Mean Value Theorem

Study Guide

1. Rolle's Theorem

Rolles theorem is helpful for showing that a function has a critical point on a certain interval. The hypotheses of the theorem are:

(a) $f(x)$ is continuous on the interval $[a, b]$,

(b) $f(x)$ is differentiable on the interval (a, b), and

(c) $f(a) = f(b)$.

In this case, the theorem states that there exists a c in (a, b) such that $f'(c) = 0$.

Hypotheses (1) and (2) are more complicated than they need to be. In most cases, the function $f(x)$ is differentiable on $[a, b]$, from which both (1) and (2) follow.

Problems: 1, (2), (3)

2. Mean Value Theorem

Thy hypotheses of the Mean Value Theorem are that:

(a) $f(x)$ is continuous on the interval $[a, b]$, and

(b) $f(x)$ is differentiable on the interval (a, b).

In this case, the theorem states that there exists a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

If you happen to know something about $f'(x)$, then you can apply this theorem to get information about $f(b) - f(a)$. For example, if you know that $f'(x) < 5$ for all x, then it follows that $f'(c) < 5$, and therefore

$$\frac{f(b) - f(a)}{b - a} < 5.$$

Problems: 4, 5, 6, (7), 8, (9), (10)

Exercises: Mean Value Theorem

1–3 ■ Use Rolle's Theorem to show the following statements.

1. Show that the function
$$f(x) = \left(1 - 2x^2\right)\cos(\pi x)$$
has a critical point on the interval $(0,1)$.

2. Let $f(x)$ be a differentiable function, and suppose that
$$f(1) = f(2) = f(3).$$
Show that $f(x)$ has at least two critical points on the interval $[1,3]$.

3. Let $f(x)$ and $g(x)$ be differentiable functions, and suppose that
$$f(2) = 6,\;\; g(2) = 4,\;\; f(5) = 3,\;\; \text{and}\;\; g(5) = 1.$$
Show that $f'(c) = g'(c)$ for some c in the interval $(2,5)$.
Hint: Consider the function $h(x) = f(x) - g(x)$.

4–10 ■ Use the Mean Value Theorem to show the following statements.

4. Let $f(x)$ be a differentiable function, and suppose that
$$f(1) = 2 \quad \text{and} \quad f(4) = 8.$$
Show that there exists a c in the interval $(1,4)$ such that $f'(c) = 2$.

5. Let $f(x) = (x+1)^2 e^x$. Show that $f'(c) = 2e$ for some c in the interval $(-1,1)$.

6. Let $f(x)$ be a differentiable function, and suppose $f(1) = 2$ and $f'(x) > 0$ for all x. Show that $f(5) > 2$.

7. Let $f(x)$ be a differentiable function, and suppose $f'(x) = 0$ for all x in the interval $(1,3)$. Show that $f(1) = f(3)$.

8. Suppose that f is a function and
$$f'(x) = \sin\left(x^2\right)$$
for all x. Given that $f(0) = 6$, show that $4 \leq f(2) \leq 8$.

9. Let $f(x)$ be a differentiable function, and suppose that
$$f(x+3) = f(x) + 6$$
for all x. Show that there exists a c in the interval $(1,4)$ such that $f'(c) = 2$.

10. Let $f(x)$ be a differentiable function, and suppose that $f'(x) < 3$ for all x in the interval $[0,4]$. Show that $f(4) - f(0) < 12$.

Answers to the Exercises

1. Note that $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. Since $f(0) = f(1) = 1$, it follows from Rolle's Theorem that $f'(c) = 0$ for some c in the interval $(0,1)$.

2. Since f is differentiable, it is continuous on the intervals $[1,2]$ and $[2,3]$ and differentiable on the intervals $(1,2)$ and $(2,3)$. By Rolle's Theorem, there exists a c_1 in $(1,2)$ so that $f'(c_1) = 0$ and a c_2 in $(2,3)$ so that $f'(c_2) = 0$. Then c_1 and c_2 are two different critical points for f in the interval $[1,3]$.

3. Let $h(x) = f(x) - g(x)$. Since f and g are differentiable, h is differentiable as well, so h is continuous on the interval $[2,5]$ and differentiable on the interval $(2,5)$. By Rolle's Theorem, there exists a c in the interval $(2,5)$ so that $h'(c) = 0$. But $h'(c) = f'(c) - g'(c)$, so $f'(c) - g'(c) = 0$, and therefore $f'(c) = g'(c)$.

4. Since f is differentiable, it is continuous on the interval $[1,4]$ and differentiable on the interval $(1,4)$. By the Mean Value Theorem, there exists a c in the interval $(1,4)$ so that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{8 - 2}{4 - 1} = 2.$$

5. Observe that $f(x)$ is differentiable, so in particular f is continuous on $[-1,1]$ and differentiable on $(-1,1)$. By the Mean Value Theorem, there exists a c in the interval $(-1,1)$ so that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{4e - 0}{2} = 2e.$$

6. Since $f(x)$ is differentiable, it is continuous on the interval $[1,5]$ and differentiable on the interval $(1,5)$. By the Mean Value Theorem, there exists a c in the interval $(1,5)$ so that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{f(5) - 2}{4}.$$

But $f'(c) > 0$, so

$$\frac{f(5) - 2}{4} > 0$$

which means that $f(5) - 2 > 0$ and therefore $f(5) > 2$.

7. Since $f(x)$ is differentiable, it is continuous on the interval $[1,3]$ and differentiable on the interval $(1,3)$. By the Mean Value Theorem, there exists a c in the interval $(1,3)$ so that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{f(3) - f(1)}{2}.$$

But $f'(c) = 0$, which means that $f(3) - f(1) = 0$, and therefore $f(1) = f(3)$.

8. Since $f'(x) = \sin\big(x^2\big)$ for all x, we know that f is differentiable, so f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. By the Mean Value Theorem, it follows that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) - 6}{2}$$

for some c in $(0, 2)$. But $f'(c) = \sin\big(c^2\big)$, which means that $-1 \leq f'(c) \leq 1$. We conclude that

$$-1 \leq \frac{f(2) - 6}{2} \leq 1$$

so

$$-2 \leq f(2) - 6 \leq 2$$

so

$$4 \leq f(2) \leq 8.$$

9. Since f is differentiable, it is continuous on $[1, 4]$ and differentiable on $(1, 4)$. By the Mean Value Theorem, there exists a c in $(1, 4)$ so that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - f(1)}{3}.$$

But $f(4) = f(1 + 3) = f(1) + 6$, so $f(4) - f(1) = 6$ and therefore $f'(C) = 2$.

10. Since f is differentiable, it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. By the Mean Value Theorem, there exists a c in the interval $(0, 4)$ so that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - f(0)}{4}.$$

Since $f'(c) < 3$, it follows that

$$\frac{f(4) - f(0)}{4} < 3$$

so $f(4) - f(0) < 12$.