Julia Sets and the Mandelbrot Set

Julia sets are certain fractal sets in the complex plane that arise from the dynamics of complex polynomials. These notes give a brief introduction to Julia sets and explore some of their basic properties.

1. The Filled Julia Set

Consider a polynomial map $f\colon \mathbb{C} \to \mathbb{C}$, such as $f(z) = z^2 - 1$. What are the dynamics of such a map? Certainly many orbits under this map diverge to infinity:

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 8, \quad p_4 = 63, \quad p_5 = 3968, \quad \ldots.$$

On the other hand, some orbits manage to remain bounded:

$$p_1 = 0.5, \quad p_2 = -0.75, \quad p_3 = -0.4375, \quad p_4 \approx -0.8086, \quad \ldots$$

Definition: Types of Orbits
Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial map, and let $\{p_1, p_2, p_3, \ldots\}$ be an orbit under f.

1. We say that the orbit is **bounded** if all the points are contained in some disk of finite radius centered at the origin. That is, the orbit is bounded if there exists a constant $R > 0$ so that $|p_n| \leq R$ for all $n \in \mathbb{N}$.

2. We say that the orbit **escapes to infinity** if $|p_n| \to \infty$ as $n \to \infty$.

This leads to the question: for what initial points p_1 will the orbit under f remain bounded, and for what initial points will the orbit escape to infinity?

Definition: Filled Julia Set/Basin of Infinity
Let $f: \mathbb{C} \to \mathbb{C}$ be a polynomial map.

1. The **filled Julia set** for f is the set
$$\{p_1 \in \mathbb{C} \mid \text{the orbit of } p_1 \text{ is bounded}\}.$$

2. The **basin of infinity** for f is the set
$$\{p_1 \in \mathbb{C} \mid \text{the orbit of } p_1 \text{ escapes to infinity}\}.$$

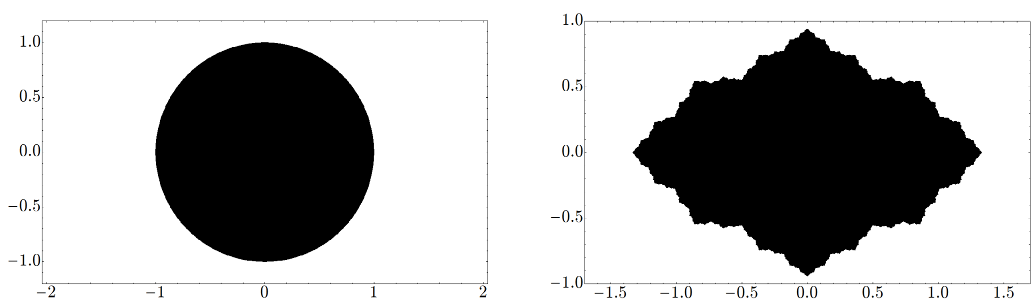
EXAMPLE 1 Repeated Squaring
Let $f: \mathbb{C} \to \mathbb{C}$ be the map $f(z) = z^2$. As we have seen, f squares the radius and doubles the angle of any complex number:

$$f\left(re^{i\theta}\right) = r^2 e^{i(2\theta)}.$$

Given an initial point p_1, the orbit $\{p_1, p_2, p_3, \ldots\}$ has the following behavior, depending on the relationship of p_1 to the unit circle:

1. If $|p_1| < 1$, then the orbit converges to 0, which is an attracting fixed point for f
2. If $|p_1| = 1$, then the entire orbit lies on the unit circle.
3. If $|p_1| > 1$, then the orbit escapes to infinity.

The orbits remain bounded in the first two cases, so the filled Julia set for f is the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, shown in Figure 1a. The basin of infinity is the complement of this disk. ■



(a) (b)

Figure 1: (a) Filled Julia set for z^2. (b) Filled Julia set for $z^2 - 4/9$.

EXAMPLE 2 The Function $z^2 - 4/9$
Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2 - 4/9$. Since

$$f(-1/3) \;=\; -1/3 \qquad \text{and} \qquad |f'(-1/3)| \;=\; |-2/3| \;<\; 1,$$

this function has an attracting fixed point at $-2/3$.

The filled Julia set for f is shown in Figure 1b. The dynamics of f are very similar to the dynamics of the squaring map: the orbit of any point in the interior of the filled Julia set converges to the fixed point, while the orbit of any point outside the filled Julia set escapes to infinity. Points that lie precisely on the boundary of the filled Julia set remain on the boundary as the map is iterated. ■

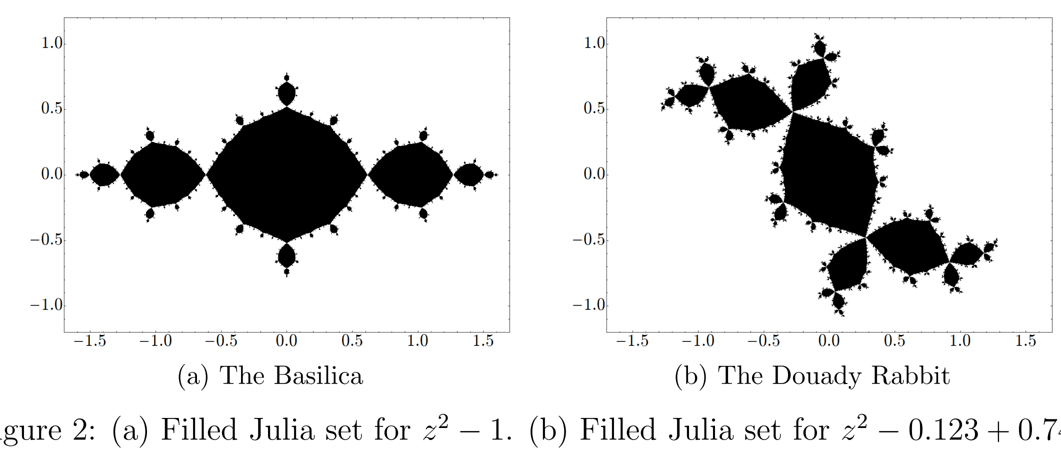
EXAMPLE 3 The Basilica and the Rabbit
Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2 - 1$. Although this function does not have an attracting fixed point, it is easy to check that

$$f(0) \;=\; -1, \qquad f(-1) \;=\; 0, \qquad \text{and} \qquad |f'(0)\, f'(-1)| \;=\; |(0)(-2)| \;=\; 0 \;<\; 1$$

so $\{-1, 0\}$ is an attracting two-cycle.

The filled Julia set for this function is shown in Figure 2b. This Julia set is known as the **Basilica**, because of its resemblance to St. Peter's Basilica. Orbits of points in the interior of the filled Julia set are attracted to the 2-cycle, while orbits of points outside escape to infinity. As with the previous examples, the orbit of any point precisely on the boundary of the filled Julia set remains on the boundary.

Another function with similar dynamics is the quadratic $f(z) = z^2 - 0.123 + 0.754i$. This polynomial has an attracting 3-cycle, and most orbits either asymptotically approach the 3-cycle or escape to infinity. The filled Julia set for this function is shown in Figure 2b. It is known as the **Douady rabbit**, named for the French mathematician Adrien Douady. ■



(a) The Basilica (b) The Douady Rabbit

Figure 2: (a) Filled Julia set for $z^2 - 1$. (b) Filled Julia set for $z^2 - 0.123 + 0.745i$.

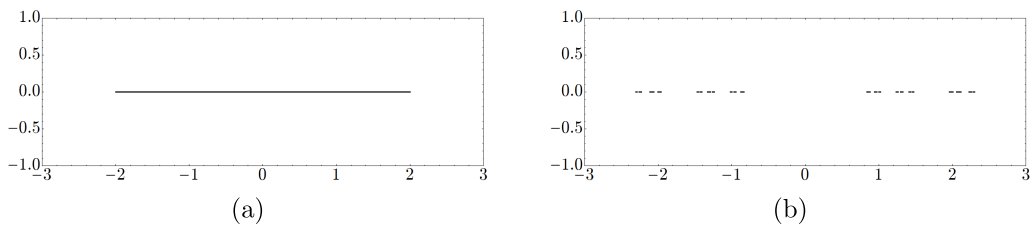


Figure 3: (a) Filled Julia set for $z^2 - 2$. (b) Filled Julia set for $z^2 - 3$.

EXAMPLE 4 A Line Segment and a Cantor Set

Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2 - 2$. Though it may not be obvious, this function has no attracting cycles—or attractor of any kind—so almost every orbit for this function escapes to infinity.

However, this function still has a few bounded orbits. For one thing, -1 and 2 are fixed points for this function. Though neither fixed point is attracting, the orbits of the fixed points themselves remain bounded, and therefore belong to the filled Julia set. Similarly, it can be shown that the function f has a 2-cycle, two 3-cycles, three 4-cycles, six 5-cycles, and so forth, all of them repelling. All of these periodic points must lie in the filled Julia set.

It turns out that the filled Julia set for this function is the line segment $[-2, 2]$ on the real axis, as shown in Figure 3a. This line segment maps to itself under the function f, so any orbit that begins on this line segment remains on the line segment for the entire duration. Any orbit that does not begin on the line segment escapes to infinity.

Another function with similar behavior is $f(z) = z^2 - 3$. Again, this function has no attractor, so almost every orbit diverges to infinity. In this case however, the filled Julia set is a fractal subset of the real line that resembles the Cantor set, as shown in Figure 3b. ■

All of the examples we have seen share a few basic traits, which are universal for all filled Julia sets of polynomials.

Theorem 1 Properties of the Filled Julia Set

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial function. Let B be the basin of infinity for f, and let J be the filled Julia set for f. Then

1. *B and J are disjoint, and $B \cup J = \mathbb{C}$.*

2. *Both B and J are invariant under f, i.e. $f(B) = B$ and $f(J) = J$.*

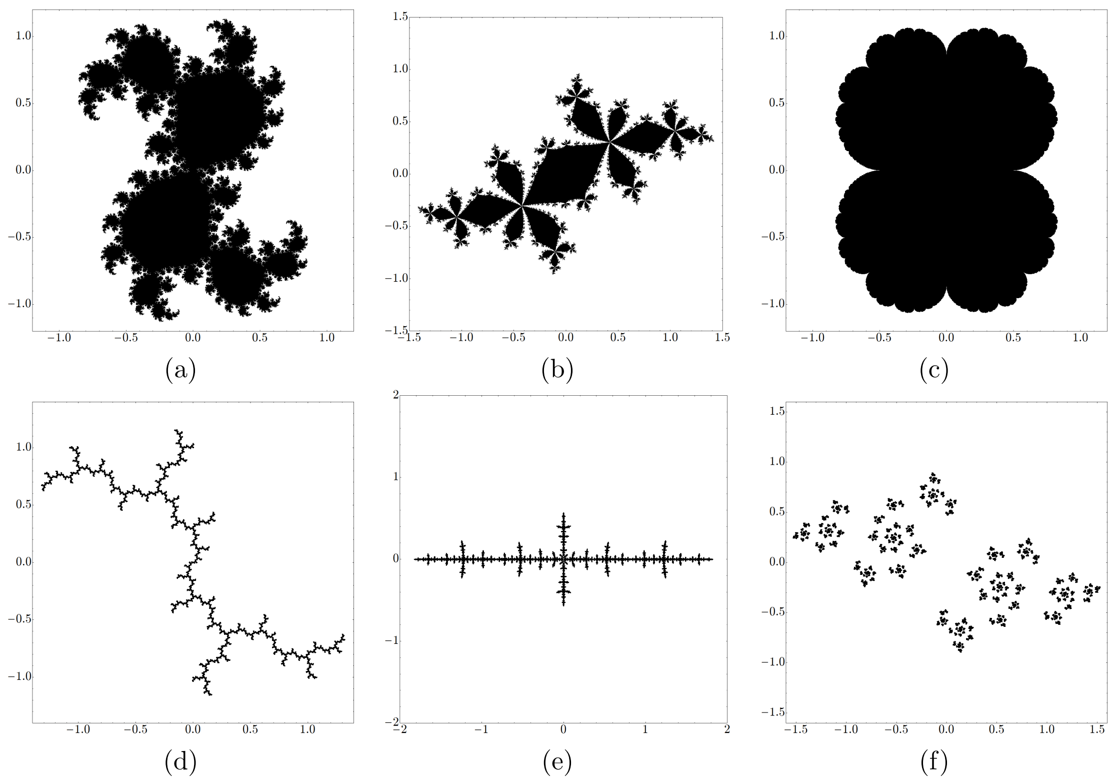


Figure 4: Filled Julia sets for the maps $z^2 + c$, where (a) $c = 0.37 + 0.16i$ (b) $c = -0.50 - 0.56i$ (c) $c = -0.25$ (d) $c = i$ (e) $c = -1.5$ (f) $c = -0.75 + 0.25i$

These examples only touch the surface of possible Julia sets, even for quadratic maps. Several more filled Julia sets for quadratic polynomials are shown in Figure 4. All of these are for polynomials of the form $f(z) = z^2 + c$ for some constant c, a class of polynomials known as the **quadratic family**.

2. Connectedness

Most of the filled Julia sets we have seen have been connected, meaning that they have only one component or piece. The exceptions are the Cantor set (Figure 3b) and the filled Julia set in Figure 4f, each of which has infinitely many different pieces. The following definition makes these notions precise:

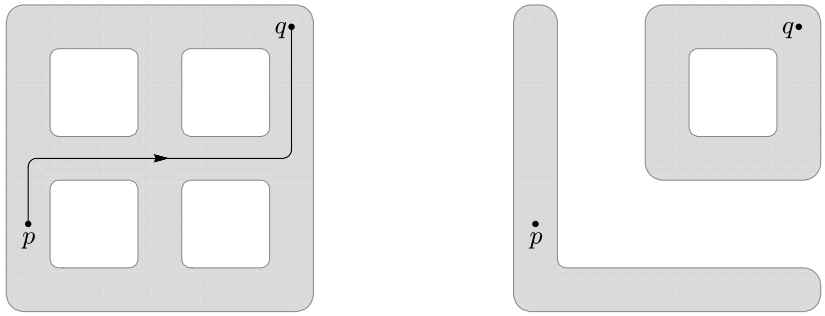


Figure 5: (a) A path-connected subset of $\mathbb{C}$. (b) A disconnected subset of $\mathbb{C}$.

Definition: Connectedness
Let S be a subset of the complex plane.

1. Two points $p, q \in S$ are **connected by a path in *S*** if there exists a continuous path lying entirely in S that begins at p and ends at q.

2. We say that S is **path-connected** if every pair of points in S are connected by a path in S. Otherwise, S is **disconnected**.

The word "path" in this definition refers to a parametric path, i.e. a continuous function $\gamma\colon [a, b] \to \mathbb{C}$ from a closed interval to the complex plane. Such a path begins at the point $\gamma(a)$ and ends at the point $\gamma(b)$.

The notion of path-connectedness is illustrated in Figure 5. Part (a) of this figure shows a path-connected subset of the complex plane, together with a sample path between two points p and q. Part (b) shows a subset of $\mathbb{C}$ that is not path-connected, since there is no path connecting the points p and q.

Intuitively, a set is path-connected if it only has one piece. The following definition makes the notion of "piece" precise:

Definition: Path Component
Let S be a subset of the complex plane, and let $p \in S$. The **path component** of S containing p is the set

$$\{q \in S \mid p \text{ and } q \text{ are connected by a path in } S\}.$$

That is, the path component of S containing p is the "piece" of S that contains p. For example, the set in Figure 5b has two path components, one containing the point p and one containing the point q.

The two examples we have seen of disconnected Julia sets are the Cantor set

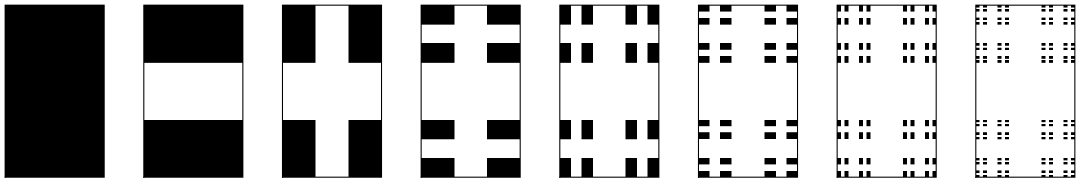


Figure 6: The first eight stages in the construction of Cantor dust.

(Figure 3b) and the spiral smattering of points shown in Figure 4f. Obviously, both of these sets have an infinite number of path components. In fact, these sets are as disconnected as possible, in the following sense:

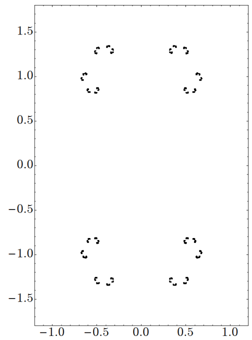
Definition: Totally Disconnected
A subset S of the complex plane is **totally disconnected** if no pair of distinct points in S can be connected by a path in S.

That is, a set is totally disconnected if each of its path components is a single point. For example, the Cantor set is totally disconnected, since it is not possible to connect any pair of points in the Cantor set with a path that lies entirely in the Cantor set. The following example describes a totally disconnected fractal in the plane:

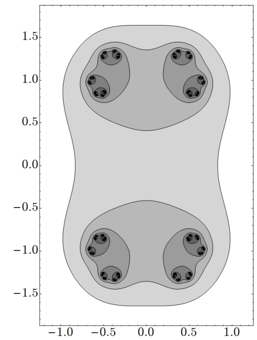
EXAMPLE 5 Cantor Dust
Figure 6 shows the first eight stages in the construction of totally disconnected fractal in the plane known as **Cantor dust**. This fractal can be thought of as a planar version of the Cantor set, with rectangles instead of intervals. The structure of this fractal is very similar to the structure of the Cantor set: it has two main pieces, each of which has two main pieces, each of which has two main pieces, and so forth. ■

Many disconnected Julia sets have a structure similar to the Cantor set or the Cantor dust fractal. For example, Figure 7a shows the filled Julia set for the function $f(z) = z^2 + 1$. The structure of this Julia set is illustrated in Figure 7b. As you can see, the Julia set is totally disconnected, with structure very similar to that of the Cantor dust fractal.

The following theorem was proved in 1919 independently by the mathematicians Gaston Julia and Pierre Fatou:



(a)



(b)

Figure 7: (a) Julia set for $z^2 + 1$. (b) Structure of the Julia set.

Theorem 2 Fundamental Dichotomy Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial function, and let J be the Julia set for f. Then the set J is either path-connected or totally disconnected.

Incredibly, Julia and Fatou discovered and proved this theorem without having any pictures of filled Julia sets to work with. Though this represented a tremendous achievement, the work of Julia and Fatou was largely forgotten until it was rediscovered by Benoît Mandelbrot in the 1960's.

3. The Mandelbrot Set

Because Julia sets of polynomials are so complicated, much of the work in complex dynamics has focused on the simplest case, namely Julia sets for quadratic polynomials. Dynamicists prefer to work with a certain family of quadratic polynomials:

Definition: The Quadratic Family
The **quadratic family** is the family of quadratic polynomials of the form

$$f(z) \;=\; z^2 + c,$$

where c is a complex constant called the **parameter**.

Though it may seem restrictive to limit ourselves to members of the quadratic family, it turns out that any quadratic polynomial must have similar dynamics to some element of the quadratic family. This is the content of the following theorem:

Theorem 3 Conjugation of Quadratic Polynomials

Every quadratic polynomial is conjugate to some member of the quadratic family.

PROOF Let $f\colon \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial, say

$$f(z) \;=\; pz^2 + qz + r.$$

Let $C\colon \mathbb{C} \to \mathbb{C}$ be the function

$$C(z) \;=\; \frac{2z - q}{2p}.$$

Then C is a continuous bijection, with inverse function

$$C^{-1}(z) \;=\; pz + \frac{q}{2}.$$

Furthermore,

$$\begin{aligned}(C^{-1} \circ f \circ C)(z) \;&=\; p\left(p\left(\frac{2z-q}{2p}\right)^2 + q\left(\frac{2z-q}{2p}\right) + r\right) + \frac{q}{2} \\ &=\; z^2 \;+\; \frac{q(2-q) + 4pr}{4},\end{aligned}$$

which is the desired member of the quadratic family. ■

As a consequence of this theorem, the Julia set for any quadratic polynomial must be similar to the Julia set for some member of the quadratic family. Thus, it makes sense to restrict our investigation of quadratic Julia sets to elements of this family.

We are now ready to define the Mandelbrot set. This set was first studied extensively in 1979 by the mathematician Benoît Mandelbrot, who discovered many of its incredible properties. It has since become one of the most famous objects in mathematics:

Definition: The Mandelbrot Set
The **Mandelbrot set** M is the following subset of the complex plane:

$$M \;=\; \{c \in \mathbb{C} \mid \text{the filled Julia set for } z^2 + c \text{ is path-connected.}\}$$

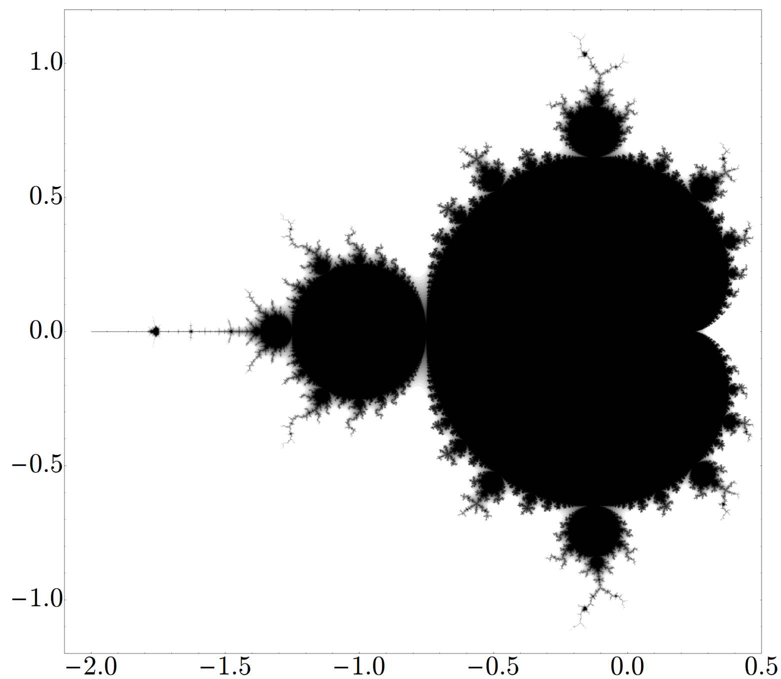


Figure 8: The Mandelbrot set.

The Mandelbrot set is shown in Figure 8. Note that each point in this figure corresponds to a filled Julia set, with the black points being path-connected Julia sets, and the white points being disconnected Julia sets. Figure 9 shows the filled Julia sets corresponding to three points in the Mandelbrot set.

Unlike other fractals we have seen, the structure of the Mandelbrot set is not self-similar. Instead, it has completely different local structures at different places. Figure 10 shows the successive stages of a zoom into the Mandelbrot set, with each row representing two orders of magnitude. As you can see, new structure is revealed at each scale, and the pattern does not seem to repeat. The same thing would happen at almost any point in the Mandelbrot set, with different structures revealed at each point.

You may be wondering how we construct pictures of portions of the Mandelbrot set. According to our definition, a point $c \in \mathbb{C}$ lies in the Mandelbrot set if the filled Julia set for c is path-connected. Though this is often obvious visually, how can we use a computer to test for the connectedness of the filled Julia set?

Mandelbrot gave an elegant solution to this problem:

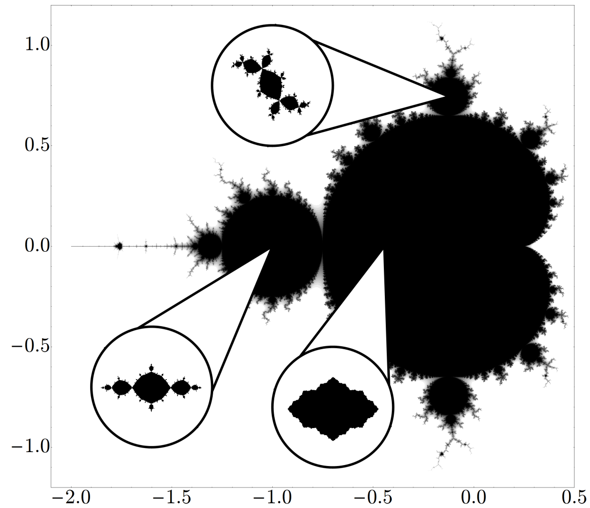


Figure 9: Filled Julia sets for three points in the Mandelbrot set.

Theorem 4 Mandelbrot's Criterion

Let $f(z) = z^2 + c$, where c is a complex constant. Then the filled Julia set for f is path-connected if and only if the orbit of 0 under f is bounded.

Corollary 5 Computing the Mandelbrot Set

The Mandelbrot set M can be defined as follows:

$$M = \{c \in \mathbb{C} \mid \text{the orbit of } 0 \text{ under } f(z) = z^2 + c \text{ is bounded}\}.$$

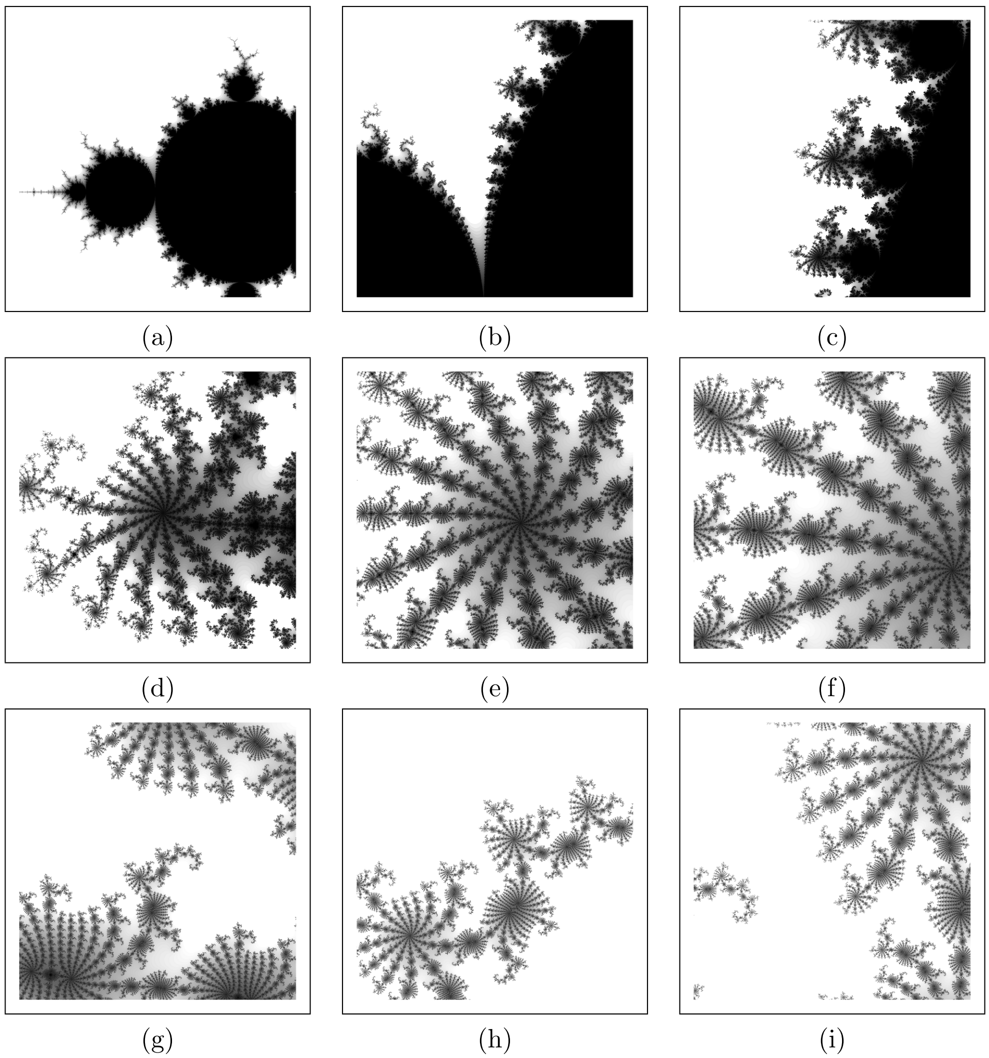


Figure 10: A zoom into the Mandelbrot set.