Hilbert Spaces

Recall that any inner product space V has an associated norm defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Thus an inner product space can be viewed as a special kind of normed vector space. In particular, every inner product space V has a metric defined by

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| = \sqrt{\langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle}.$$

Definition: Hilbert space
A **Hilbert space** is an inner product space whose associated metric is complete.

That is, a Hilbert space is an inner product space that is also a Banach space. For example, $\mathbb{R}^n$ is a Hilbert space under the usual dot product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n.$$

More generally, a finite-dimensional inner product space is a Hilbert space. The following theorem provides examples of infinite-dimensional Hilbert spaces.

Theorem 1 L^2 is a Hilbert Space

For any measure space (X, μ), the associated L^2-space $L^2(X)$ forms a Hilbert space under the inner product

$$\langle f, g \rangle = \int_X f g \, d\mu.$$

PROOF The norm associated to the given inner product is the L^2-norm:

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_X f^2 \, d\mu} = \|f\|_2.$$

We have already proven that $L^2(X)$ is complete with respect to this norm, and hence $L^2(X)$ is a Hilbert space. ■

In the case where $X = \mathbb{N}$, this gives us the following.

Corollary 2 ℓ^2 is a Hilbert Space

The space ℓ^2 of all square-summable sequences is a Hilbert space under the inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{n \in \mathbb{N}} v_n w_n.$$

ℓ^2-Linear Combinations

We now turn to some general theory for Hilbert spaces. First, recall that two vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space are called **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Proposition 3 Convergence of Orthogonal Series

Let $\{\mathbf{v}_n\}$ be a sequence of orthogonal vectors in a Hilbert space. Then the series

$$\sum_{n=1}^{\infty} \mathbf{v}_n$$

converges if and only if

$$\sum_{n=1}^{\infty} \|\mathbf{v}_n\|^2 < \infty.$$

PROOF Let $\mathbf{s}_n$ be the sequence of partial sums for the given series. By the Pythagorean theorem,

$$\|\mathbf{s}_i - \mathbf{s}_j\|^2 = \left\| \sum_{n=i+1}^{j} \mathbf{v}_n \right\|^2 = \sum_{n=i+1}^{j} \|\mathbf{v}_n\|^2.$$

for all $i \leq j$. It follows that $\{\mathbf{s}_n\}$ is a Cauchy sequence if and only if $\sum_{n=1}^{\infty} \|\mathbf{v}_n\|^2$ converges. ■

We wish to apply this proposition to linear combinations of orthonormal vectors. First recall that a sequence $\{\mathbf{u}_n\}$ of vectors in an inner product space is called **orthonormal** if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle \;=\; \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all i and j.

Corollary 4 ℓ^2-Linear Combinations

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let $\{a_n\}$ be a sequence of real numbers. Then the series

$$\sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

converges if and only if the sequence $\{a_n\}$ lies in ℓ^2.

In general, if $\{a_n\}$ is an ℓ^2 sequence, then the sum

$$\sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

is called a **$\boldsymbol{\ell^2}$-linear combination** of the vectors $\{\mathbf{u}_n\}$. By the previous corollary, every ℓ^2-linear combination orthonormal vectors in a Hilbert space converges

Proposition 5 Inner Product Formula

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$\mathbf{v} \;=\; \sum_{n=1}^{\infty} a_n \mathbf{u}_n \qquad \textit{and} \qquad \mathbf{w} \;=\; \sum_{n=1}^{\infty} b_n \mathbf{u}_n$$

be ℓ^2-linear combinations of the vectors $\{\mathbf{u}_n\}$. Then

$$\langle \mathbf{v}, \mathbf{w} \rangle \;=\; \sum_{n=1}^{\infty} a_n b_n.$$

PROOF Let $\mathbf{s}_N = \sum_{n=1}^{N} a_n \mathbf{u}_n$ and $\mathbf{t}_N = \sum_{n=1}^{N} b_n \mathbf{u}_n$, and note that $\mathbf{s}_N \to \mathbf{v}$ and $\mathbf{t}_N \to \mathbf{w}$ as $N \to \infty$. Since the inner product $\langle -, - \rangle$ is a continuous function, it follows that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \lim_{N\to\infty} \langle \mathbf{s}_N, \mathbf{t}_N \rangle = \lim_{N\to\infty} \sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{\infty} a_n b_n. \qquad \blacksquare$$

In the case where $\mathbf{v} = \mathbf{w}$, this gives the following.

Corollary 6 Norm Formula

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

be an ℓ^2-linear combination of these vectors. Then

$$\|\mathbf{v}\| = \sqrt{\sum_{n=1}^{\infty} a_n^2}.$$

We can also use the inner product formula to find a nice formula for the coefficients of an ℓ^2-linear combination.

Corollary 7 Formula for the Coefficients

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

be an ℓ^2-linear combination of these vectors. Then for all $n \in \mathbb{N}$,

$$a_n = \langle \mathbf{u}_n, \mathbf{v} \rangle.$$

PROOF Given an $n \in \mathbb{N}$, we can write $\mathbf{u}_n = \sum_{k=1}^{\infty} b_k \mathbf{u}_k$, where $b_n = 1$ and $b_k = 0$

for all $k \neq n$. By the inner product formula, it follows that

$$\langle \mathbf{u}_n, \mathbf{v} \rangle \;=\; \sum_{k=1}^{\infty} a_k b_k \;=\; a_n. \qquad \blacksquare$$

In general, we say that a vector $\mathbf{v}$ is in the **$\boldsymbol{\ell}^{\mathbf{2}}$-span** of $\{\mathbf{u}_n\}$ if $\mathbf{v}$ can be expressed as an ℓ^2-linear combination of the vectors $\{\mathbf{u}_n\}$. According to the previous corollary, any vector $\mathbf{v}$ in the ℓ^2-span of $\{\mathbf{u}_n\}$ can be written as

$$\mathbf{v} \;=\; \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \mathbf{u}_n.$$

It follows that

$$\|\mathbf{v}\| \;=\; \sqrt{\sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle^2}$$

and

$$\langle \mathbf{v}, \mathbf{w} \rangle \;=\; \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \langle \mathbf{u}_n, \mathbf{w} \rangle$$

for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in the ℓ^2-span of $\{\mathbf{u}_n\}$.

Projections

Definition: Projection Onto a Subspace
Let V be an inner product space, let S be a linear subspace of V, and let $\mathbf{v} \in V$. A vector $\mathbf{p} \in S$ is called the **projection of v onto *S*** if

$$\langle \mathbf{s}, \mathbf{v} - \mathbf{p} \rangle \;=\; 0$$

for all $\mathbf{s} \in S$.

It is easy to see that the projection $\mathbf{p}$ of $\mathbf{v}$ onto S, if it exists, must be unique. In particular, if $\mathbf{p}_1$ and $\mathbf{p}_2$ are two possible projections, then

$$\|\mathbf{p}_1 - \mathbf{p}_2\|^2 \;=\; \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2 \rangle \;=\; \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{v} - \mathbf{p}_2 \rangle - \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{v} - \mathbf{p}_1 \rangle,$$

and both of the inner products on the right are zero since $\mathbf{p}_1 - \mathbf{p}_2 \in S$.

It is always possible to project onto a finite-dimensional subspace.

Proposition 8 Projection Onto Finite-Dimensional Subspaces

Let V be an inner product space, let S be a finite-dimensional subspace of V, and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for S. Then for any $\mathbf{v} \in V$, the vector

$$\mathbf{p} \;=\; \sum_{k=1}^{n} \langle \mathbf{u}_k, \mathbf{v}\rangle \mathbf{u}_k$$

is the projection of $\mathbf{v}$ onto S.

PROOF Observe that $\langle \mathbf{u}_k, \mathbf{p}\rangle = \langle \mathbf{u}_k, \mathbf{v}\rangle$ for each k, and hence $\langle \mathbf{u}_k, \mathbf{v} - \mathbf{p}\rangle = 0$ for each k. By linearity, it follows that $\langle \mathbf{s}, \mathbf{v} - \mathbf{p}\rangle = 0$ for all $\mathbf{s} \in S$, and hence $\mathbf{p}$ is the projection of $\mathbf{v}$ onto S. ■

Our goal is to generalize this proposition to the ℓ^2-span of an orthonormal sequence.

Lemma 9 Bessel's Inequality

Let V be a Hilbert space, let $\{\mathbf{u}_n\}$ be an orthonormal sequence in V, and let $\mathbf{v} \in V$. Then

$$\sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v}\rangle^2 \;\leq\; \|\mathbf{v}\|^2.$$

PROOF Let $N \in \mathbb{N}$, and let

$$\mathbf{p}_N \;=\; \sum_{n=1}^{N} \langle \mathbf{u}_n, \mathbf{v}\rangle \mathbf{u}_n$$

be the projection of $\mathbf{v}$ onto $\mathrm{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$. Then $\langle \mathbf{p}_N, \mathbf{v} - \mathbf{p}_N\rangle = 0$, so by the Pythagorean theorem

$$\|\mathbf{v}\|^2 \;=\; \|\mathbf{p}_N\|^2 + \|\mathbf{v} - \mathbf{p}_N\|^2 \;\geq\; \|\mathbf{p}_N\|^2 \;=\; \sum_{n=1}^{N} \langle \mathbf{u}_n, \mathbf{v}\rangle^2.$$

This holds for all $N \in \mathbb{N}$, so the desired inequality follows. ■

Proposition 10 Projection Formula

Let V be a Hilbert space, and let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in V. Then for any $\mathbf{v} \in V$, the sequence $\{\langle \mathbf{u}_n, \mathbf{v} \rangle\}$ is ℓ^2, and the vector

$$\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \mathbf{u}_n$$

is the projection of $\mathbf{v}$ onto the ℓ^2-span of $\{\mathbf{u}_n\}$.

PROOF Bessel's inequality shows that the sequence $\{\langle \mathbf{u}_n, \mathbf{v} \rangle\}$ is ℓ^2, and thus the sum for $\mathbf{p}$ converges. By the coefficient formula (Corollary 7), we have that

$$\langle \mathbf{u}_n, \mathbf{p} \rangle = \langle \mathbf{u}_n, \mathbf{v} \rangle$$

for all $n \in \mathbb{N}$, and hence $\langle \mathbf{u}_n, \mathbf{v} - \mathbf{p} \rangle = 0$ for all $n \in \mathbb{N}$. By the continuity of $\langle -, - \rangle$, it follows that $\langle \mathbf{s}, \mathbf{v} - \mathbf{p} \rangle = 0$ for any $\mathbf{s}$ in the ℓ^2-span of $\{\mathbf{u}_n\}$, and hence $\mathbf{p}$ is the projection of $\mathbf{v}$ onto this subspace. ■

Hilbert Bases

Definition: Hilbert Basis
Let V be a Hilbert space, and let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in V. We say that $\{\mathbf{u}_n\}$ is a **Hilbert basis** for V if for every $\mathbf{v} \in V$ there exists a sequence $\{a_n\}$ in ℓ^2 so that

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n.$$

That is, $\{\mathbf{u}_n\}$ is a Hilbert basis for V if every vector in V is in the ℓ^2-span of $\{\mathbf{u}_n\}$. For convenience, we are requiring all Hilbert bases to be countably infinite, but in the more general theory of Hilbert spaces a Hilbert basis may have any cardinality.

Note that a Hilbert basis $\{\mathbf{u}_n\}$ for V is not actually a basis for V in the sense of linear algebra. In particular, if $\{a_n\}$ is any ℓ^2 sequence with infinitely many nonzero terms, then the vector

$$\sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

cannot be expressed as a finite linear combination of Hilbert basis vectors. Of course, it is clearly much more useful to allow ℓ^2-linear combinations, and in the context of Hilbert spaces it is common to use the word **basis** to mean Hilbert basis, while a standard linear-algebra-type basis is referred to as a **Hamel basis**.

EXAMPLE 1 The Standard Basis for ℓ^2

Consider the following orthonormal sequence in ℓ^2:

$$\mathbf{e}_1 = (1,0,0,0,\ldots), \qquad \mathbf{e}_2 = (0,1,0,0,\ldots), \qquad \mathbf{e}_3 = (0,0,1,0,\ldots), \qquad \ldots$$

If $\mathbf{v} = (v_1, v_2, \ldots)$ is a vector in ℓ^2, it is easy to show that

$$\mathbf{v} = \sum_{n=1}^{\infty} v_n \mathbf{e}_n,$$

and therefore $\{\mathbf{e}_n\}$ is a Hilbert basis for ℓ^2. ■

This example is in some sense quite general, as shown by the following proposition.

Proposition 11 Isomorphism With ℓ^2

Let V be a Hilbert space, and suppose that V has a Hilbert basis $\{\mathbf{u}_n\}$. Then there exists an isometric isomorphism $T\colon \ell^2 \to V$ such that $T(\mathbf{e}_n) = \mathbf{u}_n$ for each n.

PROOF Define a function $T\colon \ell^2 \to V$ by

$$T(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} a_n \mathbf{u}_n.$$

Clearly T is linear. Note also that T is a bijection, with inverse given by

$$T^{-1}(\mathbf{v}) = \big(\langle \mathbf{u}_1, \mathbf{v}\rangle, \langle \mathbf{u}_2, \mathbf{v}\rangle, \ldots\big),$$

and hence T is a linear isomorphism. Finally, we have

$$\big\|T(a_1, a_2, \ldots)\big\| = \left\| \sum_{n=1}^{\infty} a_n \mathbf{u}_n \right\| = \sqrt{\sum_{n=1}^{\infty} a_n^2} = \big\|(a_1, a_2, \ldots)\big\|_2$$

for all $(a_1, a_2, \ldots) \in \ell^2$, so T is isometric. ■

Proposition 12 Characterization of Hilbert Bases

Let V be a Hilbert space, and let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in V. Then the following are equivalent:

1. *The sequence $\{\mathbf{u}_n\}$ is a Hilbert basis for V.*

2. *The set of all finite linear combinations of elements of $\{\mathbf{u}_n\}$ is dense in V.*

3. *For every nonzero $\mathbf{v} \in V$, there exists an $n \in \mathbb{N}$ so that $\langle \mathbf{u}_n, \mathbf{v} \rangle \neq 0$.*

PROOF Let S be the set of all finite linear combinations of elements of $\{\mathbf{u}_n\}$, i.e. the linear span of $\{\mathbf{u}_n\}$. We prove that (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2) Suppose that $\{\mathbf{u}_n\}$ is a Hilbert basis, and let $\mathbf{v} \in V$. Then

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

for some ℓ^2 sequence $\{a_n\}$. Then $\mathbf{v}$ is the limit of the sequence of partial sums

$$\mathbf{s}_N = \sum_{n=1}^{N} a_n \mathbf{u}_n,$$

so $\mathbf{v}$ lies in the closure of S.

(2) ⇒ (3) Suppose that S is dense in V, and let $\mathbf{v}$ be a nonzero vector in V. Let $\{\mathbf{s}_n\}$ be a sequence in S that converges to $\mathbf{v}$. Then there exists an $n \in \mathbb{N}$ so that $\|\mathbf{s}_n - \mathbf{v}\| < \|\mathbf{v}\|$, and it follows that

$$\langle \mathbf{s}_n, \mathbf{v} \rangle = \frac{\|\mathbf{s}_n\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{s}_n - \mathbf{v}\|^2}{2} > \frac{\|\mathbf{s}_n\|^2}{2} \geq 0.$$

But since $\mathbf{s}_n \in S$, we know that $\mathbf{s}_n \in \mathrm{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for some $k \in \mathbb{N}$, and it follows that $\langle \mathbf{u}_i, \mathbf{v} \rangle \neq 0$ for some $i \leq k$.

(3) ⇒ (1) Suppose that condition (3) holds, let $\mathbf{v} \in V$, and let

$$\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \mathbf{u}_n$$

be the projection of $\mathbf{v}$ onto the ℓ^2-span of $\{\mathbf{u}_n\}$ (by Proposition 10). Then $\langle \mathbf{u}_n, \mathbf{p} - \mathbf{v} \rangle = 0$ for all $n \in \mathbb{N}$, so by condition (3) the vector $\mathbf{p} - \mathbf{v}$ must be zero. Then $\mathbf{v} = \mathbf{p}$, so $\mathbf{v}$ lies in the ℓ^2-span of $\{\mathbf{u}_n\}$, which proves that $\{\mathbf{u}_n\}$ is a Hilbert basis. ■

Fourier Series

The theory of Hilbert spaces lets us provide a nice theory for Fourier series on the interval $[-\pi, \pi]$. We begin with the following theorem.

Theorem 13 Density of Continuous Functions

For any closed interval $[a, b] \subseteq \mathbb{R}$, the continuous functions on $[a, b]$ are dense in $L^2\big([a, b]\big)$.

PROOF See Homework 7, Problem 2 for a proof in the L^1 case. The L^2 case is quite similar. ■

It follows that any closed subset of $L^2\big([a, b]\big)$ that contains the continuous functions must be all of $L^2\big([a, b]\big)$.

Theorem 14 The Fourier Basis

The sequence

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \quad \frac{\cos 2x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \frac{\cos 3x}{\sqrt{\pi}}, \quad \frac{\sin 3x}{\sqrt{\pi}}, \quad \ldots$$

is a Hilbert basis for $L^2\big([-\pi, \pi]\big)$.

PROOF It is easy to check that the given functions are orthonormal. Let S be the set of all finite linear combinations of the basis elements, i.e. the set of all finite trigonometric polynomials. By Proposition 12, it suffices to prove that S is dense in $L^2\big([-\pi, \pi]\big)$.

Let $C(T)$ be the set of all continuous functions f on $[-\pi, \pi] \to \mathbb{R}$ for which $f(-\pi) = f(\pi)$. By Homework 10, every function in $C(T)$ is the uniform limit (and hence the L^2 limit) of trigonometric polynomials, so the closure of S contiains $C(T)$. But clearly every continuous function on $[a, b]$ is the L^2 limit of functions in $C(T)$, and hence the closure of S contains every continuous function. By Theorem 13, we conclude that the closure of S is all of $L^2\big([-\pi, \pi]\big)$. ■

In general, an orthogonal sequence $\{f_n\}$ of nonzero L^2 functions on $[a, b]$ is called a

complete orthogonal system for $[a, b]$ if the sequence $\{f_n/\|f_n\|_2\}$ of normalizations is a Hilbert basis for $L^2([a, b])$. According to the above theorem, the sequence

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \sin 3x, \quad \ldots$$

is a complete orthogonal system for the interval $[-\pi, \pi]$.

Definition: Fourier Coefficients
Let $f \colon [-\pi, \pi] \to \mathbb{R}$ be an L^2 function. Then the **Fourier coefficients** of f are defined as follows:

$$a = \frac{\langle f, 1\rangle}{2\pi} = \frac{1}{2\pi}\int_{[-\pi,\pi]} f\,dm,$$

$$b_n = \frac{\langle f, \cos nx\rangle}{\pi} = \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\cos nx\,dm(x),$$

$$c_n = \frac{\langle f, \sin nx\rangle}{\pi} = \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\sin nx\,dm(x).$$

Note that the Fourier coefficients are the coefficients for the functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \sin 3x, \quad \ldots,$$

which are not unit vectors. The actual coefficients of the Hilbert basis vectors are

$$a\sqrt{2\pi}, \qquad \{b_n\sqrt{\pi}\}, \qquad \text{and} \qquad \{c_n\sqrt{\pi}\}.$$

Corollary 15 Riesz-Fischer Theorem

Let $f \colon [-\pi, \pi] \to \mathbb{R}$ be an L^2 function with Fourier coefficients a, $\{b_n\}$ and $\{c_n\}$. Then $\{b_n\}$ and $\{c_n\}$ are ℓ^2 sequences, and the Fourier series

$$a + \sum_{n=1}^{\infty}\bigl(b_n \cos nx + c_n \sin nx\bigr)$$

converges to f in L^2.

PROOF This follows from Theorem 14 and the coefficient formula (Corollary 7). ■

Corollary 16 Parseval's Theorem

Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be an L^2 function with Fourier coefficients a, $\{b_n\}$, $\{c_n\}$, and let $g\colon [-\pi,\pi] \to \mathbb{R}$ be an L^2 function with fourier coefficients A, $\{B_n\}$, and $\{C_n\}$. Then

$$\frac{1}{\pi}\int_{[-\pi,\pi]} fg\,dm \;=\; 2aA + \sum_{n=1}^{\infty}(b_nB_n + c_nC_n).$$

PROOF By the inner product formula (Proposition 5), we have

$$\langle f,g\rangle \;=\; \big(a\sqrt{2\pi}\big)\big(A\sqrt{2\pi}\big) + \sum_{n=1}^{\infty}\Big(\big(b_n\sqrt{\pi}\big)\big(B_n\sqrt{\pi}\big) + \big(c_n\sqrt{\pi}\big)\big(C_n\sqrt{\pi}\big)\Big),$$

and dividing through by π gives the desired formula. ■

In the case where $g = f$, this theorem yields **Parseval's identity**:

$$\frac{1}{\pi}\int_{[-\pi,\pi]} f^2\,dm \;=\; 2a^2 + \sum_{n=1}^{\infty}\big(b_n^2 + c_n^2\big).$$

Corollary 17 Isomorphism of L^2 and ℓ^2

If $a < b$, then $L^2([a,b])$ and ℓ^2 are isometrically isomorphic.

PROOF Since

$$\frac{1}{\sqrt{2\pi}},\quad \frac{\cos x}{\sqrt{\pi}},\quad \frac{\sin x}{\sqrt{\pi}},\quad \frac{\cos 2x}{\sqrt{\pi}},\quad \frac{\sin 2x}{\sqrt{\pi}},\quad \frac{\cos 3x}{\sqrt{\pi}},\quad \frac{\sin 3x}{\sqrt{\pi}},\quad \ldots$$

is a Hilbert basis for $L^2\big([-\pi,\pi]\big)$, it follows from Proposition 11 that the linear transformation $T\colon \ell^2 \to L^2\big([-\pi,\pi]\big)$ defined by

$$T(a_1,a_2,a_3,\ldots) \;=\; \frac{a_1}{\sqrt{2\pi}} + \frac{a_2\cos x}{\sqrt{\pi}} + \frac{a_3\sin x}{\sqrt{\pi}} + \frac{a_4\cos 2x}{\sqrt{\pi}} + \frac{a_5\sin 2x}{\sqrt{\pi}} + \cdots$$

is an isometric isomorphism. ■

Other Orthogonal Systems

The Fourier basis is not the only Hilbert basis for $L^2\big([a,b]\big)$. Indeed, many such families of orthogonal functions are known. In this section, we derive an orthonormal sequence of polynomials that is a Hilbert basis for $L^2\big([a,b]\big)$.

Consider the sequence of functions

$$1, \quad x, \quad x^2, \quad x^3, \quad \ldots$$

on the interval $[-1,1]$. These functions are not a Hilbert basis for $L^2\big([-1,1]\big)$, since they are not orthonormal. However, it is possible to use these functions to make a Hilbert basis of polynomials via the **Gram-Schmidt process**. We start by making the the constant function 1 into a unit vector:

$$p_0(x) \;=\; \frac{1}{\|1\|_2} \;=\; \frac{1}{\sqrt{2}}.$$

The function x is already orthogonal to p_0 on the interval $[-1,1]$, so we normalize x as well:

$$p_1(x) \;=\; \frac{x}{\|x\|_2} \;=\; x\sqrt{\frac{3}{2}}.$$

Now we want a quadratic polynomial orthogonal to p_0 and p_1. The function x^2 is already orthogonal to p_1, but not to p_0. However, if we subtract from x^2 the projection of x^2 onto p_0, then we get a quadratic polynomial orthogonal to p_0:

$$x^2 - \langle p_0, x^2\rangle\, p_0(x) \;=\; x^2 - \frac{1}{3}.$$

Normalizing gives:

$$p_2(x) \;=\; \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)$$

Continuing in this fashion, we obtain an orthonormal sequence $\{p_n\}$ of polynomials, where each p_n is obtained from x^n by subtracting the projections of x^n onto $p_0, \ldots, p_{n-1}$ and then normalizing.

Definition: Legendre Polynomials

The **normalized Legendre polynomials** are the sequence of polynomial functions $p_n \colon [-1,1] \to \mathbb{R}$ defined recursively by $p_0(x) = 1/\sqrt{2}$ and

$$p_n(x) \;=\; c_n\left(x^n \;-\; \sum_{k=0}^{n-1} \big\langle p_k, x^n \big\rangle\, p_k(x)\right)$$

for $n \geq 1$, where the constant $c_n > 0$ is chosen so that $\|p_n\|_2 = 1$.

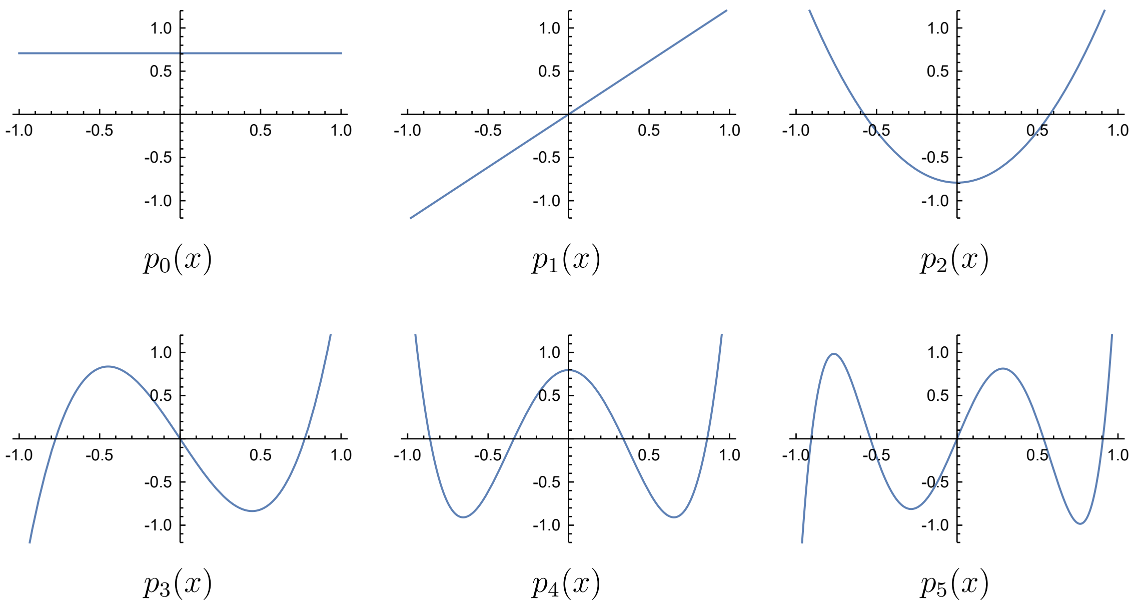

Figure 1: The normalized Legendre polynomials $p_0, \ldots, p_5$.

By design, each normalized Legendre polynomial $p_n(x)$ has degree n, and the sequence $\{p_n\}_{n \geq 0}$ is orthonormal. The next few such polynomials are

$$p_3(x) = \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5}x\right), \qquad p_4(x) = \frac{105}{8\sqrt{2}}\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right), \qquad \ldots$$

Figure 1 shows the graphs of the first six normalized Legendre polynomials.

Theorem 18 The Legendre Basis

The sequence $p_0, p_1, p_2, \ldots$ of normalized Legendre polynomials is a Hilbert basis for $L^2\big([-1,1]\big)$.

PROOF Let S be the linear span of $p_0, p_1, p_2, \ldots$. Since

$$x^n = \frac{p_n(x)}{c_n} + \sum_{k=0}^{n-1} \langle p_k, x^n \rangle \, p_k(x),$$

the subspace S contains each x^n, and hence contains all polynomials. By the Weierstrass approximation theorem, every continuous function on $[-1,1]$ is a uniform limit (and hence and L^2 limit) of a sequence of polynomials. It follows that the closure of S contains all the continuous functions, and hence contains all L^2 functions by Theorem 13. ■

Thus every L^2 function f on $[-1, 1]$ can be written as the sum of an infinite **Legendre series**

$$f = \sum_{n=0}^{\infty} \langle p_n, f \rangle p_n.$$

These behave much like Fourier series, with analogs of Parseval's theorem and Parseval's identity.

Legendre polynomials are important in partial differential equations. For the following definition, recall that a **harmonic function** on a closed region in $\mathbb{R}^3$ is any continuous function that satisfies Laplace's equation $\nabla^2 f = 0$ on the interior of the region.

Definition: Dirichlet Problem on a Ball

Let B^3 denote the closed unit ball on $\mathbb{R}^3$, and let S^2 denote the unit sphere. The **Dirichlet problem** on B^3 can be stated as follows:

> Given a continuous function $f\colon S^2 \to \mathbb{R}$, find a harmonic function $F\colon B^3 \to \mathbb{R}$ that agrees with f on S^2.

Since we are working on the ball, it makes sense to use **spherical coordinates** (ρ, θ, ϕ), which are defined by the formulas

$$x = \rho \cos\theta \sin\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho\cos\phi.$$

Using spherical coordinates, one family of solutions to Laplace's equation on the ball can be written as follows:

$$F(\rho, \theta, \phi) = \rho^n \, p_n(\cos\phi)$$

where p_n is the nth Legendre polynomial. These solutions are all **axially symmetric** around the z-axis, meaning that they have no explicit dependence on θ.

Since the Legendre polynomials are a Hilbert basis, we can use these solutions to solve the Dirichlet problem for any axially symmetric function $f\colon S^2 \to \mathbb{R}$. All we do is write f as the sum of a Legendre series

$$f(\theta, \phi) = \sum_{n=0}^{\infty} a_n \, p_n(\cos\phi),$$

and then the corresponding harmonic function F will be defined by the formula

$$F(\rho, \theta, \phi) = \sum_{n=0}^{\infty} a_n \rho^n p_n(\cos\phi).$$

EXAMPLE 1 Let $f\colon S^2 \to \mathbb{R}$ be the function defined by

$$f(x,y,z) \;=\; z^2.$$

Find a harmonic function $F\colon B^3 \to \mathbb{R}$ that agrees with f on S^2.
SOLUTION Note that $z = \cos\phi$ on S^2, so we can write f as

$$f(\theta,\phi) \;=\; \cos^2\phi.$$

Since

$$p_0(x) \;=\; c_0 \qquad \text{and} \qquad p_2(x) \;=\; c_2\big(x^2 - \tfrac{1}{3}\big)$$

where $c_0 = 1/\sqrt{2}$ and $c_2 = \sqrt{45/8}$, we can write f as

$$f(\theta,\phi) \;=\; \frac{1}{3c_0}p_0(\cos\phi) + \frac{1}{c_2}p_2(\cos\phi).$$

Then the corresponding harmonic function $F\colon B^3 \to \mathbb{R}$ is given by

$$F(\rho,\theta,\phi) \;=\; \frac{1}{3c_0}p_0(\cos\phi) + \frac{\rho^2}{c_2}p_2(\cos\phi) \;=\; \frac{1}{3} + \rho^2\left(\cos^2\phi - \frac{1}{3}\right). \qquad \blacksquare$$

The functions $p_n(\cos\phi)$ on the unit sphere can be generalized to the family of **spherical harmonics** $Y_{\ell,m}(\theta,\phi)$, which are a Hilbert basis for $L^2(S^2)$. The Legendre polynomials defined above correspond to the $m = 0$ case:

$$Y_{\ell,0}(\theta,\phi) \;=\; \frac{1}{\sqrt{2\pi}}\, p_\ell(\cos\phi).$$

Every L^2 function f on the sphere has a Fourier decomposition in terms of spherical harmonics:

$$f(\theta,\phi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_{\ell,m}(\theta,\phi).$$

In quantum mechanics, these spherical harmonics give rise to the eigenfunctions of the square of the angular momentum operator. These are known as **atomic orbitals**, and can be used to describe the quantum wave functions of electrons in an atom.