Defining the Integral

In these notes we provide a careful definition of the Lebesgue integral and we prove each of the three main convergence theorems. For the duration of these notes, let $(X, \mathcal{M}, \mu)$ be a measure space.

Simple Functions

Definition: Simple Function
A **simple function** $s\colon X \to \mathbb{R}$ is any function of the form

$$s = c_1\,\chi_{E_1} + c_2\,\chi_{E_2} + \cdots + c_n\,\chi_{E_n}$$

where $c_1, \ldots, c_n \in \mathbb{R}$ are distinct and $\{E_1, \ldots, E_n\}$ is a partition of X into nonempty measurable sets.

Equivalently, a simple function is any measurable function $s\colon X \to \mathbb{R}$ whose range is a finite set. In particular, if

$$s = c_1\,\chi_{E_1} + c_2\,\chi_{E_2} + \cdots + c_n\,\chi_{E_n},$$

then the range of s is $\{c_1, \ldots, c_n\}$, and $E_i = s^{-1}(c_i)$ for each i.

Definition: Integral of a Simple Function
The **Lebesgue integral** of a non-negative simple function $s = \sum_{i=1}^{n} c_i\,\chi(E_i)$ is defined as follows:

$$\int_X s\,d\mu = \sum_{i=1}^{n} c_i\,\mu(E_i).$$

Here and in the future, we adopt the convention that $0 \cdot \infty = \infty \cdot 0 = 0$. In particular, if $c_i = 0$ and $\mu(E_i) = \infty$ for some i, then the corresponding term $c_i\,\mu(E_i)$ in the definition of the integral should be interpreted as 0.

Note that we have required the coefficients $c_1, \ldots, c_n$ in an expansion

$$s \;=\; c_1\,\chi_{E_1} \;+\; c_2\,\chi_{E_2} \;+\; \cdots \;+\; c_n\,\chi_{E_n}$$

of a simple function to all be distinct. The purpose is to avoid any ambiguity: if we require $c_1, \ldots, c_n$ to be distinct, then any simple function has only one such expansion, so the integral is well-defined.

However, it is often helpful to express a simple function as a sum

$$s \;=\; c_1\,\chi_{E_1} \;+\; c_2\,\chi_{E_2} \;+\; \cdots \;+\; c_n\,\chi_{E_n}$$

where the coefficients $c_1, \ldots, c_n$ are not necessarily distinct. The following lemma lets us evaluate the integral in this case.

Lemma 1 Integrating Simple Functions

Let

$$s \;=\; c_1\,\chi_{E_1} \;+\; \cdots \;+\; c_n\,\chi_{E_n}$$

be a non-negative simple function, where $c_1, \ldots, c_n \in [0, \infty)$ are not necessarily distinct, and $\{E_1, \ldots, E_n\}$ is a partition of X into nonempty measurable sets. Then

$$\int_X s\,d\mu \;=\; c_1\,\mu(E_1) \;+\; \cdots \;+\; c_n\,\mu(E_n).$$

PROOF We proceed by induction on n. The base case is that $n = |\mathrm{range}(s)|$, in which case $c_1, \ldots, c_n$ are all distinct, and the lemma holds by definition.

Now suppose that $n > |\mathrm{range}(s)|$. Then there must exist two coefficients that are the same, say c_{n-1} and c_n. Then

$$s \;=\; c_1\,\chi_{E_1} \;+\; \cdots \;+\; c_{n-2}\,\chi_{E_{n-2}} \;+\; c_{n-1}\,\chi_{E_{n-1} \uplus E_n}$$

so by our induction hypothesis

$$\int_X s\,d\mu \;=\; c_1\,\mu(E_1) \;+\; \cdots \;+\; c_{n-2}\,\mu(E_{n-2}) \;+\; c_{n-1}\,\mu(E_{n-1} \uplus E_n).$$

But $\mu(E_{n-1} \uplus E_n) = \mu(E_{n-1}) + \mu(E_n)$, so $c_{n-1}\,\mu(E_{n-1} \uplus E_n) = c_{n-1}\,\mu(E_{n-1}) + c_n\,\mu(E_n)$, and hence

$$\int_X s\,d\mu \;=\; c_1\,\mu(E_1) \;+\; \cdots \;+\; c_{n-2}\,\mu(E_{n-2}) \;+\; c_{n-1}\,\mu(E_{n-1}) \;+\; c_n\,\mu(E_n). \qquad \blacksquare$$

The next lemma is extremely helpful for proving things about simple functions.

Lemma 2 Common Partitions

Let $s, t\colon X \to \mathbb{R}$ be simple functions. There there exists a partition $\{E_1, \ldots, E_n\}$ of X into nonempty measurable sets such that

$$s = c_1\,\chi_{E_1} + \cdots + c_n\,\chi_{E_n} \qquad \text{and} \qquad t = d_1\,\chi_{E_1} + \cdots + d_n\,\chi_{E_n}$$

for some constants $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{R}$.

PROOF Suppose that

$$s = c_1\,\chi_{E_1} + \cdots + c_m\,\chi_{E_m} \qquad \text{and} \qquad t = d_1\,\chi_{F_1} + \cdots + d_n\,\chi_{F_n}$$

for some constants $c_1, \ldots, c_m, d_1, \ldots, d_n \in \mathbb{R}$ and some partitions $\{E_1, \ldots, E_m\}$ and $\{F_1, \ldots, F_n\}$ of X into nonempty measurable sets.

Let $\mathcal{P}$ be the set of all pairs (i, j) for which $E_i \cap F_j \neq \emptyset$. Then $\{E_i \cap F_j\}_{(i,j)\in\mathcal{P}}$ is a partition of X into nonempty measurable sets which satisfies the given criterion. In particular,

$$s = \sum_{(i,j)\in\mathcal{P}} c_i\,\chi_{E_i\cap F_j} \qquad \text{and} \qquad t = \sum_{(i,j)\in\mathcal{P}} d_j\,\chi_{E_i\cap F_j}. \qquad \blacksquare$$

As a first application of these two lemmas, we give a quick proof of the following proposition.

Proposition 3 Monotone Property for Simple Functions

If $s, t\colon X \to [0, \infty)$ are simple functions and $s \leq t$, then $\displaystyle\int_X s\,d\mu \leq \int_X t\,d\mu$.

PROOF By the common partitions lemma, there exists a partition $\{E_1, \ldots, E_n\}$ of X into nonempty measurable sets so that

$$s = c_1\,\chi_{E_1} + \cdots + c_n\,\chi_{E_n} \qquad \text{and} \qquad t = d_1\,\chi_{E_1} + \cdots + d_n\,\chi_{E_n}$$

for some constants $c_1, \ldots, c_n, d_1, \ldots, d_n \in [0, \infty)$. Then $c_i \leq d_i$ for each i, so

$$\int_X s\,d\mu = \sum_{i=1}^{n} c_i\,\mu(E_i) \leq \sum_{i=1}^{n} d_i\,\mu(E_i) = \int_X t\,d\mu. \qquad \blacksquare$$

The Lebesgue Integral

We are now ready to define the Lebesgue integral for arbitrary measurable functions.

> **Definition: The Lebesgue Integral (Non-Negative Case)**
> If $f\colon X \to [0,\infty]$ is a non-negative measurable function, the **Lebesgue integral** of f is defined by
> $$\int_X f\,d\mu \;=\; \sup\left\{ \int_X s\,d\mu \;\middle|\; s\colon X \to \mathbb{R} \text{ is a simple function and } 0 \le s \le f \right\}.$$

Note that if f is itself a simple function, then the definition above agrees with our earlier definition for the integral of f. Specifically, since f is simple and $0 \le f \le f$, we know that

$$\underset{\text{(new)}}{\int_X f\,d\mu} \;\ge\; \underset{\text{(old)}}{\int_X f\,d\mu}.$$

But by Proposition 3, we know that

$$\int_X s\,d\mu \;\le\; \underset{\text{(old)}}{\int_X f\,d\mu}$$

for any simple function $s\colon X \to \mathbb{R}$ such that $0 \le s \le f$, and hence

$$\underset{\text{(new)}}{\int_X f\,d\mu} \le \underset{\text{(old)}}{\int_X f\,d\mu}$$

which proves that the two integrals are equal.

Of course, the definition above only works for non-negative measurable functions, but we can extend to measurable functions $f\colon X \to [-\infty,\infty]$ by the formula

$$\int_X f\,d\mu \;=\; \int_X f^+\,d\mu - \int_X f^-\,d\mu,$$

where f^+ and f^- are the positive and negative parts of f. We can also integrate on any measurable set $E \subseteq X$ by the formula

$$\int_E f\,d\mu \;=\; \int_X f\,\chi_E\,d\mu.$$

Elementary Properties

We now turn to proving some of the elementary properties of the Lebesgue integral. As a general rule, there are three steps to proving anything about the Lebesgue integral:

1. Prove the proposition for non-negative simple functions.
2. Use the definition of the Lebesgue integral to extend to non-negative measurable functions.
3. Use positive an negative parts to extend to all measurable functions.

The following proposition illustrates this technique.

Proposition 4 Monotone Property

If $f, g\colon X \to [-\infty, \infty]$ are Lebesgue integrable functions and $f \leq g$, then

$$\int_X f\, d\mu \;\leq\; \int_X g\, d\mu.$$

PROOF We already proved this for non-negative simple functions is Proposition 3. In the case where f and g are non-negative measurable functions, observe that any simple function s satisfying $0 \leq s \leq f$ also satisfies $0 \leq s \leq g$, and hence

$$\int_X f\, d\mu \;=\; \sup_{0 \leq s \leq f} \int_X s\, d\mu \;\leq\; \sup_{0 \leq s \leq g} \int_X s\, d\mu \;=\; \int_X g\, d\mu.$$

Finally, suppose that $f, g\colon X \to [-\infty, \infty]$ are Lebesgue integrable functions and $f \leq g$. Then $f^+ \leq g^+$ and $g^- \leq f^-$, so

$$\int_X f\, d\mu \;=\; \int_X f^+\, d\mu - \int_X f^-\, d\mu \;\leq\; \int_X g^+\, d\mu - \int_x g^-\, d\mu \;=\; \int_X g\, d\mu. \qquad \blacksquare$$

Proposition 5 Equal Almost Everywhere

Let $f, g\colon X \to [-\infty, \infty]$ be measurable functions, where f is Lebesgue integrable, and suppose that $f = g$ almost everywhere. Then g is Lebesgue integrable, and

$$\int_X f\, d\mu \;=\; \int_X g\, d\mu.$$

PROOF Suppose first that f and g are non-negative simple functions. By the common partitions lemma, there exists a partition $\{E_1, \ldots, E_n\}$ of X into nonempty measurable sets so that

$$f = c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n} \qquad \text{and} \qquad g = d_1 \chi_{E_1} + \cdots + d_n \chi_{E_n}$$

for some constants $c_1, \ldots, c_n, d_1, \ldots, d_n \in [0, \infty)$. Then for each i either $c_i = d_i$ or $\mu(E_i) = 0$, and in either case $c_i\, \mu(E_i) = d_i\, \mu(E_i)$, so

$$\int_X f\, d\mu = \sum_{i=1}^{n} c_i\, \mu(E_i) = \sum_{i=1}^{n} d_i\, \mu(E_i) = \int_X g\, d\mu.$$

Next, suppose that f and g are non-negative measurable functions, and let $E \subseteq X$ be a measurable set such that $f|_E = g|_E$ and $\mu(E^c) = 0$. If $s\colon X \to \mathbb{R}$ is any simple function for which $0 \leq s \leq f$, then $s\chi_E$ is a simple function, $0 \leq s\chi_E \leq g$, and $s\chi_E = s$ almost everywhere, so

$$\int_X s\, d\mu = \int_X s\chi_E\, d\mu \leq \int_X g\, d\mu.$$

Since s was arbitrary, it follows that

$$\int_X f\, d\mu \leq \int_X g\, d\mu$$

and a similar argument shows the opposite inequality.

Finally, in the general case observe that $f^+ = g^+$ almost everywhere and $f^- = g^-$ almost everywhere, so

$$\int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu = \int_X g^+\, d\mu - \int_x g^-\, d\mu = \int_X g\, d\mu. \qquad \blacksquare$$

Proposition 6 Scalar Multiplication

If $f\colon X \to [-\infty, \infty]$ is a Lebesgue integrable function and $k \in \mathbb{R}$, then kf is Lebesgue integrable, and

$$\int_X kf\, d\mu = k \int_X f\, d\mu.$$

PROOF This proposition is trivial in the case where $k = 0$. For $k > 0$, it clearly holds for simple functions, so suppose that f is a non-negative measurable function.

If $s\colon X \to \mathbb{R}$ is a simple function and $0 \le s \le f$, then ks is simple and $0 \le ks \le kf$, so

$$\int_X kf\, d\mu \;\ge\; \int_X ks\, d\mu \;=\; k\int_X s\, d\mu.$$

Since s was arbitrary, it follows that

$$\int_X kf\, d\mu \;\ge\; k\int_X f\, d\mu,$$

and a similar argument proves the opposite inequality.

If $k > 0$ and $f\colon X \to [-\infty,\infty]$, then $(kf)^+ = kf^+$ and $(kf)^- = kf^-$. If f is Lebesgue integrable, it follows that kf is Lebesgue integrable, with

$$\int_X kf\, d\mu \;=\; \int_X kf^+\, d\mu - \int_X kf^-\, d\mu \;=\; k\int_X f^+\, d\mu - k\int_x f^-\, d\mu \;=\; k\int_X f\, d\mu.$$

Finally, for $k < 0$ observe that $(kf)^+ = |k|\, f^-$ and $(kf)^- = |k|\, f^+$, so

$$\int_X kf\, d\mu \;=\; \int_X |k|\, f^-\, d\mu - \int_X |k|\, f^+\, d\mu \;=\; k\int_X f\, d\mu. \qquad \blacksquare$$

The Monotone Convergence Theorem

We are now ready to prove our first major convergence theorem. We begin with a special case involving simple functions.

Lemma 7 Expanding the Domain

Let $s\colon X \to [0,\infty)$ be a non-negative simple function, let

$$E_1 \;\subseteq\; E_2 \;\subseteq\; E_3 \;\subseteq\; \cdots$$

be a sequence of measurable subsets of X, and let $E = \bigcup_{n\in\mathbb{N}} E_n$. Then

$$\int_E s\, d\mu \;=\; \lim_{n\to\infty} \int_{E_n} s\, d\mu.$$

PROOF Suppose that $s = \sum_{i=1}^m c_i\, \chi_{F_i}$, where $c_1,\ldots,c_m \in [0,\infty)$ and $\{F_1,\ldots,F_m\}$ is a partition of X into nonempty measurable sets. Then for any measurable set G, the product $s\chi_G$ is a simple function, with

$$\int_X s\chi_G\, d\mu \;=\; c_1\, \mu(F_1\cap G) \,+\, \cdots \,+\, c_m\, \mu(F_m\cap G).$$

In particular

$$\int_{E_n} s\,d\mu \;=\; \int_X s\chi_{E_n}\,d\mu \;=\; c_1\,\mu(F_1 \cap E_n) \,+\, \cdots \,+\, c_m\,\mu(F_m \cap E_n).$$

for each n and

$$\int_E s\,d\mu \;=\; \int_X s\chi_E\,d\mu \;=\; c_1\,\mu(F_1 \cap E) \,+\, \cdots \,+\, c_m\,\mu(F_m \cap E).$$

But $\mu(F_i \cap E_n) \to \mu(F_i \cap E)$ for each i as $n \to \infty$, so $\displaystyle\int_{E_n} s\,d\mu \to \int_E s\,d\mu$. ■

Theorem 8 Lebesgue's Monotone Convergence Theorem

Let

$$0 \,\le\, f_1 \,\le\, f_2 \,\le\, f_3 \,\le\, \cdots$$

be a sequence of measurable functions on X. Then

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;=\; \int_X \lim_{n\to\infty} f_n\,d\mu.$$

PROOF Let f be the limit of $\{f_n\}$. Since $f_n \le f$ for each n, we know that $\int_X f_n\,d\mu \le \int_X f\,d\mu$ for each n, and hence

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;\le\; \int_x f\,d\mu.$$

For the opposite inequality, let s be a non-negative simple function for which $s \le f$. It suffices to prove that

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;\ge\; \int_X s\,d\mu. \tag{$*$}$$

Let $\epsilon > 0$, and for each n let

$$E_n \;=\; \{x \in X \mid f_n(x) > (1-\epsilon)s\}.$$

Note that $\bigcup_{n\in\mathbb{N}} E_n \,=\, X$, and that each E_n is measurable, being the preimage of $(0,\infty]$ under the measurable function $f_n - (1-\epsilon)s$. But

$$\int_X f_n\,d\mu \;\ge\; \int_{E_n} f_n\,d\mu \;\ge\; (1-\epsilon)\int_{E_n} s\,d\mu$$

for each n. Taking the limit as $n \to \infty$ and applying the lemma, we conclude that

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;\ge\; (1-\epsilon)\int_X s\,d\mu.$$

Since ϵ was arbitrary, the desired inequality $(*)$ follows. ■

Corollary 9 Infinite Sums

Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. Then

$$\int_X \sum_{n\in\mathbb{N}} f_n \, d\mu \;=\; \sum_{n\in\mathbb{N}} \int_X f_n \, d\mu.$$

Corollary 10 Integral on an Ascending Union

Let $f\colon X \to [0,\infty]$ be a non-negative measurable function, let

$$E_1 \;\subseteq\; E_2 \;\subseteq\; E_3 \;\subseteq\; \cdots$$

be a sequence of measurable subsets of X, and let $E = \bigcup_{n\in\mathbb{N}} E_n$. Then

$$\int_E f \, d\mu \;=\; \lim_{n\to\infty} \int_{E_n} f \, d\mu.$$

Corollary 11 Integral on a Disjoint Union

Let $f\colon X \to [0,\infty]$ be a non-negative measurable function, let $\{E_n\}$ be a sequence of pairwise disjoint measurable subsets of X, and let $E = \biguplus_{n\in\mathbb{N}} E_n$. Then

$$\int_E f \, d\mu \;=\; \sum_{n\in\mathbb{N}} \int_{E_n} f \, d\mu.$$

Approximation by Simple Functions

Among other consequences, the monotone convergence theorem allows us to use sequences of simple functions to evaluate certain Lebesgue integrals. In particular, suppose that $f\colon X \to [0,\infty]$ is a measurable function, and suppose that

$$0 \;\leq\; s_1 \;\leq\; s_2 \;\leq\; s_3 \;\leq\; \cdots$$

is a sequence of simple functions that converges pointwise to f. Then according to the monotone convergence theorem,

$$\int_X f \, d\mu \;=\; \lim_{n\to\infty} \int_X s_n \, d\mu.$$

As we will see, this technique can be used with any non-negative measurable function.

Theorem 12 Uniform Approximation by Simple Functions

Let $f\colon X \to [0,\infty)$ be a bounded function. Then f is measurable if and only if f is a uniform limit of simple functions.

PROOF We have already proven that any pointwise limit of measurable functions is measurable, and hence any uniform limit of simple functions is measurable.

For the converse, let $\{s_n\}$ be the sequence of functions

$$s_n = \frac{\lfloor nf \rfloor}{n}.$$

That is, for each integer k,

$$s_n(x) = \frac{k}{n} \qquad \text{if and only if} \qquad \frac{k}{n} \leq f(x) < \frac{k+1}{n}.$$

Since f is bounded, each s_n has finite range. Since f is measurable, each $s_n^{-1}(k/n)$ is a measurable set, and hence s_n is a simple function. But clearly $|s_n - f| < 1/n$ for each n, so $s_n \to f$ uniformly. ■

Theorem 13 Pointwise Approximation by Simple Functions

If $f\colon X \to [-\infty,\infty]$, then f is measurable if and only if f is a pointwise limit of simple functions.

PROOF Again, we have already proven that any pointwise limit of meaurable functions is measurable, and hence any pointwise limit of simple functions is measurable.

For the converse, let $\{f_n\}$ be the sequence of functions

$$f_n(x) = \begin{cases} -n & \text{if } x < -n, \\ f(x) & \text{if } |f(x)| \leq n, \\ n & \text{if } f(x) > n, \end{cases}$$

and note that each f_n is bounded and measurable. By the previous theorem, there exists for each n a simple function s_n so that $|s_n - f_n| < 1/n$. Then it follows easily that $s_n \to f$ pointwise. ■

Corollary 14 Combining Measurable Functions

If $f_1, f_2 \colon X \to \mathbb{R}$ are measurable and $G \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, then the function $h \colon X \to \mathbb{R}$ defined by

$$h(x) = G(f_1(x), f_2(x))$$

is measurable. In particular, the sum, difference, product, maximum, or minimum of any two measurable functions is measurable.

PROOF Let $\{s_n\}$ be a sequence of simple functions that converges pointwise to f_1, and let $\{t_n\}$ be a sequence of simple functions that converge pointwise to f_2. Then by the common partitions lemma, for each n the function

$$u_n(x) = G(s_n(x), t_n(x))$$

is simple. Since G is continuous, $\{u_n\}$ converges pointwise to h, and hence h is measurable. ■

We can improve on Theorem 13 in the case of non-negative functions.

Theorem 15 Pointwise Approximation by Simple Functions

Let $f \colon X \to [0, \infty]$ be a measurable function. Then there exists a sequence

$$0 \leq s_1 \leq s_2 \leq s_3 \leq \cdots$$

of simple functions that converges pointwise to f.

PROOF For any real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and define

$$\lfloor x \rfloor_n = 2^{-n} \lfloor 2^n x \rfloor$$

for each n. That is, $\lfloor x \rfloor_n$ is the number obtained by truncating the binary expansion of x after n digits. It is clear that

$$\lfloor x \rfloor_1 \leq \lfloor x \rfloor_2 \leq \lfloor x \rfloor_3 \leq \cdots$$

and that $\lfloor x \rfloor_n \to x$ as $n \to \infty$.

Now, it is easy to check that $\lfloor f \rfloor$ is measurable for any measurable function f, and hence $\lfloor f \rfloor_n$ is measurable for any measurable function f and any $n \in \mathbb{N}$. Let

$f\colon X \to [0,\infty]$ be a measurable function, and let $\{s_n\}$ be the sequence

$$s_n \;=\; \min\bigl(\lfloor f \rfloor_n, n\bigr).$$

Then each s_n is measurable, with range contained in the finite set

$$\left\{ \frac{k}{2^n} \;\middle|\; 0 \leq k \leq n^2 \right\},$$

so each s_n is simple. Then $\{s_n\}$ is the desired sequence of simple functions. ■

We now have the tools to prove the following important property of Lebesgue integrals.

Proposition 16 Integral of a Sum

Let $f, g\colon X \to [0,\infty]$ be Lebesgue integrable functions. Then

$$\int_X (f+g)\,d\mu \;=\; \int_X f\,d\mu + \int_X g\,d\mu,$$

assuming the sum on the right is defined.

PROOF The proposition clearly holds for non-negative simple functions by the common partitions lemma. For non-negative measurable functions, let $\{s_n\}$ and $\{t_n\}$ be non-decreasing sequences of non-negative simple functions that converge pointwise to f and g, respectively. Then $\{s_n + t_n\}$ is a non-decreasing sequence of non-negative simple functions that converges pointwise to $f+g$, so by the monotone convergence theorem

$$\begin{aligned}\int_X (f+g)\,d\mu \;&=\; \lim_{n\to\infty} \int_X (s_n + t_n)\,d\mu \\ &=\; \lim_{n\to\infty} \int_X s_n\,d\mu + \lim_{n\to\infty} \int_X t_n\,d\mu \;=\; \int_X f\,d\mu + \int_X g\,d\mu.\end{aligned}$$

Finally, for the general case, let $h = f + g$ and observe that

$$h^+ - h^- \;=\; f^+ - f^- + g^+ - g^-.$$

Rearranging gives

$$h^+ + f^- + g^- \;=\; f^+ + g^+ + h^-$$

and hence

$$\int_X h^+ \, d\mu + \int_X f^- \, d\mu + \int_X g^- \, d\mu \;=\; \int_X f^+ \, d\mu + \int_X g^+ \, d\mu + \int_X h^- \, d\mu.$$

It follows that $\int_X h \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$. ■

The Bounded Convergence Theorem

As a prelude to the bounded convergence theorem, we prove the following theorem named for Russian mathematician Dmitri Egorov. Egorov published his proof in 1911, but in fact this theorem first appears in a 1910 publication by Italian mathematician Carlo Severini, though this was not noticed until some time later.

Theorem 17 Egorov's Theorem

Suppose that $\mu(X) < \infty$, and let $\{f_n\}$ be a sequence of measurable functions converging pointwise to a measurable function f. Then for every $\epsilon > 0$ there exists a measurable set $E \subseteq X$ with $\mu(E^c) < \epsilon$ such that $f_n \to f$ uniformly on E.

PROOF For each N and k, let

$$E_{N,k} \;=\; \bigl\{x \in X \;\big|\; |f_n(x) - f(x)| < 1/k \text{ for all } n \geq N\bigr\}.$$

Note that each $E_{N,k}$ is measurable, and that $E_{1,k} \subseteq E_{2,k} \subseteq \cdots$. Since $f_n \to f$ pointwise, we know that $\bigcup_{N\in\mathbb{N}} E_{N,k} = X$ for each k, so choose an $N_k \in \mathbb{N}$ such that

$$\mu(E_{N_k,k}) \;\geq\; \mu(X) - \frac{\epsilon}{2^k}.$$

Let $E = \bigcap_{k\in\mathbb{N}} E_{N_k,k}$. Then $E^c = \bigcup_{k\in\mathbb{N}} E^c_{N_k,k}$, so

$$\mu(E^c) \;\leq\; \sum_{k\in\mathbb{N}} \mu\bigl(E^c_{N_k,k}\bigr) \;\leq\; \sum_{k\in\mathbb{N}} \frac{\epsilon}{2^k} \;=\; \epsilon.$$

But $|f_n - f| < 1/k$ on E for all $n \geq N_k$, and thus $f_n \to f$ uniformly on E. ■

Theorem 18 Lebesgue's Bounded Convergence Theorem

Suppose that $\mu(X) < \infty$, and let $\{f_n\}$ be a uniformly bounded, pointwise convergent sequence of measurable functions on X. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu \;=\; \int_X \lim_{n\to\infty} f_n \, d\mu.$$

PROOF Let $\epsilon > 0$, and let $M > 0$ so that $|f_n| \leq M$ for all n. By Egorov's theorem, there exists a measurable set $E \subseteq X$ so that $\mu(E^c) < \epsilon/(4M)$ and $f_n \to f$ uniformly on E. Let $N \in \mathbb{N}$ so that $|f_n - f| < \epsilon/\big(2\mu(E)\big)$ on E for all $n \geq N$. Then for all $n \geq N$, we have

$$\begin{aligned}\left|\int_X f_n \, d\mu - \int_X f \, d\mu\right| \leq \int_X |f_n - f| \, d\mu &= \int_E |f_n - f| \, d\mu + \int_{E^c} |f_n - f| \, d\mu \\ &\leq \frac{\epsilon}{2\mu(E)}\mu(E) + 2M\mu(E^c) = \epsilon.\end{aligned}$$

■

The Dominated Convergence Theorem

To prove the dominated convergence theorem, we first introduce an important family of measures.

Proposition 19 Weighted Measures

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $g\colon X \to [0,\infty]$ be a measurable function. Then the function $\nu\colon \mathcal{M} \to [0,\infty]$ defined by

$$\nu(E) \;=\; \int_E g \, d\mu$$

is a measure on X.

PROOF Clearly $\nu(\emptyset) = 0$. For countable additivity, let $\{E_n\}$ be a sequence of pairwise disjoint measurable sets, and let $E = \bigcup_{n=1}^{\infty} E_n$. Then

$$\nu(E) = \int_X g\,\chi_E \, d\mu = \int_X \sum_{n\in\mathbb{N}} g\,\chi_{E_n} \, d\mu = \sum_{n\in\mathbb{N}} \int_X g\,\chi_{E_n} \, d\mu = \sum_{n\in\mathbb{N}} \nu(E_n).$$

■

The measure ν defined above is sometimes called the **weighted measure** obtained from μ with **density function** g. The relationship between ν, μ, and g is usually denoted

$$d\nu = g\,d\mu.$$

The following proposition justifies this notation.

Proposition 20 Integration Using Weighted Measures

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $g\colon X \to [0,\infty]$ be a measurable function, and let $d\nu = g\,d\mu$. Then for any measurable function $f\colon X \to [-\infty,\infty]$,

$$\int_X f\,d\nu = \int_X fg\,d\mu.$$

(That is, the integral on the left exists if and only if the integral on the right exists, in which case their values are the same.)

PROOF Suppose first that $s = \sum_{k=1}^{n} a_k\,\chi_{E_k}$ is a non-negative simple function. Then

$$\int_X s\,d\nu = \sum_{k=1}^{n} a_k\,\nu(E_k) = \sum_{k=1}^{n} a_k \int_{E_k} g\,d\mu = \sum_{k=1}^{n} a_k \int_X \chi_{E_k} g\,d\mu = \int_X sg\,d\mu.$$

Next, if $f\colon X \to [0,\infty]$ is a non-negative measurable function, let $s_1 \le s_2 \le \cdots$ be a sequence of simple functions that converge to f pointwise. Then $s_1 g \le s_2 g \le \cdots$ and $s_n g \to fg$ pointwise as $n \to \infty$, so by the monotone convergence theorem

$$\int_X f\,d\nu = \lim_{n\to\infty} \int_X s_n\,d\nu = \lim_{n\to\infty} \int_X s_n g\,d\mu = \int_X fg\,d\mu.$$

Finally, if $f\colon X \to [-\infty,\infty]$ is any measurable function, then

$$\int_X f^+ d\nu = \int_X f^+ g\,d\mu = \int_X (fg)^+\,d\mu$$

and similarly $\int_X f^-\,d\nu = \int_X (fg)^-\,d\mu$, so the desired conclusion follows. ∎

Theorem 21 Lebesgue's Dominated Convergence Theorem

Let $\{f_n\}$ be a pointwise convergent sequence of measurable functions on X, and suppose that there exists a non-negative L^1 function g on X such that $|f_n| \le g$ for all n. Then

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X \lim_{n\to\infty} f_n\,d\mu.$$

PROOF Note first that $g(x) < \infty$ almost everywhere, so we may assume that $g\colon X \to [0,\infty)$. Let $d\nu = g\,d\mu$, and note that $\nu(X) = \int_X g\,d\mu < \infty$. Let f be the pointwise limit of the sequence $\{f_n\}$, let $E = \{x \in X \mid g(x) > 0\}$, and note that $f_n = 0$ on E^c for each n, and $f = 0$ on E^c as well. Then

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;=\; \lim_{n\to\infty} \int_E f_n\,d\mu \;=\; \lim_{n\to\infty} \int_E \frac{f_n}{g}\,d\nu.$$

But $|f_n/g| \leq 1$ for all n, so by the bounded convergence theorem we can switch the limit and the integral to get

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;=\; \int_E \lim_{n\to\infty} \frac{f_n}{g}\,d\nu \;=\; \int_E \frac{f}{g}\,d\nu \;=\; \int_E f\,d\mu \;=\; \int_X f\,d\mu. \qquad \blacksquare$$