L^p Functions

Given a measure space (X, μ) and a real number $p \in [1, \infty)$, recall that the $\boldsymbol{L^p}$**-norm** of a measurable function $f\colon X \to \mathbb{R}$ is defined by

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$$

Note that the L^p-norm of a function f may be either finite or infinite. The L^p **functions** are those for which the p-norm is finite.

Definition: L^p Function
Let (X, μ) be a measure space, and let $p \in [1, \infty)$. An $\boldsymbol{L^p}$ **function** on X is a measurable function f on X for which

$$\int_X |f|^p \, d\mu < \infty.$$

Like any measurable function, and L^p function is allowed to take values of $\pm\infty$. However, it follows from the definition of an L^p function that it must take finite values almost everywhere, so there is not harm in restricting to L^p functions $X \to \mathbb{R}$.

It is easy to see that any scalar multiple of an L^p is again L^p. Moreover, if f and g are L^p functions, then by Minkowski's inequality

$$\|f + g\|_p \le \|f\|_p + \|g\|_p < \infty$$

so $f + g$ is an L^p function. Thus the set of L^p functions forms a vector space.

EXAMPLE 1 L^p Functions on $[0, 1]$
Any bounded function on $[0, 1]$ is automatically L^p for every value of p. However it is possible for the p-norm of a measurable function on $[0, 1]$ to be infinite. For example,

let $f\colon [0,1] \to \mathbb{R}$ be the function

$$f(x) \;=\; \frac{1}{x}$$

where the value of $f(0)$ is immaterial. Then by the monotone convergence theorem,

$$\int_{[0,1]} |f|\, dm \;=\; \lim_{a\to 0^+} \int_{[a,1]} \frac{1}{x}\, dm(x) \;=\; \lim_{a\to 0^+} \bigl[\log x\bigr]_a^1 \;=\; \infty$$

so f is not L^1. Indeed, it is easy to check that f is not L^p for any $p \in [1,\infty)$.

A function with a vertical asymptote does not automatically have infinite p-norm. For example, if

$$f(x) \;=\; \frac{1}{\sqrt{x}}$$

then f has a vertical asymptote at $x = 0$, but

$$\int_{[0,1]} |f|\, dm \;=\; \lim_{a\to 0^+} \int_{[a,1]} \frac{1}{\sqrt{x}}\, dm(x) \;=\; \lim_{a\to 0^+} \Bigl[2\sqrt{x}\Bigr]_a^1 \;=\; 2.$$

In general,

$$\int_{[0,1]} \frac{1}{x^r}\, dm(x) \;=\; \begin{cases} \infty & \text{if } r \geq 1 \\ 1/(1-r) & \text{if } r < 1. \end{cases}$$

It follows that the function $f(x) = 1/x^r$ is L^p if and only if $pr < 1$, i.e. if and only if $p < 1/r$. For example, $f(x) = 1/\sqrt{x}$ is L^p for all $p \in [1,2)$, but is not L^p for any $p \in [2,\infty)$. ■

The last example suggests that it should be *harder* for a function to be L^p the larger we make p. The following proposition confirms this intuition.

Proposition 1 Relation Between L^p and L^q

Let (X,μ) be a measure space, and let $1 \leq p \leq q < \infty$. If $\mu(X) = 1$, then

$$\|f\|_p \;\leq\; \|f\|_q$$

for every measurable function f. More generally, if $0 < \mu(X) < \infty$, then

$$\|f\|_p \;\leq\; \mu(X)^r\, \|f\|_q$$

for every measurable function f, where $r = (1/p) - (1/q)$, and hence every L^q function is also L^p.

PROOF The case where $\mu(X) = 1$ is the generalized mean inequality for the p-mean and the q-mean. For $0 < \mu(X) < \infty$, let $C = \mu(X)$, and let ν be the measure

$$d\nu \;=\; \frac{1}{C}\, d\mu.$$

Then $\nu(X) = 1$, so by the generalized mean inequality

$$\begin{aligned}\left(\int_X |f|_p \, d\mu\right)^{1/p} &= C^{1/p}\left(\int_X |f|^p \, d\nu\right)^{1/p} \\ &\le C^{1/p}\left(\int_X |f|^q \, d\nu\right)^{1/q} = C^{1/p}C^{-1/q}\left(\int_X |f|^q \, d\mu\right)^{1/q}. \qquad \blacksquare\end{aligned}$$

Note that this proposition only applies in the case where $\mu(X)$ is finite. As the following example shows, the relationship between L^p and L^q functions can be more complicated when $\mu(X) = \infty$.

EXAMPLE 2 Horizontal Asymptotes
Let $f\colon [1,\infty) \to \mathbb{R}$ be the function

$$f(x) \;=\; \frac{1}{x}.$$

Then f is not L^1, since by the monotone convergence theorem

$$\int_{[1,\infty)} |f| \, dm \;=\; \lim_{b\to\infty} \int_{[1,b]} \frac{1}{x}\, dm(x) \;=\; \lim_{b\to\infty} \big[\log x\big]_1^b \;=\; \infty.$$

However f is L^2, since

$$\int_{[1,\infty)} |f|^2 \, dm \;=\; \lim_{b\to\infty} \int_{[1,b]} \frac{1}{x^2}\, dm(x) \;=\; \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b \;=\; 1.$$

In general,

$$\int_{[1,\infty)} \frac{1}{x^r}\, dm(x) \;=\; \begin{cases} 1/(r-1) & \text{if } r > 1 \\ \infty & \text{if } r \le 1. \end{cases}$$

Thus $f(x) = 1/x^r$ is L^p if and only if $pr > 1$, i.e. if and only if $p > 1/r$. $\blacksquare$

Thus, for horizontal asymptotes it is *easier* for a function to be L^p the larger the value of p. Intuitively, this is because numbers close to 0 get smaller when taken to a larger power, so $|f|^p$ will be closer to the x-axis the larger the value of p.

ℓ^p Sequences

An important special case of L^p functions is for the measure space $(\mathbb{N}, \mu)$, where μ is counting measure on $\mathbb{N}$. In this case, a measurable function f on $\mathbb{N}$ is just a *sequence*

$$f(1), \quad f(2), \quad f(3), \quad \ldots$$

and the Lebesgue integral is the same as the sum of the series

$$\int_{\mathbb{N}} f\, d\mu \;=\; \sum_{n\in\mathbb{N}} f(n).$$

The definition of an L^p function on $\mathbb{N}$ takes the following form.

> **Definition: ℓ^p-Norm and ℓ^p Sequences**
> If $p \in [1,\infty)$, the ***ℓ^p*-norm** of a sequence $\{a_n\}$ of real numbers is defined by the formula
> $$\|\{a_n\}\|_p \;=\; \left(\sum_{n\in\mathbb{N}} |a_n|^p\right)^{1/p}.$$
> An ***ℓ^p* sequence** is a sequence $\{a_n\}$ of real numbers for which
> $$\sum_{n\in N} |a_n|^p \;<\; \infty.$$

Sequences behave in a similar manner to functions with horizontal asymptotes.

EXAMPLE 3 P-series

Recall that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. It follows that the sequence $\{1/n^p\}$ is ℓ^1 if and only if $p > 1$. For example,

$$\left\{\frac{1}{n^2}\right\} \text{ is } \ell^1 \qquad \text{but} \qquad \left\{\frac{1}{n}\right\} \text{ and } \left\{\frac{1}{\sqrt{n}}\right\} \text{ are not.}$$

Moreover, since $(1/n^r)^p = 1/n^{rp}$, we find that $\{1/n^r\}$ is ℓ^p if and only if $p > 1/r$. Thus

$$\left\{\frac{1}{n}\right\} \text{ is } \ell^2 \text{ but not } \ell^1,$$

and

$$\left\{\frac{1}{\sqrt{n}}\right\} \text{ is } \ell^3 \text{ but not } \ell^2. \qquad \blacksquare$$

All of this is very similar to our analysis of the function $1/x^p$ on $[1,\infty]$. Indeed, it follows from the integral test that

$$\int_1^\infty \frac{1}{x^p}\,dx < \infty \qquad \text{if and only if} \qquad \sum_{n=1}^\infty \frac{1}{n^p} < \infty$$

so there is a strong theoretical relationship between these two cases.

Proposition 2 Relationship Between ℓ^p and ℓ^q

If $1 \le p < q < \infty$, then every ℓ^p sequence is also ℓ^q.

PROOF Let $\{a_n\}$ be an ℓ^p sequence. Then

$$\sum_{n\in\mathbb{N}} |a_n|^p$$

converges, so it must be the case that $a_n \to 0$ as $n \to \infty$. In particular, there exists an $N \in \mathbb{N}$ such that $|a_n| < 1$ for all $n \ge N$. Then $|a_n|^q < |a_n|^p$ for all $n \ge N$, so

$$\sum_{n\in\mathbb{N}} |a_n|^q$$

converges by the comparison test. $\blacksquare$

Incidentally, Hölder's inequality is very interesting for sequences, since it essentially functions as a new convergence test for series.

Theorem 3 Hölder's Inequality for Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $p, q \in [1, \infty)$ so that $1/p + 1/q = 1$. If the series

$$\sum_{n=1}^{\infty} |a_n|^p \qquad \text{and} \qquad \sum_{n=1}^{\infty} |b_n|^p$$

both converge, then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges absolutely, and

$$\left| \sum_{n=1}^{\infty} a_n b_n \right| \leq \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}.$$

Corollary 4 Cauchy-Schwarz Inequality for Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If the series

$$\sum_{n=1}^{\infty} a_n^2 \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n^2$$

both converge, then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges absolutely, and

$$\left(\sum_{n=1}^{\infty} a_n b_n \right)^2 \leq \left(\sum_{n=1}^{\infty} a_n^2 \right) \left(\sum_{n=1}^{\infty} b_n^2 \right).$$

L^p Completeness

It is possible to generalize the completeness theorem to L^p.

Definition: L^p Sequences
Let (X, μ) be a measure space, let $\{f_n\}$ be a sequence of measurable functions on X, and let $p \in [1, \infty)$.

1. We say that $\{f_n\}$ is an **L^p Cauchy sequence** if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that
$$i, j \geq N \quad \Rightarrow \quad \|f_i - f_j\|_p < \epsilon.$$
2. We say that $\{f_n\}$ has **bounded L^p-variation** if
$$\sum_{n \in \mathbb{N}} \|f_{n+1} - f_n\|_p < \infty.$$
3. We say that $\{f_n\}$ **converges in L^p** to a measurable function f if
$$\lim_{n \to \infty} \|f_n - f\|_p = 0.$$

Theorem 5 L^p Convergence Criterion

Let (X, μ) be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on X with bounded L^p-variation. Then $\{f_n\}$ converges pointwise almost everywhere to a measurable function f, and $f_n \to f$ in L^p.

PROOF Let
$$M = \sum_{n \in \mathbb{N}} \|f_{n+1} - f_n\|_p < \infty.$$
and let
$$g = \sum_{n=1}^{\infty} |f_{n+1} - f_n| \qquad \text{and} \qquad g_N = \sum_{n=1}^{N} |f_{n+1} - f_n|$$
for each $N \in \mathbb{N}$. By Minkowski's inequality,
$$\|g_N\|_p \leq \sum_{n=1}^{N} \|f_{n+1} - f_n\|_p \leq M$$

for all $N \in \mathbb{N}$. By the monotone convergence theorem, it follows that

$$\int_X g^p\,d\mu \;=\; \int_X \lim_{N\to\infty} g_N^p\,d\mu \;=\; \lim_{N\to\infty}\int_X g_N^p\,d\mu \;=\; \lim_{N\to\infty} \|g_N\|_p^p \;\leq\; M^p \;<\; \infty.$$

From this we conclude that $g(x) < \infty$ for almost all $x \in X$, so $\{f_n(x)\}$ has bounded variation for almost all $x \in X$, and hence $\{f_n(x)\}$ converges pointwise almost everywhere.

Let f be the pointwise limit of the sequence $\{f_n\}$, and note that for each $n \in \mathbb{N}$,

$$f - f_n \;=\; \lim_{N\to\infty} f_{N+1} - f_n \;=\; \lim_{N\to\infty} \sum_{k=n}^{N} (f_{k+1} - f_k) \;=\; \sum_{k=n}^{\infty} (f_{k+1} - f_k)$$

almost everywhere. Then

$$|f - f_n|^p \;=\; \left|\sum_{k=n}^{\infty} \big(f_{k+1} - f_k\big)\right|^p \;\leq\; \left(\sum_{k=n}^{\infty} |f_{k+1} - f_k|\right)^p \;\leq\; g^p$$

almost everywhere, so by the dominated convergence theorem

$$\lim_{n\to\infty} \int_X |f - f_n|^p\,d\mu \;=\; \int_X \lim_{n\to\infty} |f - f_n|^p\,d\mu \;=\; 0.$$

Thus $f_n \to f$ in L^p. ■

L^p completeness follows easily. We leave the proof to the reader.

Theorem 6 L^p Completeness

Let (X, μ) be a measure space, and let $\{f_n\}$ be an L^p Cauchy sequence on X. Then $\{f_n\}$ converges in L^p to some measurable function f on X.

The L^∞ Norm

It is possible to extend the L^p norms in a natural way to the case $p = \infty$.

Definition: L^∞-Norm
Let (X, μ) be a measure space, and let f be a measurable function on X. The **L^∞-norm** of f is defined as follows:

$$\|f\|_\infty \,=\, \min\bigl\{M \in [0,\infty] \bigm| |f| \leq M \text{ almost everywhere}\bigr\}.$$

We say that f is an **L^∞ function** if $\|f\|_\infty < \infty$.

Note that the set

$$\bigl\{M \in [0,\infty] \bigm| |f| \leq M \text{ almost everywhere}\bigr\}$$

really does have a minimum element, for if $|f| \leq M + 1/n$ almost everywhere for all $n \in \mathbb{N}$, then it follows that $|f| \leq M$ almost everywhere.

The L^∞-norm $\|f|_\infty$ is sometimes called the **essential supremum** of $|f|$, and L^∞ functions are sometimes said to be **essentially bounded** or **bounded almost everywhere**. Note that a continuous function on $\mathbb{R}$ is L^∞ if and only if it is bounded, in which case $\|f\|_\infty$ is equal to the supremum of $|f|$.

Much of what we have done for $p \in [1,\infty)$ also works for $p = \infty$. We list some of the results, and leave the proofs to the reader:

Minkowski's Inequality. If f and g are L^∞ functions, then $f + g$ is L^∞, and

$$\|f + g\|_\infty \,\leq\, \|f\|_\infty + \|g\|_\infty.$$

Hölder's Inequality. If f is an L^1 function and g is an L^∞ function, then fg is Lebesgue integrable and

$$|\langle f, g\rangle| \,\leq\, \|f\|_1\,\|g\|_\infty.$$

L^∞ Convergence. If $\{f_n\}$ is a sequence of functions, we say that $\{f_n\}$ converges **in L^∞** to a function f if

$$\lim_{n\to\infty} \|f_n - f\|_\infty \,=\, 0.$$

This turns out to be the same as uniform convergence almost everywhere, i.e. $f_n \to f$ in L^∞ if and only if there exists a set Z of measure zero such that $f_n \to f$ uniformly on Z^c.

L^∞ Completeness. If $\{f_n\}$ is an L^∞ Cauchy sequence of measurable functions, then $\{f_n\}_\infty$ converges in L^∞ to some measurable function f.

Relation Between L^∞ and L^p If $\mu(X) = 1$, then $\|f\|_p \leq \|f\|_\infty$ for any measurable function f on X. More generally, if $0 < \mu(X) < \infty$ then

$$\|f\|_p \;\leq\; \mu(X)^{1/p}\,\|f\|_\infty$$

for all p, so any L^∞ function on X is also L^p for all $p \in [1,\infty)$.

In the case of sequences, the L^∞ norm takes the following form.

Definition: ℓ^∞-Norm
Let $\{a_n\}$ be a sequence of real numbers. The **ℓ^∞-norm** of $\{a_n\}$ is defined as follows:

$$\|\{a_n\}\|_\infty \;=\; \sup_{n\in\mathbb{N}} |a_n|$$

Thus an ℓ^∞ sequence is the same as a bounded sequence. Note that if $p \in [1,\infty)$, then any ℓ^p sequence must be ℓ^∞, since any ℓ^p sequence must converge to zero.

Exercises

For the following exercises, let (X,μ) be a measure space.

1. Let $f\colon [0,\infty) \to \mathbb{R}$ be the function $f(x) = e^{-x}$. For what values of p is f an L^p function?

2. Let $f\colon (0,\infty) \to \mathbb{R}$ be the function

$$f(x) \;=\; \begin{cases} x^{-1/3} & 0 < x < 1, \\ x^{-1/2} & 1 \leq x < \infty. \end{cases}$$

   For what values of p is f an L^p function?

3. Let $f\colon [0,1] \to [0,\infty]$ be the function $f(x) = -\log x$, with $f(0) = \infty$.

   (a) Show that f is L^1.

   (b) Show that f is L^p for all $p \in [1,\infty)$. (*Hint:* Substitute $u = 1/x$.)

4. For what values of p is
$$\left\{\frac{1}{(n^2+1)^{1/3}}\right\}$$
an ℓ^p sequence?

5. For what values of p is
$$\left\{\frac{1}{\sqrt{n}\,\log n}\right\}$$
an ℓ^p sequence?

6. Prove that every L^p Cauchy sequence has a subsequence of bounded L^p-variation.

7. Prove the L^p completeness theorem (Theorem 6).

8. If f and g are measurable functions on X, prove that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

9. If f is an L^1 function on X and g is an L^∞ function on X, prove that fg is Lebesgue integrable and $|\langle f, g\rangle| \leq \|f\|_1\,\|g\|_\infty$.

10. Let $\{f_n\}$ be a sequence of measurable functions on X, and let f be a measurable function on X. Prove that $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere.

11. If $0 < \mu(X) < \infty$ and f is a measurable function on X, prove that
$$\|f\|_p \;<\; \mu(X)^{1/p}\,\|f\|_\infty$$
for all $p \in [1, \infty)$.

12. Prove that every L^∞ Cauchy sequence of measurable functions converges uniformly almost everywhere.