition: Lyapunov Number
Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic map. The **Lyapunov number** of an orbit $\{p_1, p_2, p_3, \ldots\}$ for f is the limit

$$L(p_1) \;=\; \lim_{n\to\infty} |f'(p_1) \cdots f'(p_n)|^{1/n}.$$

If $L(p_1) > 1$ and p_1 is not pre-periodic, then the orbit of p_1 is said to be **chaotic**.

Exercises

1. Evaluate each of the following expressions. Write your answer in the form $a+bi$.

   a) $\dfrac{1+2i}{3-5i}$.

   b) $e^{2i\pi/3}$.

   c) $(1-i)^9$.

2. Use Euler's formula and the Binomial Theorem to find a formula for $\cos(7x)$.

3. Find all roots of the polynomial z^6-1. Express your answers in the form $a+bi$.

4. Let $f(z)=z^2+\frac{15}{32}i$. Find and classify the fixed points of f as sources or sinks.

5. Find and classify the period-two orbit of the map $f(z)=1+iz^2$.