Riemann Sums

Study Guide

Problems in parentheses are for extra practice.

1. Basic Idea

A **Riemann sum** is a way of approximating an integral by summing the areas of vertical rectangles. A Riemann sum approximation has the form

$$\int_a^b f(x)\,dx \;\approx\; f(x_1)\,\Delta x \,+\, f(x_2)\,\Delta x \,+\, \cdots \,+\, f(x_n)\,\Delta x$$

Here Δx represents the width of each rectangle. This is given by the formula

$$\Delta x \;=\; \frac{b-a}{n}$$

where n is the number of rectangles. The x-values $x_1, x_2, \ldots, x_n$ are chosen from the rectangles according to some rule. The three most common rules are:

1. Use the right endpoint of each recangle.
2. Use the left endpoint of each rectangle.
3. Use the midpoint of each rectangle.

You can find the x-values for one of these rules by partitioning the interval $[a, b]$ into subintervals of width Δx and then choosing the x-values. For example, if $[a, b] = [1, 3]$ and $n = 4$ then $\Delta x = (3 - 1)/4 = 0.5$, so the subintervals would be

$$[1, 1.5], \quad [1.5, 2], \quad [2, 2.5], \quad \text{and} \quad [2.5, 3].$$

In this case:

1. The right endpoint rule would give $f(1.5) \cdot 0.5 + f(2) \cdot 0.5 + f(2.5) \cdot 0.5 + f(3) \cdot 0.5$,
2. The left endpoint rule would give $f(1) \cdot 0.5 + f(1.5) \cdot 0.5 + f(2) \cdot 0.5 + f(2.5) \cdot 0.5$, and
3. The midpoint rule would give $f(1.25) \cdot 0.5 + f(1.75) \cdot 0.5 + f(2.25) \cdot 0.5 + f(2.75) \cdot 0.5$.

It is usually easy to tell from the graph whether left endpoints or right endpoints give an overestimate or underestimate of the true integral. In particular:

- If $f(x)$ is increasing then the left endpoint rule gives an underestimate and the right endpoint rule gives an overestimate.
- If $f(x)$ is decreasing then the left endpoint rule gives an overestimate and the right endpoint rule gives an underestimate.

Problems: 1–8 odd, (1–8 even)

2. Summations

Summation notation lets us describe large sums by giving a formula for each term. For example,

$$\sum_{k=1}^{10} k^2 \qquad \text{means} \qquad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2.$$

The idea is that the variable k takes all of the values between 1 and 10. For each k we compute k^2 and then add the values together.

We can write summations for a Riemann sum using each of the three rules:

Right endpoint rule $\displaystyle\int_a^b f(x)\,dx \;\approx\; \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right)\frac{b-a}{n}$

Left endpoint rule $\displaystyle\int_a^b f(x)\,dx \;\approx\; \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}(k-1)\right)\frac{b-a}{n}$

Midpoint rule $\displaystyle\int_a^b f(x)\,dx \;\approx\; \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}\left(k-\tfrac{1}{2}\right)\right)\frac{b-a}{n}$

Taking the limit of Riemann sums as the number of rectangles goes to infinity yields the actual value of the integral. The simplest case is to use right endpoints:

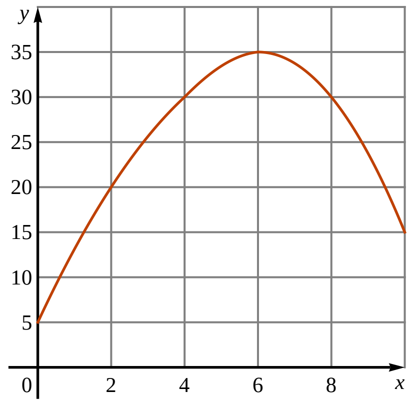
$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\sum_{k=1}^{n} f\left(x + \frac{b-a}{n}k\right)\frac{b-a}{n}$$

This is called the **limit definition of the integral**.

Problems: 9–14 odd, (9–14 even)

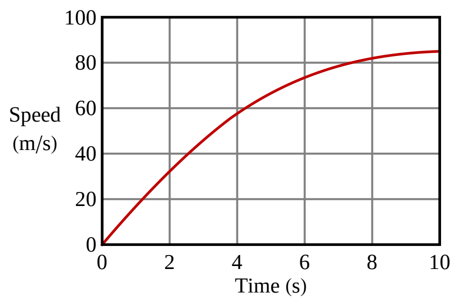
Exercises: Riemann Sums

1. The following figure shows the graph of a function $f(x)$.



Estimate the value of $\int_0^{10} f(x)\,dx$ using five rectangles and left endpoints.

2. The following graph shows the speed of a racecar for the first ten seconds of a race.



(a) Estimate the distance traveled by the racecar using five rectangles and right endpoints.

(b) Is your estimate in part (a) an underestimate or overestimate?

3–4 ■ Approximate the given integral using the given number of rectangles and the indicated rule.

3. $\int_0^2 x^2\,dx$; four rectangles; right endpoints

4. $\int_0^\pi \sin^2 x\,dx$; three rectangles; midpoints

5. Suppose we approximate the integral $\int_1^5 \sqrt{x}\,dx$ using 100 rectangles and right endpoints.

(a) What is the area of the first rectangle?

(b) What is the area of the last rectangle?

(c) Will the approximation be an overestimate or an underestimate? Explain.

6. The following table shows some data for a function $f(x)$.

x	0.0	0.1	0.2	0.3	0.4	0.5
$f(x)$	3.6	2.6	1.8	1.2	0.8	0.6

(a) Estimate $\int_0^{0.5} f(x)\,dx$ using five rectangles and right endpoints.

(b) Would you expect that your estimate from part (a) is an overestimate or an underestimate? Explain.

7–8 ■ Write an integral that can be approximated using the given Riemann sum.

7. $(5.01)^3(0.02) + (5.03)^3(0.02) + \cdots + (5.99)^3(0.02)$

8. $\sqrt{3.01}\,(0.01) + \sqrt{3.02}\,(0.01) + \cdots + \sqrt{7.00}\,(0.01)$

9. Evaluate $\sum_{k=1}^{4} (k^2 + 2k)$.

10. Write the sum

$$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + 5 \cdot 8 + 6 \cdot 9$$

as a summation.

11. Write a summation that approximates the integral $\int_1^3 \sin x\,dx$ using 50 rectangles and right endpoints.

12. Write an integral that is approximated by the following Riemann sum:

$$\sum_{k=1}^{200} (3 + 0.02k)^3(0.02).$$

13. Use the limit definition of the integral to write the integral

$$\int_2^6 x^5\,dx$$

as the limit of a summation.

14. Write the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{2k}{n}\right) \frac{2}{n}$$

as an integral.

Answers to the Exercises

1. 240 **2.** (a) roughly 660 m (b) overestimate **3.** $\dfrac{15}{4} = 3.75$ **4.** $\dfrac{\pi}{2}$

5. (a) $\sqrt{1.04}\,(0.04)$ (b) $\sqrt{5}\,(0.04)$ (c) overestimate, since $\sqrt{x}$ is increasing on $[1, 5]$

6. (a) 0.7 (b) underestimate, since $f(x)$ appears to be decreasing on $[0, 0.5]$

7. $\displaystyle\int_5^6 x^3\,dx$ **8.** $\displaystyle\int_3^7 \sqrt{x}\,dx$ **9.** 50 **10.** $\displaystyle\sum_{k=1}^{6} k(k+3)$ **11.** $\displaystyle\sum_{k=1}^{50} \sin\left(1 + \frac{k}{25}\right)\frac{1}{25}$

12. $\displaystyle\int_3^7 x^3\,dx$ **13.** $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left(2 + \frac{4k}{n}\right)^2 \frac{4}{n}$ **14.** $\displaystyle\int_1^3 \ln x\,dx$