# $L^1$ Completeness

We are now ready for one of the most important theorems in analysis, arguably more important than even the dominated convergence theorem. We refer to it as the **$L^1$ completeness theorem**, though it is also known as the **$L^1$ Riesz-Fischer theorem**.

## Theorem 1 $L^1$ Completeness

*Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on $X$. Suppose that, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that*

$$i, j \geq N \quad \Rightarrow \quad \int_X |f_i - f_j| \, d\mu \, < \, \epsilon.$$

*Then there exists a measurable function $f$ on $X$ so that*

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu \, = \, 0.$$

We will put off the proof of this theorem for a little while in favor of discussing its meaning.

This theorem is essentially a Cauchy criterion for sequences of measurable functions. In particular, recall that a sequence $\{x_n\}$ of real numbers is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad |x_i - x_j| < \epsilon.$$

The **Cauchy criterion** states that any Cauchy sequence of real numbers converges.

The $L^1$ completeness theorem can be viewed as a version of the Cauchy criterion for measurable functions. The hypothesis of the theorem says that $\{f_n\}$ is something like a Cauchy sequence of measurable functions, and the conclusion says that $\{f_n\}$ converges to some measurable function $f$ in a certain sense. The following definition makes both of these notions precise.

**Definition: $L^1$ Cauchy sequence, $L^1$ Convergence**
Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on $X$.

1. We say that $\{f_n\}$ is an ***$L^1$* Cauchy sequence** if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad \int_X |f_i - f_j| \, d\mu \, < \, \epsilon.$$

2. We say that $\{f_n\}$ **converges in *$L^1$*** to a measurable function $f$ if

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu \, = \, 0.$$

Using these definitions, the $L^1$ completeness theorem (Theorem 1) is simply the statement that every $L^1$ Cauchy sequence converges in $L^1$.

We can simplify the notation in these definitions using the $L^1$ norm. If $f$ is a measurable function on $X$, recall that the ***$L^1$*-norm** of $f$ is defined by

$$\|f\|_1 \, = \, \int_X |f| \, d\mu.$$

Then a sequence $\{f_n\}$ of measurable functions is a Cauchy sequence if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad \|f_i - f_j\|_1 < \epsilon.$$

Similarly, a sequence $\{f_n\}$ converges in $L^1$ to a function $f$ if and only if

$$\lim_{n\to\infty} \|f_n - f\|_1 \, = \, 0.$$

## Proof of Completeness

We begin by stating and proving a useful criterion for convergence of sequences of functions, which is based on the idea of bounded variation. A sequence $\{x_n\}$ of real numbers is said to have **bounded variation** if

$$\sum_{n=1}^{\infty} |x_{n+1} - x_n| \,<\, \infty.$$

That is, $\{x_n\}$ has bounded variation if the path on the number line that visits every term of the sequence has finite total length.

## Proposition 2 Bounded Variation Test

*Any sequence $\{x_n\}$ of real numbers with bounded variation converges.*

**PROOF** Since $\{x_n\}$ has bounded variation, the series

$$\sum_{n=1}^{\infty} (x_{n+1} - x_n)$$

converges absolutely. Then

$$\lim_{n\to\infty} x_n \,=\, \lim_{n\to\infty} \left( x_1 + \sum_{k=1}^{n-1} (x_{k+1} - x_k) \right) \,=\, x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k).$$ ■

The following theorem gives a nice criterion for when a sequence $\{f_n\}$ of measurable functions converges.

## Theorem 3 $L^1$ Convergence Criterion

*Let $(X, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that*

$$\sum_{n=1}^{\infty} \int_X |f_{n+1} - f_n| \, d\mu \,<\, \infty.$$

*Then $\{f_n\}$ converges pointwise almost everywhere to a measurable function $f$, and $f_n \to f$ in $L^1$.*

In general, we say that a sequence $\{f_n\}$ of measurable functions has **bounded $L^1$-variation** if

$$\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_1 \,<\, \infty.$$

The condition is precisely the hypothesis of the present theorem.

**PROOF** Let

$$g \,=\, \sum_{n=1}^{\infty} |f_{n+1} - f_n|.$$

By the monotone convergence theorem, we know that

$$\int_X g\, d\mu \,=\, \int_X \sum_{n=1}^{\infty} |f_{n+1} - f_n|\, d\mu \,=\, \sum_{n=1}^{\infty} \int_X |f_{n+1} - f_n|\, d\mu \,<\, \infty.$$

so $g$ is $L^1$. In particular $g(x) < \infty$ for almost all $x \in X$, so $\{f_n(x)\}$ has bounded variation for almost all $x \in X$, and hence $\{f_n(x)\}$ converges pointwise almost everywhere.

Let $f$ be the pointwise limit of the sequence $\{f_n\}$, and note that for each $n \in \mathbb{N}$,

$$f - f_n \,=\, \lim_{N\to\infty} f_{N+1} - f_n \,=\, \lim_{N\to\infty} \sum_{k=n}^{N} (f_{k+1} - f_k) \,=\, \sum_{k=n}^{\infty} (f_{k+1} - f_k)$$

almost everywhere. Then

$$|f - f_n| \,=\, \left| \sum_{k=n}^{\infty} \big(f_{k+1} - f_k\big) \right| \,\leq\, \sum_{k=n}^{\infty} |f_{k+1} - f_k| \,\leq\, g$$

almost everywhere, so by the dominated convergence theorem

$$\lim_{n\to\infty} \int_X |f - f_n|\, d\mu \,=\, \int_X \lim_{n\to\infty} |f - f_n|\, d\mu \,=\, 0.$$

Thus $f_n \to f$ in $L^1$. ■

We will use this theorem to prove that every $L^1$ Cauchy sequence converges. We begin with the following proposition.

## Proposition 4 Subsequences of Bounded Variation

*Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be an $L^1$ Cauchy sequence of measurable functions on $X$. Then there exists a subsequence of $\{f_n\}$ that has bounded $L^1$-variation.*

**PROOF** Since $\{f_n\}$ is an $L^1$ Cauchy sequence, there exists an increasing sequence $N_1 < N_2 < \cdots$ of positive integers so that

$$i, j \geq N_k \quad \Rightarrow \quad \|f_i - f_j\|_1 < \frac{1}{2^k}.$$

Since $N_{k+1}, N_k \geq N_k$ for all $k$, it follows that $\|f_{N_{k+1}} - f_{N_k}\|_1 < 1/2^k$, so

$$\sum_{k=1}^{\infty} \|f_{N_{k+1}} - f_{N_k}\| \;\leq\; \sum_{k=1}^{\infty} \frac{1}{2^k} \;=\; 1.$$

Thus $\{f_{N_k}\}$ is a subsequence of $\{f_n\}$ with bounded $L^1$-variation. ■

We are now ready to prove the $L^1$ completeness theorem.

**PROOF OF THEOREM 1** Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be an $L^1$ Cauchy sequence of measurable functions on $X$. Let $\{f_{n_k}\}$ be a subsequence of bounded $L^1$-variation, where $n_1 < n_2 < \cdots$ is an increasing sequence of positive integers. By Theorem 3, there exists a measurable function $f$ so that $f_{n_k} \to f$ in $L^1$. We claim that $f_n \to f$ in $L^1$.

Let $\epsilon > 0$. Since $\{f_n\}$ is an $L^1$ Cauchy sequence, there exists an $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad \|f_i - f_j\|_1 \;<\; \frac{\epsilon}{2}.$$

Since $f_{n_k} \to f$ in $L^1$, there exists a $k$ so that $n_k \geq N$ and $\|f_{n_k} - f\|_1 < \epsilon/2$. Then for all $n \geq N$, we have

$$\|f_n - f\|_1 \;\leq\; \|f_n - f_{n_k}\|_1 + \|f_{n_k} - f\|_1 \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$ ■

## Properties of $L^1$ Convergence

$L^1$ convergence is a new kind of convergence for functions, different from both pointwise convergence and uniform convergence. In this section we explore the properties of this new kind of convergence.

First, we observe that the $L^1$ limit of a sequence isn't quite uniquely determined.

**Proposition 5** Almost Uniqueness of the Limit

*Let $(X, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable functions on $X$, and let $f$ and $g$ be measurable functions on $X$. If*

$$f_n \to f \text{ in } L^1 \qquad \text{and} \qquad f_n \to g \text{ in } L^1$$

*then $f = g$ almost everywhere.*

**PROOF** Suppose $f_n \to f$ in $L^1$ and $f_n \to g$ in $L^1$. By the triangle inequality

$$\|f - g\|_1 \;\le\; \|f_n - f\|_1 + \|f_n - g\|_1$$

for each $n$. Since $\|f - f_n\|_1 \to 0$ and $\|g - f_n\|_1 \to 0$ as $n \to \infty$, it follows that $\|f - g\|_1 = 0$, and hence $f = g$ almost everywhere. ■

Note also that if $\{f_n\}$ and $\{g_n\}$ are sequences of measurable functions and $f_n = g_n$ almost everywhere for each $n$, then $f_n \to f$ in $L^1$ if and only if $g_n \to f$ in $L^1$. Thus $L^1$ convergence really doesn't care about the behavior of the functions on sets of measure zero.

Now, whenever we learn about a new kind of convergence for functions, one important question to ask is how "strong" the convergence is. For example, uniform convergence is stronger than pointwise convergence, since every sequence of functions that converges uniformly also converges pointwise. The following example shows that uniform convergence is not stronger than $L^1$ convergence.

**EXAMPLE 1** Uniform Convergence Does Not Imply $L^1$ Convergence

Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be the sequence of measurable functions

$$f_n \;=\; \frac{\chi_{[0,n]}}{n}.$$

Then $|f_n| \le 1/n$ for all $n$, so $f_n \to 0$ uniformly. However,

$$\lim_{n\to\infty} \|f_n - 0\|_1 \;=\; \lim_{n\to\infty} \int_{[0,n]} \frac{1}{n}\, dm \;=\; 1 \;\ne\; 0$$

and thus $f_n \not\to 0$ in $L^1$. ■

Despite this example, there is a large class of measure spaces on which uniform convergence is stronger than $L^1$ convergence.

## **Proposition 6** Uniform Convergence vs. $L^1$ Convergence

*Let $(X, \mu)$ be a measure space with $\mu(X) < \infty$, let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that $f_n \to f$ uniformly almost everywhere. Then $f_n \to f$ in $L^1$.*

**PROOF** By hypothesis, $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. But

$$\|f_n - f\|_1 \,=\, \int |f_n - f|\, d\mu \,\leq\, \|f_n - f\|_\infty\, \mu(X)$$

for all $n$, and hence $\|f_n - f\|_1 \to 0$ as $n \to \infty$. ■

The next question is whether $L^1$ convergence is stronger or weaker than pointwise convergence. The answer turns out to be neither.

**EXAMPLE 2** Pointwise Convergence Does Not Imply $L^1$ Convergence
Let $E_n$ be the following sequence of measurable subsets of $[0, 1]$:

$$[0,1], \quad \left[0,\tfrac{1}{2}\right], \ \left[\tfrac{1}{2},1\right], \quad \left[0,\tfrac{1}{3}\right], \ \left[\tfrac{1}{3},\tfrac{2}{3}\right], \ \left[\tfrac{2}{3},1\right], \quad \ldots$$

Then $\{\chi_{E_n}\}$ is a sequence of measurable functions on $[0, 1]$, and

$$\lim_{n\to\infty} \|\chi_{E_n} - 0\|_1 \,=\, \lim_{n\to\infty} m(E_n) \,=\, 0,$$

so $\chi_{E_n} \to 0$ in $L^1$. However, the sequence $\{\chi_{E_n}(x)\}$ does not converge to 0 at any point $x \in [0, 1]$, since each $x$ lies in infinitely many of the sets $E_n$. ■

**EXAMPLE 3** $L^1$ Convergence Does Not Imply Pointwise Convergence
Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the sequence of measurable functions

$$f_n \,=\, n\,\chi_{(0,1/n)}.$$

Then $f_n \to 0$ pointwise, but

$$\lim_{n\to\infty} \|f_n - 0\|_1 \,=\, \lim_{n\to\infty} \int_{(0,1/n)} n\, dm \,=\, 1 \,\neq\, 0$$

and thus $f_n \not\to 0$ in $L^1$. ■

Despite the previous example, there is a certain sense in which $L^1$ convergence implies pointwise convergence.

## Proposition 7 Pointwise Convergence on a Subsequence

*Let $(X, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that $f_n \to f$ in $L^1$. Then there exists a subsequence of $\{f_n\}$ that converges pointwise to $f$ almost everywhere.*

**PROOF** It is easy to prove that any $L^1$ convergent sequence must be an $L^1$ Cauchy sequence (see Exercise 1). In particular, $\{f_n\}$ is an $L^1$ Cauchy sequence, so by Proposition 4 there must be a subsequence $\{f_{n_k}\}$ of bounded $L^1$-variation. By Theorem 3, this subsequence converges pointwise almost everywhere to some function $g$, and indeed $f_{n_k} \to g$ in $L^1$. But $f_{n_k} \to f$ in $L^1$ as well, so $f = g$ almost everywhere, and hence $f_{n_k} \to f$ pointwise almost everywhere. ■

Finally, there are some circumstances under which pointwise convergence implies $L^1$ convergence.

## Proposition 8 $L^1$ Dominated Convergence Theorem

*Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of $L^1$ functions on $X$ converging pointwise to a measurable function $f$. Suppose there exists an $L^1$ function $g$ on $X$ such that*

$$|f_n| \leq g$$

*for all $n \in \mathbb{N}$. Then $f_n \to f$ in $L^1$.*

**PROOF** Since $|f_n| \leq g$ for all $n$ and $f_n \to f$ pointwise, it follows that $|f| \leq g$, and hence $|f_n - f| \leq |f_n| + |f| = 2g$ for all $n$. Therefore, by the dominated convergence theorem

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = \int_X \lim_{n\to\infty} |f_n - f| \, d\mu = 0,$$

so $f_n \to f$ in $L^1$. ■

The conclusion "$f_n \to f$ in $L^1$" here is actually stronger than the usual conclusion that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \;=\; \int_X f \, d\mu,$$

as the following proposition shows.

## Proposition 9 Integrals of $L^1$ Limits

*Let $(X, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of $L^1$ functions on $X$, and suppose $f_n \to f$ in $L^1$. Then $f$ is $L^1$, and*

$$\lim_{n\to\infty} \int_X f_n \, d\mu \;=\; \int_X f \, d\mu.$$

**PROOF** Since $f_n \to f$ in $L^1$, there exists a $k$ so that $\|f_k - f\|_1 < \infty$. Then by the triangle inequality

$$\|f\|_1 \;\leq\; \|f_k - f\|_1 + \|f_k\|_1 \;<\; \infty + \infty \;=\; \infty$$

which proves that $f$ is $L^1$. Moreover, we have that

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \;=\; \left| \int_X (f_n - f) \, d\mu \right| \;\leq\; \int_X |f_n - f| \, d\mu \;=\; \|f_n - f\|_1$$

for each $n$. Since $\|f_n - f\|_1 \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \;=\; \int_X f \, d\mu.$$

■

# Exercises

For the following exercises, let $(X, \mu)$ be a measure space.

1. Let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that $\{f_n\}$ converges in $L^1$ to a measurable function $f$. Prove that $\{f_n\}$ is an $L^1$ Cauchy sequence.

2. Let $\{f_n\}$ be the sequence of measurable functions on $[0, 1]$ defined by

$$f_n(x) \;=\; \sqrt{n}\, x^n.$$

   Prove that $f_n \to 0$ in $L^1$.

3. Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on $X$, let $f$ and $g$ be measurable functions on $X$, and suppose that $f_n \to f$ in $L^1$ and $g_n \to g$ in $L^1$. Prove that $f_n + g_n \to f + g$ in $L^1$.

4. Let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that $\{f_n\}$ has bounded $L^1$ variation. Give a direct proof that $\{f_n\}$ is an $L^1$ Cauchy sequence.

5. Let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that

$$\sum_{n\in\mathbb{N}} \|f_n\|_1 \;<\; \infty.$$

   Prove that the sequence of partial sums $S_n = \sum_{k=1}^n f_k$ converges in $L^1$.

6. Let $f$ be an $L^1$ function on $X$. Prove that there exists a sequence of simple functions that converges to $f$ in $L^1$.

7. Find a sequence $\{f_n\}$ of analytic functions on $\mathbb{R}$ such that $f_n \to 0$ uniformly but $f_n \not\to 0$ in $L^1$.

8. Let $\{f_n\}$ be a sequence of measurable functions on $X$, and suppose that

$$\lim_{n\to\infty} \int_X |f_n|\, d\mu \;=\; 0.$$

   Prove that there exists a subsequence of $\{f_n\}$ that converges to 0 almost everywhere.