Antiderivatives

Study Guide

Problems in parentheses are for extra practice.

1. Antiderivatives and indefinite integrals

An **antiderivative** for a given function $f(x)$ is a function whose derivative is $f(x)$. For example,

$$F(x) = \frac{1}{3}x^3 \quad \text{is an antiderivative for} \quad f(x) = x^2.$$

Any two antiderivatives for the same function differ by a constant. Thus the most general antiderivative for the function $f(x) = x^2$ is

$$F(x) = \frac{1}{3}x^3 + C$$

where C is an arbitrary constant. This is also known as an **indefinite integral**, and is written using the integral sign:

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

Problems: 1–16 odd, (1–16 even)

2. Initial Value Problems

An **initial value problem** tells you the derivative of a function as well as one value, and asks you to find the function. For example, we might might be asked to find a function $f(x)$ so that

$$f'(x) = x^2 \qquad \text{and} \qquad f(1) = 2.$$

Here are the steps for solving this problem:

1. Since $f'(x) = x^2$, we know that $f(x) = \frac{1}{3}x^3 + C$ for some constant C.
2. Plugging in $x = 1$ gives the equation $2 = \frac{1}{3}(1)^3 + C$, and solving for C yields $C = \frac{5}{3}$.

Thus $f(x) = \frac{1}{3}x^3 + \frac{5}{3}$.

One variation on this kind of problem gives you the *second* derivative as well as two values of x. For example, suppose we are asked to find a function $f(x)$ so that

$$f''(x) = 3x, \qquad f(1) = 4, \qquad \text{and} \qquad f(2) = 6.$$

The steps to solving this are as follows:

1. Since $f''(x) = 3x$, we know that $f'(x) = \frac{3}{2}x^2 + C$ for some constant C, so $f(x) = \frac{1}{2}x^3 + Cx + D$ for some constant D.
2. Plugging in $x = 1$ and $x = 2$ gives two equations:

   $$4 = \frac{1}{2} + C + D \qquad \text{and} \qquad 6 = 4 + 2C + D$$

   and solving gives $C = -\frac{3}{2}$ and $D = 5$.

Thus $f(x) = \frac{1}{2}x^3 - \frac{3}{2}x + 5$.

Problems: 17–22 odd, (17–22 even)

Exercises: Antiderivatives

1–6 ■ Find the most general form of the antiderivative for the given function.

1. $f(x) = 6x^2 + 2$

2. $f(x) = 8x^3 - \dfrac{x}{3}$

3. $f(x) = 4\sin x$

4. $f(x) = 3\cos x + \dfrac{1}{1+x^2}$

5. $f(x) = 3\sqrt{x} + x^{-2}$

6. $f(x) = e^{3x} + 4\sin(2x)$

7–10 ■ Evaluate the given indefinite integral.

7. $\displaystyle\int (5x^2+1)\,dx$

8. $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$

9. $\displaystyle\int \csc x \cot x\,dx$

10. $\displaystyle\int \frac{1}{e^x}\,dx$

11. Find the most general antiderivative for $x\cos(x^2)$.

12. Find the most general antiderivative for $x^2\cos x + 2x\sin x$.

13. Find the most general antiderivative for $\sin^2 x\cos x$.
(*Hint:* Use a power of $\sin x$.)

14. Find the most general antiderivative for $\dfrac{3x^2}{1+x^6}$.
(*Hint:* Use the inverse tangent.)

15. Given that $f'(x) = \sqrt[3]{x}$, find the value of $f(8) - f(1)$.

16. If $g(x)$ and $h(x)$ are differentiable functions, find the most general antiderivative of the function

$$f(x) = g'(h(x))\,h'(x) + 3\,g(x)^2\,g'(x).$$

17–22 ■ Find a function $f(x)$ that satisfies the given conditions.

17. $f'(x) = 6x^2 + 3x$ and $f(0) = 5$.

18. $f'(x) = \cos(2x) + \sec^2 x$ and $f(\pi/4) = 2$.

19. $f'(x) = e^x + e^{2x}$ and $f(\ln 4) = 15$.

20. $f''(x) = 3x$, $f'(0) = 2$, and $f(1) = 4$.

21. $f''(x) = -2$, $f(1) = 4$, and $f(3) = 2$.

22. $f''(x) = \dfrac{3}{\sqrt{x}}$, $f(1) = 8$, and $f(4) = 45$.

Answers to the Exercises

1. $2x^3 + 2x + C$ **2.** $2x^4 - \frac{x^2}{6} + C$ **3.** $-4\cos x + C$ **4.** $3\sin x + \tan^{-1} x + C$

5. $2x^{3/2} - x^{-1} + C$ **6.** $\frac{1}{3}e^{3x} - 2\cos(2x) + C$ **7.** $\frac{5}{3}x^3 + x + C$ **8.** $2\sqrt{x} + C$ **9.** $-\csc x + C$

10. $-e^{-x} + C$ **11.** $\frac{1}{2}\sin(x^2) + C$ **12.** $x^2 \sin x + C$ **13.** $\frac{1}{3}\sin^3 x + C$ **14.** $\tan^{-1}(x^3) + C$

15. $\frac{45}{4}$ **16.** $g(h(x)) + g(x)^3 + C$ **17.** $2x^3 + \frac{3}{2}x^2 + 5$ **18.** $\frac{1}{2}\sin(2x) + \tan x + \frac{1}{2}$

19. $e^x + \frac{1}{2}e^{2x} + 3$ **20.** $\frac{1}{2}x^3 + 2x + \frac{3}{2}$ **21.** $-x^2 + 3x + 2$ **22.** $4x^{3/2} + 3x + 1$