Measures In General

Lebesgue measure on $\mathbb{R}$ is just one of many important measures in mathematics. In these notes we introduce the general framework for measures.

Definition: σ-Algebra
Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a **σ-algebra** if it satisfies the following conditions:

1. $\emptyset \in \mathcal{M}$.
2. If $E \in \mathcal{M}$, then the complement $E^c = X - E$ also lies in $\mathcal{M}$.
3. If $\{E_n\}$ is a sequence of sets in $\mathcal{M}$, then the union $\bigcup_{n\in\mathbb{N}} E_n$ also lies in $\mathcal{M}$.

The first axiom is almost unnecessary, for if $\mathcal{M}$ contains any set E, then it follows that $\emptyset = (E \cup E^c)^c$ lies in $\mathcal{M}$. Thus, the first axiom is equivalent to the requirement that the collection $\mathcal{M}$ is nonempty.

For example, the collection of Lebesgue measurable subsets of $\mathbb{R}$ forms a σ-algebra. The following proposition states some basic properties of σ-algebras, with the proofs left to the reader.

Proposition 1

Let X be a set, and let $\mathcal{M}$ be a σ-algebra on X.

1. *If $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2$ and $E_1 \cap E_2$ lie in $\mathcal{M}$.*

2. *If $\{E_n\}$ is a sequence of sets in $\mathcal{M}$, then the intersection $\bigcap_{n\in\mathbb{N}} E_n$ also lies in $\mathcal{M}$.*

We can now define the general notion of a measure, with the elements of some σ-algebra being the collection of measurable sets.

Definition: Measure
Let X be a set. A **measure** on X is a function $\mu\colon \mathcal{M} \to [0,\infty]$, where $\mathcal{M}$ is a σ-algebra of subsets of X, satisfying the following conditions:

1. $\mu(\emptyset) = 0$.
2. (Countable Additivity) For every sequence $\{E_n\}$ of pairwise disjoint sets in $\mathcal{M}$,
$$\mu\left(\biguplus_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n).$$

Again, the first axiom is almost unnecessary. Since the sequence $\emptyset, \emptyset, \emptyset, \ldots$ is pairwise disjoint, it follows from the second axiom that

$$\mu(\emptyset) = \sum_{n\in\mathbb{N}} \mu(\emptyset)$$

and therefore $\mu(\emptyset)$ is either 0 or ∞. If $\mu(\emptyset) = \infty$, it follows that $\mu(E) = \infty$ for every set E, so the first axiom serves only to disallow this "infinity-only" measure.

The prototypical example of a measure is Lebesgue measure on $\mathbb{R}$, but many other measures are possible. For a simple example, if X is any set, then the function $\mu\colon \mathcal{P}(x) \to [0,\infty]$ defined by $\mu(E) = |E|$ (the cardinality of E) is a measure, known as **counting measure** on X. For another example, there is a measure on $\mathbb{R}^2$ that essentially measures the area of a subset, and more generally there is a measure o $\mathbb{R}^n$ that measures n-dimensional volume.

As we will see in our construction, the Lebesgue integral can be defined on any set X that has been equipped with a measure. In the case of counting measure, the Lebesgue integral turns out to be the same as the sum of the function, i.e.

$$\int_X f = \sum_{x\in X} f(x)$$

for any function $f\colon X \to [-\infty,\infty]$. In the case of area measure on $\mathbb{R}^2$, the Lebesgue integral turns out the be a Lebesgue version of the usual double integral.

Incidentally, if $\mu\colon \mathcal{M} \to [0,\infty]$ is a measure on X, then the triple $(X, \mathcal{M}, \mu)$ is known as a **measure space**. A measure space is a very general notion of a set on which one can integrate, in the same way that a topological space is a very general notion of a set on which one can take limits.

The following proposition lists some of the most important properties of any measure. The proofs are left to the reader.

Proposition 2 Properties of Measures

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. *If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.*

2. *If $E, F \in \mathcal{M}$, then $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$.*

3. *If $\{E_n\}$ is any sequence in $\mathcal{M}$, then*

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu(E_n).$$

4. *If $E_1 \subseteq E_2 \subseteq \cdots$ is a nested sequence in $\mathcal{M}$, then*

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sup_{n\in\mathbb{N}} \mu(E_n).$$

5. *If $E_1 \supseteq E_2 \supseteq \cdots$ is a nested sequence in $\mathcal{M}$ and at least one E_n has finite measure, then*

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \inf_{n\in\mathbb{N}} \mu(E_n).$$

Outer Measures

Our construction of the Lebesgue measure began by defining the Lebesgue outer measure m^*, and then using it to determine which sets are measurable. This sort of construction can be carried out in general.

Definition: Outer Measure
Let X be a set. An **outer measure** on X is a function $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ satisfying the following conditions:

1. $\mu^*(\emptyset) = 0$.

2. (Monotonicity) If $S \subseteq T \subseteq X$, then $\mu^*(S) \leq \mu^*(T)$.

3. (Countable Subadditivity) If $\{S_n\}$ is any sequence of subsets of X, then

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(S_n).$$

The following theorem generalizes our construction of Lebesgue measure. The proof of this theorem is almost word-for-word the same as our proofs in the construction of Lebesgue measure.

Theorem 3 Carathéodory's Extension Theorem

Let X be a set, and let μ^* be an outer measure on X. Let $\mathcal{M}$ be the collection of all subsets $E \subseteq X$ satisfying

$$\mu^*(T \cap E) + \mu^*(T \cap E^c) \;=\; \mu^*(T)$$

for every $T \subseteq X$. Then $\mathcal{M}$ is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a measure.

A set E that lies in the σ-algebra $\mathcal{M}$ defined in this theorem is said to be **Carathéodory measurable** with respect to μ^*. For example, a set $E \subseteq \mathbb{R}$ is Carathéodory measurable with respect to Lebesgue outer measure m^* if and only if it is Lebesgue measurable.

Carathéodory's extension theorem makes it easy to construct measures. For example, consider the function $\mu^* \colon \mathcal{P}(\mathbb{R}^2) \to [0,\infty]$ defined by the formula

$$\mu^*(S) \;=\; \inf\left\{ \sum_{D\in\mathcal{C}} A(D) \;\middle|\; \mathcal{C} \text{ is a collection of open disks that covers } S \right\}.$$

where $A(D)$ denotes the area of an open disk D. It is quite easy to show that μ^* is an outer measure on $\mathbb{R}^2$, and using the above theorem we immediately obtain a measure μ on $\mathbb{R}^2$.

What is hard to prove is that μ actually measures area on $\mathbb{R}^2$, i.e. that $\mu(D) = A(D)$ for any open disk D. This involves the geometry of disks in a significant way, similar to how we needed to use the geometry of intervals in a significant way to prove that the Lebesgue measure of an interval is its length. In general, Carathéodory's extension theorem makes it easy to construct measures, but the construction is entirely abstract, and it's often difficult to determine the actual geometric meaning of the resulting measure.

Complete Measures

For many measures, sets of measure zero have a special property which is often useful.

Definition: Complete Measure
Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that μ is **complete** if for every $Z \in \mathcal{M}$ with $\mu(Z) = 0$, every subset of Z also lies in $\mathcal{M}$.

That is, a measure is complete if every subset of a set of measure zero is measurable. A subset of a set of measure zero is sometimes called a **null set**, so a measure is complete if and only if every null set is measurable.

Proposition 4 Completeness of Carathéodory Extensions

Any measure obtained from Carathéodory's extension theorem is complete. In particular, Lebesgue measure is complete.

PROOF Let X be a set, let μ^* be an outer measure on X, and let $\mu\colon \mathcal{M} \to [0,\infty]$ be the measure obtained from μ^* via Carathéodory's extension theorem. Let $N \subseteq X$ be a null set, so $N \subseteq Z$ for some $Z \in \mathcal{M}$ with $\mu(Z) = 0$. Then $\mu^*(N) \leq \mu^*(Z) = 0$, so $\mu^*(N) = 0$. Then for any set $T \subseteq X$, we have

$$\mu^*(T \cap N) + \mu^*(T \cap N^c) \;\leq\; \mu^*(N) + \mu^*(T) \;=\; \mu^*(T).$$

But $\mu^*(T \cap N) + \mu^*(T \cap N^c) \geq \mu^*(N)$ by the subadditivity of μ^*, and therefore N is Carathéodory measurable. ■

The following proposition shows that any measure can be extended to a complete measure in a canonical way.

Proposition 5 Completion of a Measure

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let

$$\mathcal{M}' \;=\; \{E \cup N \mid E \in \mathcal{M} \text{ and } N \subseteq X \text{ is a null set}\}.$$

Then $\mathcal{M}'$ is a σ-algebra, and there exists a unique measure $\mu'\colon \mathcal{M}' \to [0,\infty]$ that agrees with μ on $\mathcal{M}$. Furthermore, this measure μ' is complete.

PROOF Note first that any countable union of null sets is a null set. For given a sequence $\{N_n\}$ of null sets, let $\{Z_n\}$ be a sequence of sets of measure zero in $\mathcal{M}$ such that $N_n \subseteq Z_n$ for each n. Then the union $\bigcup_{n\in\mathbb{N}} Z_n$ has measure zero and contains $\bigcup_{n\in\mathbb{N}} N_n$, so the latter is a null set.

We claim that $\mathcal{M}'$ is a sigma algebra. Clearly $\mathcal{M}'$ is nonempty. Next, given an element $E \cup N \in \mathcal{M}'$, let $Z \in \mathcal{M}$ be a set of measure zero that contains N. Then

$$(E \cup N)^c \;=\; (E \cup Z)^c \cup (Z - N).$$

But $(E \cup Z)^c \in \mathcal{M}$ and $Z - N$ is a null set, so $(E \cup N)^c \in \mathcal{M}'$. Finally, given a sequence $\{E_n \cup N_n\}$ in $\mathcal{M}'$, we have

$$\bigcup_{n\in\mathbb{N}} (E_n \cup N_n) \;=\; \left(\bigcup_{n\in\mathbb{N}} E_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} N_n\right).$$

The first term on the right lies in $\mathcal{M}$ and the second is a null set, so the whole union lies in $\mathcal{M}'$.

Now define a function $\mu' \colon \mathcal{M}' \to [0,\infty]$ by $\mu'(E \cup N) = \mu(E)$ for every $E \in \mathcal{M}$ and every null set N. To prove this function is well-defined, suppose that $E_1 \cup N_1 = E_2 \cup N_2$ for some $E_1, E_2 \in \mathcal{M}$ and some null sets N_1 and N_2. Then $E_1 \subseteq E_2 \cup N_2 \subseteq E_2 \cup Z_2$, where $Z_2 \in \mathcal{M}$ is a set of measure zero that contains N_2, so

$$\mu(E_1) \;\leq\; \mu(E_2 \cup Z_2) \;\leq\; \mu(E_2) + \mu(Z_2) \;=\; \mu(E_2).$$

A similar proof shows that $\mu(E_2) \leq \mu(E_1)$, so $\mu(E_1) = \mu(E_2)$.

To prove that μ' is a measure, observe first that $\mu'(\emptyset) = \mu(\emptyset) = 0$. Next, if $\{E_n \cup N_n\}$ is a sequence of pairwise disjoint elements of $\mathcal{M}'$, let $E = \biguplus_{n\in\mathbb{N}} E_n$ and $N = \biguplus_{n\in\mathbb{N}} N_n$. Then $E \in \mathcal{M}$ and N is a null set, so

$$\mu'\left(\biguplus_{n\in\mathbb{N}} (E_n \cup N_n)\right) \;=\; \mu'(E \cup N) \;=\; \mu(E) \;=\; \sum_{n\in\mathbb{N}} \mu(E_n) \;=\; \sum_{n\in\mathbb{N}} \mu'(E_n \cup N_n)$$

which proves that μ' is a measure.

To prove that μ' is unique, suppose $\mu'' \colon \mathcal{M}' \to [0,\infty]$ is any measure that agrees with μ on $\mathcal{M}$. Then for any $E \cup N \in \mathcal{M}'$, we have

$$\mu(E) \;=\; \mu''(E) \;\leq\; \mu''(E \cup N) \;\leq\; \mu''(E \cup Z) \;=\; \mu(E \cup Z) \;=\; \mu(E),$$

where $Z \in \mathcal{M}$ is a set of measure zero that contains N. It follows that $\mu''(E \cup N) = \mu(E) = \mu'(E \cup N)$, and thus $\mu'' = \mu'$.

All that remains is to prove that μ' is complete, so let S be any null set with respect to μ'. Then $S \subseteq E \cup N$ for some $E \cup N \in \mathcal{M}'$ for which $\mu'(E \cup N) = 0$. Note then that $\mu(E) = \mu'(E \cup N) = 0$. Furthermore, since N is a null set with respect to μ, we know that $N \subseteq Z$ for some set $Z \in \mathcal{M}$ with measure zero. Then $E \cup Z \in \mathcal{M}$ and has measure zero and $S \subseteq E \cup Z$, so S is a null set with respect to μ, and hence $S \in \mathcal{M}'$. ■

The measure μ' constructed in the above proposition is known as the **completion** of μ. Note that the completion of μ is entirely determined by μ, but admits more measurable sets. In most contexts, this makes the completion of a measure strictly better than the original measure, and when constructing measures it is common to immediately pass from any non-complete measure to the corresponding completion.