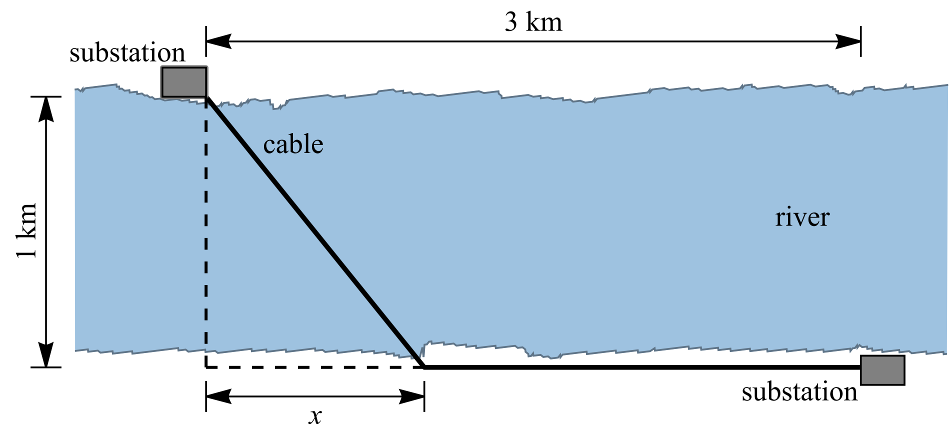
Optimization

Extra Practice Problems

1. Engineers wish to construct a power transmission cable connecting two power substations on opposite sides of a river.

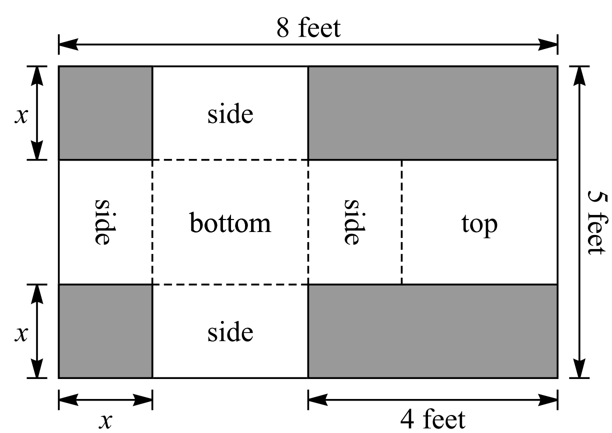


Underwater cable costs \$5 million/km, while cable on land costs \$3 million/km.

(a) Find a formula for the total cost of the cable as a function of the length x shown in the picture.

(b) Find the value of x that minimizes the total cost of the cable.

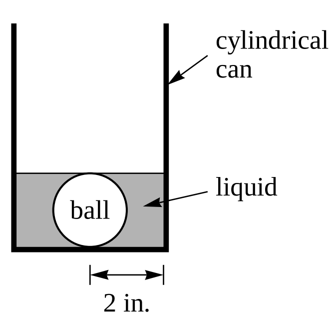
2. A closed box is to be made from a large sheet of cardboard by cutting out two squares and two rectangles, and then bending the sides and top into position.

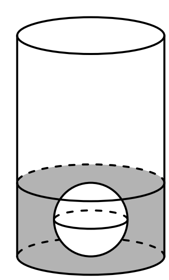


(a) Find a formula for the volume of the resulting box as a function of the length x.

(b) Find the maximum possible volume of the box.

3. A heavy metal ball is placed into a tin can, and then liquid is added until the top of the ball is just barely covered. The can has a radius of 2 inches and the ball has a radius of r and a volume of $\frac{4}{3}\pi r^3$.





(a) Find a formula for the volume of liquid in the can as a function of r.

(b) Find the value of r that maximizes the volume of liquid.

4. At 12:00 noon, a boat departs from a small island and sails north with a speed of 20 km/h. At the same time, a second boat is 30 km due east of the island, and is sailing towards the island at a speed of 15 km/h.

   (a) Let t be the time in hours after noon. Find a formula for the distance between the boats as a function of t.

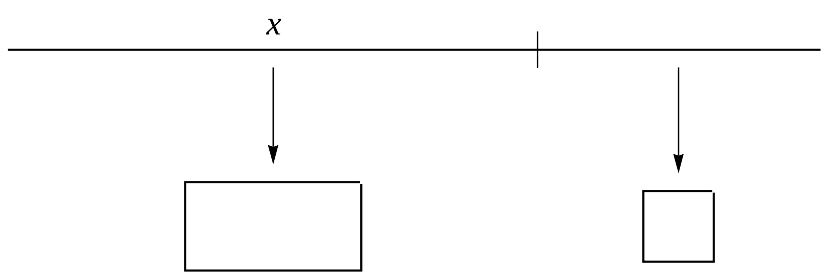
   (b) At what time are the two boats closest together?

5. A point P lies on the curve $y = \sqrt{x^2 + x + 1}$.

   (a) Find a formula for the distance from P to origin as a function of the x-coordinate of P.

   (b) Find the point on the curve $y = \sqrt{x^2 + x + 1}$ which is closest to the origin.

6. A piece of wire one meter long is cut into two parts.



   The first part is bent into the shape of a rectangle whose length is equal to twice its width, while the second part is bent into the shape of a square.

   (a) Find a formula for the total combined area of the two shapes as a function of x.

   (b) Find the value of x that minimizes the total area.

7. A manufacturer wishes to design a closed cylindrical can that uses 600 cm^2 worth of sheet metal.

   (a) Find a formula for the volume of the can as a function of the radius r.

   (b) Find the value of r that maximizes the volume of the can.

Solutions

1.

(a) The length of the underwater cable is $\sqrt{1+x^2}$, and the length of the land cable is $3-x$, so the total cost is

$$\boxed{5\sqrt{1+x^2}+3(3-x) \text{ million dollars}}$$

(b) We set the derivative equal to zero:

$$5\,\frac{1}{2\sqrt{1+x^2}}(2x)-3=0$$

which simplifies to

$$\frac{5x}{\sqrt{1+x^2}}=3.$$

Squaring both sides yields

$$\frac{25x^2}{1+x^2}=9$$

Solving for x yields $\boxed{x=3/4 \text{ km}}$.

2.

(a) The box has a length of $4-x$, a width of $5-2x$, and a height of x, so

$$V \;=\; (4-x)(5-2x)x \;=\; \boxed{2x^3-13x^2+20x}$$

(b) Setting the derivative equal to 0 gives $6x^2-26x+20=0$, and using the quadratic formula yields $x=1$ or $x=10/3$. But $x=10/3$ is impossible since the width would be negative, so the maximum volume must be when $x=1$ foot. Plugging this into our formula for volume above gives a maximum volume of $\boxed{9 \text{ feet}^3}$.

3.

(a) The liquid and the ball together have the shape of a cylinder with a radius of 2 inches and a height of $2r$. This cylinder has a volume of $\pi(2)^2(2r)=8\pi r$. Since the ball has a volume of $\frac{4}{3}\pi r^3$, the liquid must have a volume of $\boxed{8\pi r-\frac{4}{3}\pi r^3}$.

(b) Setting the derivative equal to zero gives the equation $8\pi-4\pi r^2=0$, and solving for r yields $\boxed{r=\sqrt{2} \text{ inches}}$.

4.

(a) The distance from the first boat to the island is $20t$, and the distance from the second boat to the island is $30-15t$. These distances make a right triangle whose hypotenuse is the distance between the two boats, so the distance is $\boxed{\sqrt{(20t)^2+(30-15t)^2}}$.

(b) Setting the derivative equal to zero gives the equation

$$\frac{2(20t)(20)+2(30-15t)(-15)}{2\sqrt{(20t)^2+(30-15t)^2}}=0.$$

Multiplying through by the denominator yields

$$2(20t)(20)+2(30-15t)(-15)=0.$$

Simplifying and solving for t yields $\boxed{t=18/25 \text{ hours after noon}}$ (around 12:43 pm).

5.

(a) The coordinates of the point P are $\left(x,\sqrt{x^2+x+1}\right)$, so the distance from P to the origin is

$$\sqrt{x^2+\left(\sqrt{x^2+x+1}\right)^2}=\boxed{\sqrt{2x^2+x+1}}$$

(b) Setting the derivative equal to zero gives the equation

$$\frac{4x+1}{\sqrt{2x^2+x+1}}=0.$$

Multiplying through by the denominator yields $4x+1=0$, so $x=-1/4$. Thus the closest point on the curve to the origin is $\boxed{\left(-\frac{1}{4},\frac{\sqrt{13}}{4}\right)}$.

6.

(a) The rectangle has length $x/3$ and width $x/6$, and thus has area $x^2/18$. The other piece of wire has length $1-x$, which means the square has side length $dfrac1-x4$ and area $\frac{(1-x)^2}{16}$. Thus the total area is $\boxed{\frac{x^2}{18}+\frac{(x-1)^2}{16}}$.

(b) Setting the derivative equal to zero gives

$$\frac{x}{9}+\frac{x-1}{8}=0$$

and solving yields $\boxed{x=9/17 \text{ meters}}$.

7.

(a) Let h be the height of the can. The surface area is $2\pi r^2 + 2\pi rh$, so

$$2\pi r^2 + 2\pi rh = 600.$$

Solving for h gives $h = \dfrac{300 - \pi r^2}{\pi r}$. Then

$$V \;=\; \pi r^2 h \;=\; \pi r^2 \cdot \frac{300 - \pi r^2}{\pi r} \;=\; \boxed{300r - \pi r^3}$$

(b) Setting the derivative equal to zero gives

$$300 - 3\pi r^2 = 0$$

and solving yields $\boxed{r = \dfrac{10}{\sqrt{\pi}}\text{ cm}}$.