Two Theorems

Study Guide

Problems in parentheses are for extra practice.

1. The Sandwich Theorem

The Sandwich Theorem (or Squeeze Theorem) can be used for finding certain limits. The hypotheses are:

1. $f(x) \le g(x) \le h(x)$, and
2. $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ have the same value L.

The conclusion is:

- $\lim_{x\to a} g(x) = L$ as well.

Techinically, hypothesis (1) doesn't have to be true for all x. First, it's ok if it's not true at $x = a$, since we're taking a limit as $x \to a$. Second, it only needs to be true when x is close to a, i.e. when x lies in some open interval that contains a.

For example, let's use the Sandwich Theorem to show that

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Recall that $-1 \le \sin(\theta) \le 1$ for all angles θ, so

$$-1 \;\le\; \sin(1/x) \;\le\; 1$$

for all $x \ne 0$. Since $x^2 \ge 0$, we can multiply through by x^2 to get

$$-x^2 \;\le\; x^2 \sin(1/x) \;\le\; x^2$$

for all $x \ne 0$. Since

$$\lim_{x\to 0} -x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 = 0,$$

it follows from the Sandwich Theorem that $\lim_{x\to 0} x^2 \sin(1/x) = 0$ as well.

By the way, the Sandwich Theorem works just fine for one-sided limits, as well as for limits as $x \to \infty$ or $x \to -\infty$. It also works when the limit L is either ∞ or $-\infty$.

Problems: 1, 2, 3, (4), 5, (6), (7)

2. The Intermediate Value Theorem

The Intermediate Value Theorem is useful for showing that certain equations have solutions. The hypotheses are:

1. The function f is continuous on the interval $[a, b]$, and
2. The number y_0 is between $f(a)$ and $f(b)$.

The conclusion is:

- There exists a c in $[a, b]$ so that $f(c) = y_0$.

For example, let's use the Intermediate Value Theorem to show that the equation

$$x^3 + x = 7$$

has a solution in the interval $[1, 2]$. Let $f(x) = x^3 + x$. Then f is continuous on $[1, 2]$, with $f(1) = 2$ and $f(2) = 10$. Since 7 is between 2 and 10, it follows from the Intermediate Value Theorem that $f(c) = 7$ for some c in $[1, 2]$.

Problems: 8, 9, 10, 11, (12), (13)

Exercises: Two Theorems

1–4 ■ Use the Sandwich Theorem to evaluate the following limits. Make sure to justify your answers.

1. $\lim_{x\to 0} x^4 \sin\left(1/x^2\right)$

2. $\lim_{x\to\infty} \dfrac{3+\sin x}{x}$

3. $\lim_{x\to 0^+} \dfrac{\cos(\ln x)-3}{x}$

4. $\lim_{x\to 0} x\cos(1/x)$

5. Let f be a function, and suppose that

$$2x \;\le\; f(x) \;\le\; x^2+1$$

for all real values of x. Show that

$$\lim_{x\to 1} 3f(x) \;=\; 6.$$

6. Let g be a function, and suppose that

$$0 \;\le\; g(x) \;\le x^2$$

for all real values of x. Find the value of $\lim_{x\to\infty}\dfrac{g(x)}{x^3}$, and justify your answer using the Sandwich Theorem.

7. Let

$$f(x) = \begin{cases} x^3\sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Use the limit definition of the derivative and the Sandwich Theorem to show that $f'(0)=0$.

8. Show that the equation $x^4 - 2x = 10$ has at least one solution in the interval $[1,2]$.

9. Show that the equation $e^x = 3x$ has at least one solution in the interval $[0,1]$.

10. Show that the equation $x^3 - 5x = 6$ has at least one solution.

11. Let f and g be continuous functions, and suppose that

$$f(0)=2, \quad f(2)=5, \quad g(0)=4, \quad \text{and} \quad g(2)=1.$$

Show that the equation $f(x)=g(x)$ has at least one solution on the interval $[0,2]$.

12. Suppose the function f is continuous on the interval $[1,3]$, with

$$f(1)=4 \qquad \text{and} \qquad f(3)=2.$$

Show that there exists a c in $[1,3]$ so that $c\,f(c)=5$.

13. Let f be a differentiable function. Suppose that f' is continuous on $[1,5]$, with

$$f(1)=4, \quad f(3)=2, \quad \text{and} \quad f'(5)=1.$$

Use the Mean Value Theorem and the Intermediate Value Theorem to show that f has a critical point in the interval $[1,5]$.

Answers to the Exercises

1. We know that

$$-1 \;\le\; \sin\bigl(1/x^2\bigr) \;\le\; 1$$

for all $x \neq 0$. Since $x^4 \geq 0$, it follows that

$$-x^4 \;\le\; x^4 \sin\bigl(1/x^2\bigr) \;\le\; x^4$$

for all $x \neq 0$. But

$$\lim_{x\to 0} -x^4 = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^4 = 0$$

so $\lim\limits_{x\to 0} x^4 \sin\bigl(1/x^2\bigr) = 0$ by the Sandwich Theorem.

2. Recall that

$$-1 \;\le\; \sin x \;\le\; 1$$

for all x. Adding 3 gives

$$2 \;\le\; 3 + \sin x \;\le\; 4$$

for all x. For $x > 0$, we can divide through by x to get

$$\frac{2}{x} \;\le\; \frac{3 + \sin x}{x} \;\le\; \frac{4}{x}.$$

Since

$$\lim_{x\to\infty} \frac{2}{x} = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{4}{x} = 0$$

it follows from the Sandwich Theorem that

$$\lim_{x\to\infty} \frac{3 + \sin x}{x} = 0.$$

3. We know that

$$-1 \le \cos(\ln x) \le 1$$

for all $x > 0$. Subtracting 3 gives

$$-4 \le \cos(\ln x) - 3 \le -2$$

for all $x > 0$, and dividing through by x gives

$$-\frac{4}{x} \le \frac{\cos(\ln x) - 3}{x} \le -\frac{2}{x}$$

for all $x > 0$. Since

$$\lim_{x \to 0^+} -\frac{4}{x} = -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} -\frac{2}{x} = -\infty$$

it follows from the Sandwich Theorem that

$$\lim_{x \to 0^+} \frac{\cos(\ln x) - 3}{x} = -\infty.$$

4. We know that

$$-1 \le \cos(1/x) \le 1$$

for all $x \neq 0$. It follows that

$$-|x| \le x\cos(1/x) \le |x|$$

for all $x \neq 0$. Since

$$\lim_{x \to 0} -|x| = 0 \qquad \text{and} \qquad \lim_{x \to 0} |x| = 0$$

we conclude that $\lim\limits_{x \to 0} x\cos(1/x) = 0$ by the Sandwich Theorem.

Note: In the above problem it wouldn't be right to start with

$$-1 \le \cos(1/x) \le 1$$

and then multiply through by x to get

$$-x \le x\cos(1/x) \le x.$$

The reason is that this only works when $x > 0$. If x is negative, then $-x$ is actually greater than x, so the inequality $-x \le x\cos(1/x) \le x$ makes no sense. Using absolute values as in the solution above avoids this problem.

5. We are given that

$$2x \le f(x) \le x^2 + 1$$

for all x. Multiplying through by 3 gives

$$6x \le 3\,f(x) \le 3x^2 + 3$$

for all x. Since

$$\lim_{x\to 1} 6x = 6 \qquad \text{and} \qquad \lim_{x\to 1} 3x^2 + 3 = 6$$

it follows from the Sandwich Theorem that

$$\lim_{x\to 1} 3\,f(x) = 6.$$

6. We know that

$$0 \le g(x) \le x^2$$

for all x. Assuming $x > 0$, we can divide through by x^3 to get

$$0 \le \frac{g(x)}{x^3} \le \frac{1}{x}.$$

Since

$$\lim_{x\to\infty} 0 = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{1}{x} = 0,$$

the Sandwich Theorem tells us that $\lim_{x\to 1} \dfrac{g(x)}{x^3} = 0$.

Note: We can assume that $x > 0$ above since we are taking a limit as $x \to \infty$. We couldn't make this assumption for a limit as $x \to 0$.

7. By the limit definition of the derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^3 \sin(1/x) - 0}{x} = \lim_{x\to 0} x^2 \sin(1/x).$$

But $-1 \le \sin(1/x) \le 1$ for all $x \ne 0$. Since $x^2 \ge 0$, it follows that $-x^2 \le x^2 \sin(1/x) \le x^2$ for all $x \ne 0$. But

$$\lim_{x\to 0} -x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 = 0.$$

By the Sandwich Theorem, we conclude that $\lim_{x\to 0} x^2 \sin(1/x) = 0$, so $f'(0) = 0$.

8. Let $f(x) = x^4 - 2x$. Then f is continuous on $[1, 2]$ with $f(1) = -1$ and $f(2) = 12$. Since 10 is between -1 and 12, the Intermediate Value Theorem tells us that $f(c) = 10$ for some c in $[1, 2]$, and thus $x = c$ is a solution to the given equation.

9. We can rewrite this equation as $e^x - 3x = 0$. Let $f(x) = e^x - 3x$, and note that f is continuous on the interval $[0, 1]$. Furthermore, $f(0) = 1$ and $f(1) = e-3 < 0$. Since 0 is between $f(0)$ and $f(1)$, it follows from the Intermediate Value Theorem that the given equation has a solution in $[0, 1]$.

10. This time we have to guess our own interval. Let $f(x) = x^3 - 5x$, and note that f is continuous. Let's try a few different values of x:

$$f(-1) = 4, \qquad f(0) = 0, \qquad f(1) = -4, \qquad f(2) = -2, \qquad f(3) = 12.$$

Since 6 is between -2 and 12, it follows from the Intermediate Value Theorem that $f(c) = 6$ for some c in $[2, 3]$.

11. Let $h(x) = f(x) - g(x)$, and note that h is continuous, being the difference of two continuous functions. Then

$$h(0) = f(0) - g(0) = 2 - 4 = -2 \qquad \text{and} \qquad h(2) = f(2) - g(2) = 5 - 1 = 4.$$

By the Intermediate Value Theorem, there exists a number c in the interval $[0, 2]$ such that $h(c) = 0$. Then $f(c) - g(c) = 0$, so $f(c) = g(c)$.

12. Let $g(x) = x\,f(x)$, and note that g is continuous on $[1, 3]$ since it is the product of two continuous functions. But

$$g(1) = 1\,f(1) = 4 \qquad \text{and} \qquad g(3) = 3\,f(3) = 6.$$

By the Intermediate Value Theorem, there exists a c in $[1, 3]$ such that $g(c) = 5$, and therefore $c\,f(c) = 5$.

13. Since f is differentiable, it is differentiable on $(1, 3)$ and continuous on $[1, 3]$. Therefore, by the Mean Value Theorem there exists a number a in $(1, 3)$ so that

$$f'(a) = \frac{f(3) - f(1)}{3 - 1} = \frac{2 - 4}{3 - 1} = -1.$$

Now $f'(a) = -1$ and $f'(5) = 1$. Since f' is continuous on $[a, 5]$, it follows from the Intermediate Value Theorem that $f'(c) = 0$ for some c in $[a, 5]$, so c is a critical point for f that lies in $[1, 5]$.