Integrals

Study Guide

Problems in parentheses are for extra practice.

1. Integrals and area

If $f(x) \geq 0$, the **integral** $\int_a^b f(x)\,dx$ represents the area under the graph of $f(x)$ and above the x-axis for $a \leq x \leq b$. This kind of integral is sometimes called a "definite integral", to distinguish it from an indefinite integral or antiderivative.

If the function $f(x)$ goes below the x-axis, then area above the graph of $f(x)$ and under the x-axis counts as *negative* for the integral.

In some cases, it's easiest to evaluate an integral by interpreting it as an area. For example, recall that

$$y = \sqrt{r^2 - x^2}$$

is the upper half of a circle of radius r centered at the origin. This makes it possible to compute integrals like

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

using the area formula for a circle.

Problems: 1, 2, 3, (4), 5, (6)

2. Rules for integrals

Table 5.6 on pg. 321 of the textbook lists several rules that definite integrals follow. You should make sure that you know these rules and that they make sense to you.

Problems: 7–12

3. The Fundamental Theorem of Calculus

The fundamental theorem of calculus states that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

for any differentiable function $f(x)$. That is, the area under the graph of the derivative is the change in the value of the original function.

This makes it possible to evaluate integrals by computing antiderivatives. For example,

$$\int_1^4 x^2\,dx \;=\; \left[\frac{1}{3}x^3\right]_1^4 \;=\; \frac{1}{3}(4)^3 - \frac{1}{3}(1)^3 \;=\; 21.$$

Problems: 13, 14, 15–26 odd, (15–26 even)

4. Derivatives of integrals

It follows from the Fundamental Theorem of Calculus that

$$\frac{d}{dx}\int_a^x f(t)\,dt \;=\; f(x)$$

for any constant a and any function $f(x)$.

More generally, it is always possible to evaluate the derivative of an integral using the chain rule. For example, to evaluate

$$\frac{d}{dx}\int_3^{x^2} \frac{\sin t}{t}\,dt$$

Let $F(t)$ be an antiderivative for $\dfrac{\sin t}{t}$. Then

$$\int_3^{x^2} \frac{\sin t}{t}\,dt \;=\; F(x^2) - F(3).$$

By the chain rule,

$$\frac{d}{dx}\int_3^{x^2} \frac{\sin t}{t}\,dt \;=\; \frac{d}{dx}\big[F(x^2) - F(3)\big] \;=\; F'(x^2)\,2x.$$

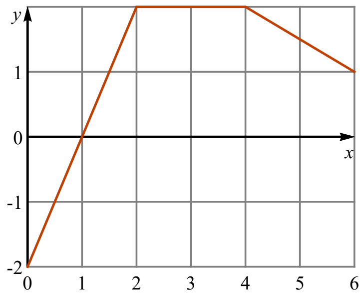
But $F'(t) = \dfrac{\sin t}{t}$, so $F'(x^2) = \dfrac{\sin(x^2)}{x^2}$, and therefore

$$\frac{d}{dx}\int_3^{x^2} \frac{\sin t}{t}\,dt \;=\; \frac{\sin(x^2)}{x^2}\,2x \;=\; \frac{2\sin(x^2)}{x}.$$

Problems: 27–32 odd, (27–32 even)

Exercises: Integration

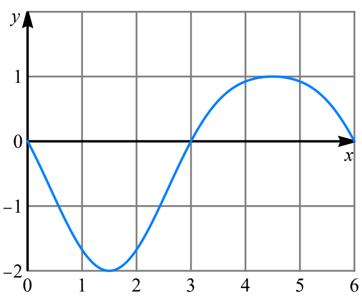
1. The following figure shows the graph of a function $f(x)$.



Use this graph to evaluate each of the following integrals.

(a) $\int_0^1 f(x)\,dx$ (b) $\int_0^2 f(x)\,dx$ (c) $\int_0^6 f(x)\,dx$

2. The following figure shows the graph of a function $f(x)$.



Determine whether each of the following integrals is positive or negative.

(a) $\int_1^3 f(x)\,dx$ (b) $\int_3^4 f(x)\,dx$ (c) $\int_0^6 f(x)\,dx$

3–6 ■ Evaluate the given integral by interpreting it as an area.

3. $\int_{-1}^{2} |x|\,dx$

4. $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

5. $\int_0^2 \sqrt{4-x^2}\,dx$

6. $\int_{0.5}^{2.5} \lceil x \rceil\,dx$

7–12 ■ Given that $\int_1^5 f(x)\,dx = 9$, $\int_3^5 f(x)\,dx = 3$, and $\int_1^5 g(x)\,dx = 2$, evaluate the following integrals.

7. $\int_1^3 f(x)\,dx$

8. $\int_4^4 g(x)\,dx$

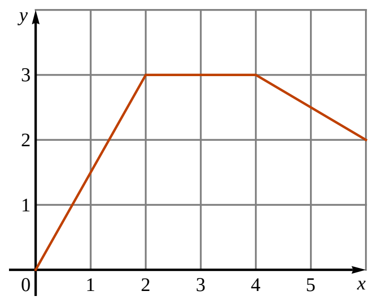
9. $\int_1^5 2f(x)\,dx$

10. $\int_1^5 \big(f(x)-g(x)\big)\,dx$

11. $\int_5^1 g(x)\,dx$

12. $\int_1^5 f(t)\,dt$

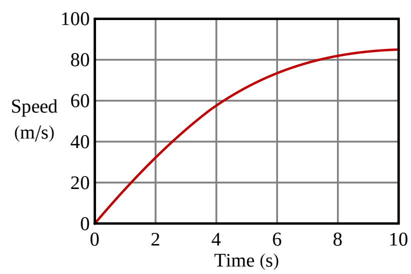
13. The following figure shows the graph of the **derivative** $f'(x)$ of a function $f(x)$.



Given that $f(2) = 1$, find each of the following values.

(a) $f(4)$ (b) $f(6)$ (c) $f(0)$

14. The following graph shows the speed of a racecar for the first ten seconds of a race.



Estimate the following quantities as accurately as you can.

(a) The speed of the racecar at $t = 6$ s.

(b) The acceleration of the racecar at $t = 2$ s.

(c) The distance traveled by the racecar over the first ten seconds of the race.

15–26 ■ Use the Fundamental Theorem of Calculus to evaluate the given integral.

15. $\int_2^4 3x^2\,dx$

16. $\int_0^3 (x^2+1)\,dx$

17. $\int_{-2}^{1} (x^3+5x+2)\,dx$

18. $\int_0^{\pi/6} \cos x\,dx$

19. $\int_0^{\pi} \sin x\,dx$

20. $\int_1^3 \frac{1}{x}\,dx$

21. $\int_0^6 2\,dx$

22. $\int_1^4 x^{-2}\,dx$

23. $\int_0^{\ln 3} e^x\,dx$

24. $\int_0^{\pi/4} \sec^2 x\,dx$

25. $\int_0^4 x^{3/2}\,dx$

26. $\int_0^1 \frac{1}{1+x^2}\,dx$

27–32 ■ Compute the given derivative.

27. $\dfrac{d}{dx}\displaystyle\int_1^x \cos(t^2)\,dt$

28. $\dfrac{d}{dx}\displaystyle\int_3^x e^{1/t}\,dt$

29. $\dfrac{d}{dx}\displaystyle\int_4^{x^3} \sin\left(\sqrt[3]{t}\right)\,dt$

30. $\dfrac{d}{dx}\displaystyle\int_{x^2}^3 \frac{1}{1+t^3}\,dt$

31. $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^x \tan(t^2)\,dt$

32. $\dfrac{d}{dx}\displaystyle\int_{\ln x}^{x^2} \sqrt{t^3+1}\,dt$

Answers

1. (a) -1 (b) 0 (c) 7 **2.** (a) negative (b) positive (c) negative **3.** $\dfrac{5}{2}$ **4.** $\dfrac{9\pi}{2}$ **5.** π **6.** 4

7. 6 **8.** 0 **9.** 18 **10.** 7 **11.** -2 **12.** 9 **13.** (a) 7 (b) 12 (c) -2

14. (a) roughly 73 m/s (b) roughly 15 m/s^2 (c) roughly 580 m **15.** 56 **16.** 12 **17.** $-21/4$ **18.** $1/2$

19. 2 **20.** $\ln(3)$ **21.** 12 **22.** $3/4$ **23.** 2 **24.** 1 **25.** $64/5$ **26.** $\pi/4$

27. $\cos(x^2)$ **28.** $e^{1/x}$ **29.** $3x^2 \sin x$ **30.** $-\dfrac{2x}{1+x^6}$ **31.** $\tan(x^2) - \dfrac{\tan x}{2\sqrt{x}}$ **32.** $2x\sqrt{x^6+1} - \dfrac{\sqrt{(\ln x)^3+1}}{x}$