L'Hôpital's Rule

Study Guide

Problems in parentheses are for extra practice.

1. The Basic Rule

If the limit of a fraction

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

has the form 0/0 or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \;=\; \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

For example:

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} = \lim_{x\to 0} \frac{3\cos(3x)}{5\cos(5x)} = \frac{3}{5}.$$

Note that the ∞/∞ case also includes situations where the numerator and/or denominator approaches $-\infty$.

Sometimes it is necessary to apply L'Hôpital's rule more than once. For example:

$$\lim_{x\to 0} \frac{1-\cos x}{e^x - 1 - x} \;=\; \lim_{x\to 0} \frac{\sin x}{e^x - 1} \;=\; \lim_{x\to 0} \frac{\cos x}{e^x} \;=\; \frac{1}{1} \;=\; 1.$$

Problems: Section 4.5 # 7, 13, 15, (21), (45)

2. Limits of the form $0 \cdot \infty$

Products of the form $0 \cdot \infty$ are also indeterminate. The usual method of handling them is to change to a fraction using

$$f(x)\,g(x) \;=\; \frac{f(x)}{g(x)^{-1}} \qquad \text{or} \qquad f(x)\,g(x) \;=\; \frac{g(x)}{f(x)^{-1}}$$

and then use L'Hôpital's rule. For example:

$$\lim_{x\to 0^+} x\,\ln x \;=\; \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \;=\; \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} \;=\; \lim_{x\to 0^+} -x \;=\; 0.$$

Problems: Section 4.5 # 25, 63, (65)

3. Limits of the form $\infty - \infty$

A difference of the form $\infty - \infty$ (or a sum of the form $-\infty + \infty$) is also indeterminate. Usually you can handle such a limit by first rearranging it using algebra.

Problems: Section 4.5 # 37, 41

4. Indeterminate Powers

Powers of the form 0^0, ∞^0, and 1^∞ are also indeterminate. To find the limit of such a power, use the identity

$$a^b = e^{b \ln a}$$

and then find the limit of the exponent. For example, the limit

$$\lim_{x \to \infty} x^{1/x}$$

has type ∞^0. Using the identity $a^b = e^{b \ln a}$, we see that $x^{1/x} = e^{\ln(x)/x}$. Since

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

we see that

$$\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\ln(x)/x} = e^0 = 1.$$

Problems: Section 4.5 # 51, (53), 59