Derivatives

Study Guide

1. Definition of the Derivative

There are two limit definitions of the derivative, each of which is useful in different circumstances. For most problems, either definition will work. The two definitions are

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad \text{and} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Problems: 1–4

2. Power Rule

The power rule states that

$$\frac{d}{dx}\left[x^n\right] = n\,x^{n-1}$$

for any constant n. Remember that

$$x^{1/p} = \sqrt[p]{x} \qquad \text{and} \qquad x^{-p} = \frac{1}{x^p}.$$

Problems: 5–12

3. More Derivative Rules

We know that

$$\frac{d}{dx}\left[e^x\right] = e^x, \qquad \frac{d}{dx}\left[\sin x\right] = \cos x, \qquad \frac{d}{dx}\left[\cos x\right] = -\sin x, \qquad \frac{d}{dx}\left[\tan x\right] = \sec^2 x.$$

Also

$$\frac{d}{dx}\left[C\right] = 0, \qquad \frac{d}{dx}\left[x\right] = 1, \qquad \frac{d}{dx}\left[Cu\right] = C\,\frac{du}{dx}, \qquad \frac{d}{dx}\left[u+v\right] = \frac{du}{dx} + \frac{dv}{dx}$$

for any u and v and any constant C. Note that

$$\frac{d}{dx}\left[\frac{u}{C}\right] = \frac{d}{dx}\left[\frac{1}{C}\,u\right] = \frac{1}{C}\frac{du}{dx}$$

for any nonzero constant C.

Problems: 13–18

4. Tangent Lines

The point–slope formula states that the equation of the line through the point (x_0, y_0) with slope m is

$$y - y_0 = m(x - x_0).$$

This makes it easy to find the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$. Just compute $y_0 = f(x_0)$ and $m = f'(x_0)$ and plug them into the equation.

Problems: 19–22

5. Product Rule

Here is the rule for taking the derivative of a product:

$$\frac{d}{dx}\left[uv\right] = u\frac{dv}{dx} + v\frac{du}{dx}$$

The order of the two terms doesn't matter—all that matters is that each term has the derivative of u multiplied by the original v, and the other term has the derivative of v multiplied by the original u.

Problems: 23–26

6. Quotient Rule

Here is the rule for taking the derivative of a quotient:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Problems: 27–30

Answers to the Exercises

1.

Solution #1

$$f'(2) = \lim_{x\to 2} \frac{(3x^2+2)-14}{x-2} = \lim_{x\to 2} \frac{3x^2-12}{x-2} = \lim_{x\to 2} \frac{3\big(x^2-4\big)}{x-2} = \lim_{x\to 2} \frac{3(x+2)(x-2)}{x-2}$$

$$= \lim_{x\to 2} 3(x+2) = 12$$

Solution #2

$$f'(2) = \lim_{h\to 0} \frac{\big(3(2+h)^2+2\big)-14}{h} = \lim_{h\to 0} \frac{\big(3(4+4h+h^2)+2\big)-14}{h} = \lim_{h\to 0} \frac{12+12h+3h^2+2-14}{h}$$

$$= \lim_{h\to 0} \frac{12h+3h^2}{h} = \lim_{h\to 0} 12+3h = 12$$

2.

Solution #1

$$f'(3) = \lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{3x\left(\frac{1}{x}-\frac{1}{3}\right)}{3x(x-3)} = \lim_{x\to 3} \frac{3-x}{3x(x-3)} = \lim_{x\to 3} \frac{-(x-3)}{3x(x-3)} = \lim_{x\to 3} \frac{-1}{3x} = -\frac{1}{9}$$

Solution #2

$$f'(3) = \lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0} \frac{3(3+h)\left(\frac{1}{3+h}-\frac{1}{3}\right)}{3(3+h)h} = \lim_{h\to 0} \frac{3-(3+h)}{3(3+h)h} = \lim_{h\to 0} \frac{-h}{3(3+h)h}$$

$$= \lim_{h\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$$

3.

Solution #1

$$f'(25) = \lim_{x\to 25} \frac{2\sqrt{x}-10}{x-25} = \lim_{x\to 25} \frac{\big(2\sqrt{x}-10\big)\big(2\sqrt{x}+10\big)}{(x-25)\big(2\sqrt{x}+10\big)} = \lim_{x\to 25} \frac{4x-100}{(x-25)\big(2\sqrt{x}+10\big)}$$

$$= \lim_{x\to 25} \frac{4(x-25)}{(x-25)\big(2\sqrt{x}+10\big)} = \lim_{x\to 25} \frac{4}{2\sqrt{x}+10} = \frac{4}{20} = \frac{1}{5}$$

Solution #2

$$f'(25) = \lim_{h\to 0} \frac{2\sqrt{25+h}-10}{h} = \lim_{h\to 0} \frac{\big(2\sqrt{25+h}-10\big)\big(2\sqrt{25+h}+10\big)}{h\big(2\sqrt{25+h}+10\big)} = \lim_{h\to 0} \frac{4(25+h)-100}{h\big(2\sqrt{25+h}+10\big)}$$

$$= \lim_{h\to 0} \frac{4h}{h\big(2\sqrt{25+h}+10\big)} = \lim_{h\to 0} \frac{4}{2\sqrt{25+h}+10} = \frac{4}{20} = \frac{1}{5}$$

4.

Solution #1

$$f'(x_0) = \lim_{x\to x_0} \frac{\big(4x^2+1\big)-\big(4x_0^2+1\big)}{x-x_0} = \lim_{x\to x_0} \frac{4x^2-4x_0^2}{x-x_0} = \lim_{x\to x_0} \frac{4\big(x^2-x_0^2\big)}{x-x_0}$$

$$= \lim_{x\to x_0} \frac{4(x+x_0)(x-x_0)}{x-x_0} = \lim_{x\to x_0} 4(x+x_0) = 8x_0, \text{ so } f'(x) = 8x.$$

Solution #2

$$f'(x) = \lim_{h\to 0} \frac{\big(4(x+h)^2+1\big)-\big(4x^2+1\big)}{h} = \lim_{h\to 0} \frac{\big(4(x^2+2xh+h^2)+1\big)-\big(4x^2+1\big)}{h}$$

$$= \lim_{h\to 0} \frac{4x^2+8xh+4h^2+1-4x^2-1}{h} = \lim_{h\to 0} \frac{8xh+4h^2}{h} = \lim_{h\to 0} 8x+4h = 8x$$

5. $3x^2$ **6.** $\dfrac{1}{2\sqrt{x}}$ **7.** $-\dfrac{1}{x^2}$ **8.** $-\dfrac{2}{x^3}$ **9.** $\dfrac{1}{3\sqrt[3]{x^2}}$ **10.** $-\dfrac{1}{2x\sqrt{x}}$ **11.** $\dfrac{5}{2}x\sqrt{x}$ **12.** $\dfrac{2}{3\sqrt[3]{x}}$

13. $20x^3 - \dfrac{3}{2\sqrt{x}}$ **14.** $\dfrac{2}{5}x + 3e^x + \cos x$ **15.** $-\dfrac{40}{x^6} + 6$ **16.** $\sqrt{2} + \dfrac{4}{3}\sqrt[3]{x}$ **17.** $3e^2x^2 + 4\sin x$

18. $\dfrac{\sec^2 x}{\pi} - \dfrac{1}{x\sqrt{x}}$ **19.** $y - 3 = -3(x-1)$ **20.** $y - 2 = \dfrac{1}{12}(x-8)$ **21.** $y - 2 = -2\sqrt{3}\left(x - \dfrac{5\pi}{6}\right)$

22. $y - 3 = 3(x - \ln 3)$ **23.** $3x^2\sin x + x^3\cos x$ **24.** $8x + \dfrac{\tan x}{2\sqrt{x}} + \sqrt{x}\sec^2 x$

25. $4e^x(\cos x + \sin x)$ **26.** $\sin x\cos x + x\big(\cos^2 x - \sin^2 x\big)$ **27.** $\dfrac{x^2(x^2+3)}{(x^2+1)^2}$

28. $\dfrac{2+\sqrt{x}}{\big(1+\sqrt{x}\big)^2}$ **29.** $\dfrac{3(x-1)\cos x - 3\sin x - 1}{(x-1)^2}$ **30.** $\dfrac{(1+\cos x+\sin x)e^x}{(1+\cos x)^2}$