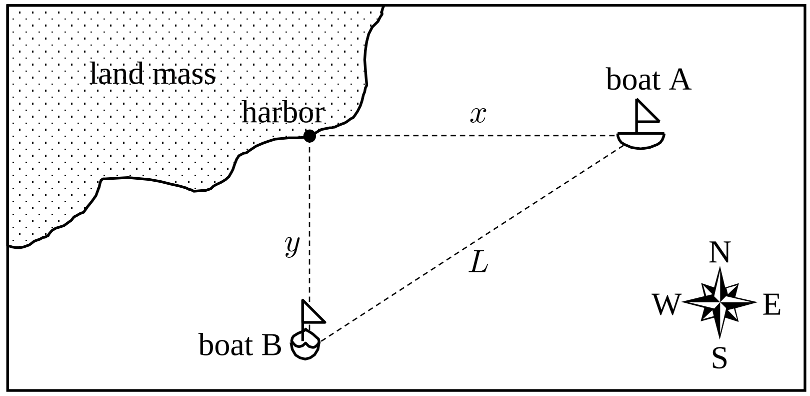
Related Rates

Extra Practice Problems

1. Two boats leave a harbor at the same time, boat A heading due east and boat B heading due south.

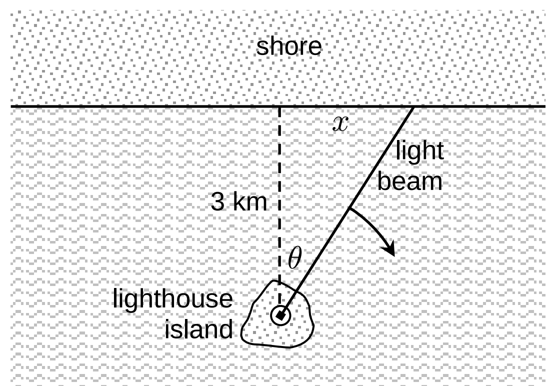
   

   (a) Find a formula relating the distances x, y, and L shown in the figure to the right.

   (b) Take the derivative of your formula from part (a) with respect to t.

   (c) Six hours after leaving, boat A is 80 miles from the harbor sailing at 15 miles/hour, and boat B is 60 miles from the harbor sailing at 20 miles/hour. How quickly is the distance between the boats increasing?

2. A lighthouse sits on a small island near a rocky shoreline, emitting a rotating beam of light. The lighthouse is 3 km from the shore, and it emits a beam of light that rotates at a rate of 8π rad/min.

   (a) Find a formula for x as a function of θ.

   (b) Take the derivative of your formula from part (a) to find a formula for $\dfrac{dx}{dt}$ in terms of θ and $\dfrac{d\theta}{dt}$.

   (c) How quickly is the end of the light beam moving along the shoreline when $\theta = \pi/6$ rad?

3. In physics, the energy stored in a stretched spring is determined by the equation

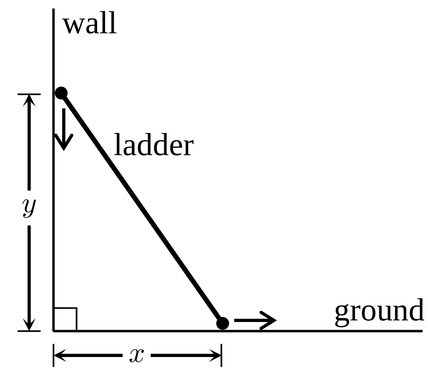
$$E = \frac{1}{2}kx^2$$

   where E is the energy, k is a constant (the "spring constant"), and x represents the distance that the spring has been stretched.

   (a) Find a formula for $\dfrac{dE}{dt}$ in terms of k, x, and $\dfrac{dx}{dt}$.

   (b) A spring with spring constant $k = 0.2$ Joules/cm^2 is being stretched at a rate of 1.5 cm/sec. How quickly is the energy stoored in the spring increasing at the moment that $x = 10$ cm?

4. A 10-foot-long ladder rests against a vertical wall.



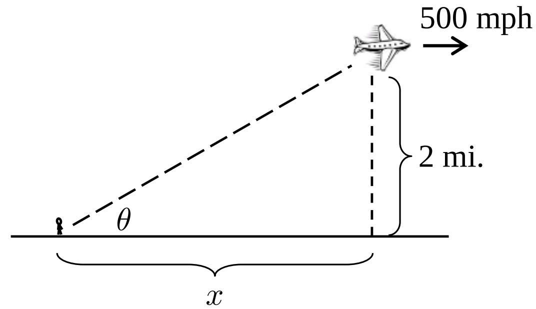
   (a) Find an equation relating the distances x and y shown in the figure to the right.

   (b) Take the derivative of your formula from part (a) to find an equation relating x, y, $\frac{dx}{dt}$, and $\frac{dy}{dt}$.

   (c) The ladder begins to slide, with the bottom end of the ladder moving away from the wall at a rate of 2 feet/second, and the top end of the ladder sliding down the wall. How quickly is the top end of the ladder sliding towards the ground at the moment that the bottom end of the ladder is 6 feet from the wall?

5. The radius of a cylinder is increasing at a rate of 2 cm/sec, while the height is decreasing at a rate of 3 cm/sec. How quickly is the volume of the cylinder increasing when the radius and height are both 10 cm?

6. An airplane flies directly over an observer standing on the ground. The picture to the right shows the position of the plane a short while later.



   (a) Find an equation relating the distance x and the angle θ.

   (b) Take the derivative of your equation from part (a) to find an equation relating $\frac{dx}{dt}$, θ, and $\frac{d\theta}{dt}$.

   (c) How quickly is the angle θ decreasing when $x = 1.5$ miles?

7. When expanding adiabatically (i.e. without gaining or losing heat), the pressure P and volume V of a sample of argon gas are related by the formula

$$PV^{5/3} = \text{constant}.$$

   (a) Take the derivative of the above equation to find a formula relating P, V, $\frac{dP}{dt}$, and $\frac{dV}{dt}$.

   (b) Suppose the volume of the sample is increasing at a rate of 0.4 liters/sec. How quickly is the pressure decreasing when the volume is 8 liters and the pressure is 60 kPa?

Solutions

1.

(a) $\boxed{x^2 + y^2 = L^2}$

(b) $\boxed{2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2L\frac{dL}{dt}}$

(c) We are given that $x = 80$, $\frac{dx}{dt} = 15$, $y = 60$, and $\frac{dy}{dt} = 20$. We can find L using the equation from part (a):

$$(80)^2 + (60)^2 = L^2.$$

Then $L = 100$. Substituting everything in to the equation from part (b) gives

$$2(80)(15) + 2(60)(20) = 2(100)\frac{dL}{dt}$$

Solving, we find that $\frac{dL}{dt} = \boxed{24 \text{ miles/hour}}$.

2.

(a) $\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{x}{3}$, so $\boxed{x = 3\tan(\theta)}$.

(b) $\boxed{\frac{dx}{dt} = 3\sec^2(\theta)\,\frac{d\theta}{dt}}$

(c) Substituting in $\theta = \pi/6$ and $\frac{d\theta}{dt} = 8\pi$ gives

$$\frac{dx}{dt} = 3\sec^2\left(\frac{\pi}{6}\right)(8\pi) = \frac{24\pi}{\big(\cos(\pi/6)\big)^2} = \frac{24\pi}{\big(\sqrt{3}/2\big)^2} = \frac{24\pi}{3/4} = \boxed{32\pi \text{ km/min}}$$

3.

(a) $\boxed{\frac{dE}{dt} = kx\frac{dx}{dt}}$

(b) Substituting in $k = 0.2$, $x = 10$, and $\frac{dx}{dt} = 1.5$ gives

$$\frac{dE}{dt} = (0.2)(10)(1.5) = \boxed{3 \text{ Joules/sec}}$$

4.

(a) Since the ladder is 10 feet long, $\boxed{x^2 + y^2 = 100}$

(b) $\boxed{2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0}$

(c) We are given that $x = 6$ and $\frac{dx}{dt} = 2$. We can find y using the equation from part (a):

$$(6)^2 + y^2 = 100$$

Then $y = 8$. Substituting everything in to the equation from part (b) gives

$$2(6)(2) + 2(8)\frac{dy}{dt} = 0$$

Solving, we find that $\frac{dy}{dt} = -1.5$ feet/sec, which is negative since y is decreasing. Thus y is decreasing at a rate of $\boxed{1.5 \text{ feet/second}}$.

5. The volume of a cylinder is

$$V = \pi r^2 h.$$

Taking the derivative with respect to t gives

$$\frac{dV}{dt} = \pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right).$$

Substituting in $\frac{dt}{dt} = 2$, $\frac{dh}{dt} = -3$ (negative since h is decreasing), and $r = h = 10$ gives

$$\frac{dV}{dt} = \pi\big(2(10)(2)(10) + (10)^2(-3)\big) = \boxed{100\pi \text{ cm}^3/\text{sec}}$$

6.

(a) $\tan(\theta) = \dfrac{\text{opp}}{\text{adj}} = \dfrac{2}{x}$, so $x\tan(\theta) = 2$. This is a pretty good answer, but it makes things slightly easier to divide through by $\tan(\theta)$ to get $\boxed{x = 2\cot(\theta)}$.

(b) $\boxed{\dfrac{dx}{dt} = -2\csc^2(\theta)\,\dfrac{d\theta}{dt}}$

(c) We are given that $x = 1.5$ miles and $\dfrac{dx}{dt} = 500$ miles/hour. Using the Pythagorean theorem, the length of the hypotenuse is

$$\sqrt{(1.5\text{ mi.})^2 + (2\text{ mi.})^2} = 2.5\text{ mi.}$$

Then $\csc(\theta) = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{2.5}{2} = \dfrac{5}{4}$. Substituting everything in gives

$$500 = -2\left(\frac{5}{4}\right)^2\frac{d\theta}{dt}.$$

Solving yields $\dfrac{d\theta}{dt} = -160$, which is negative since θ is decreasing, so θ is decreasing at a rate of $\boxed{160\text{ rad/hour}}$.

7.

(a) The product rule gives $\boxed{\dfrac{dP}{dt}V^{5/3} + \dfrac{5}{3}PV^{2/3}\dfrac{dV}{dt} = 0}$

(b) Substituting in $V = 8$, $\dfrac{dV}{dt} = 0.4$, and $P = 60$ gives

$$\frac{dP}{dt}(8)^{5/3} + \frac{5}{3}(60)(8)^{2/3}(0.4) = 0$$

Since $8^{1/3} = 2$, we know that $8^{5/3} = 2^5 = 32$ and $8^{2/3} = 2^2 = 4$, so

$$\frac{dP}{dt}(32) + \frac{5}{3}(60)(4)(0.4) = 0$$

Solving gives $\dfrac{dP}{dt} = -5$, which is negative since P is decreasing. The units for P seem to be kPa, and the units for time are seconds, so the pressure is decreasing at a rate of $\boxed{5\text{ kPa/sec}}$.