More Measure Theory

In this set of notes we sketch some results in measure theory that we don't have time to cover in full. Most of the results can be found in Rudin's *Real & Complex Analysis*.

Some of the results here require a certain technical assumption on measures.

> **Definition: σ-Finite Measure Space**
> A measure space (X, μ) is said to be **σ-finite** if X can be expressed as a countable union of measurable sets of finite measure.

For example, the real line is σ-finite with respect to Lebesgue measure, since

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} [-n, n]$$

and each set $[-n, n]$ has finite measure. Similarly, the natural numbers $\mathbb{N}$ are σ-finite with respect to counting measure.

Not every measure space is σ-finite. For example, if we put counting measure on $\mathbb{R}$, then the resulting measure space is not σ-finite, since $\mathbb{R}$ cannot be expressed as a countable union of finite sets. However, most measure spaces that are important in mathematics are σ-finite, and it is considered a very reasonable restriction to place on a measure space. Throughout these notes **we will assume that all measure spaces under consideration are σ-finite**.

Product Measures

We would like to be able to use the Lebesgue integral to integrate functions on $\mathbb{R}^2$ and $\mathbb{R}^3$. This involves defining measures on $\mathbb{R}^2$ and $\mathbb{R}^3$ that correspond to areas and volume, respectively. The following theorem treats this construction from a general point of view.

Theorem 1 Product Measures

Let $(X, \mathcal{E}, \mu)$ and $(Y, \mathcal{F}, \nu)$ be measure spaces, and let $\mathcal{E} \otimes \mathcal{F}$ be the σ-algebra on $X \times Y$ generated by the following collection of sets:

$$\{E \times F \mid E \in \mathcal{E}, F \in \mathcal{F}\}.$$

Then there exists a unique measure $\xi \colon \mathcal{E} \otimes \mathcal{F} \to [0, \infty]$ such that

$$\xi(E \times F) = \xi(E)\,\xi(F)$$

for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$.

The measure ξ is called the **product** of the measures μ and ν. This is often denoted

$$\xi \;=\; \mu \times \nu$$

or sometimes

$$d\xi \;=\; d\mu \times d\nu.$$

EXAMPLE 1 Lebesgue Measure on $\mathbb{R}^n$

If m denotes Lebesgue measure on $\mathbb{R}$, then the n-fold product

$$\mu \;=\; \underbrace{m \times \cdots \times m}_{n \text{ times}}$$

is a measure on $\mathbb{R}^n$. This measure has the property that

$$\mu(E_1 \times \cdots E_n) \;=\; m(E_1) \cdots m(E_n)$$

for any measurable subsets $E_1, \ldots, E_n \subseteq \mathbb{R}$, so μ essentially measures the n-dimensional volume of set. This measure μ is sometimes referred to as **Lebesgue measure on $\boldsymbol{R}^n$**. ■

One flaw in our definition of a product measure is that the product $\mu \times \nu$ may not be complete, even if μ and ν are themselves complete measures. For this reason, the product of μ and ν is sometimes defined to be the *completion* of the measure defined above. In particular, Lebesgue measure on $\mathbb{R}^n$ is usually defined to be the completion of the n-fold product $m \times \cdots \times m$.

Whenever we construct a new measure, a basic question to ask is what the associated Lebesgue integral looks like. For a product measure $\xi = \mu \times \nu$, the obvious guess is that

$$\int_{X \times Y} f\, d\xi \;=\; \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y) \;=\; \int_X \int_Y f(x,y)\, d\mu(y)\, d\nu(x)$$

for every measurable function f on $X \times Y$. The integrals on the right are known as **iterated integrals**, and intuitively they should always be equal. Unfortunately, there are simple examples where this is not the case.

EXAMPLE 2 Unequal Iterated Integrals

Define a piecewise constant function $f\colon \mathbb{R}^2 \to \mathbb{R}$ as follows. In the first quadrant, f is constant on each lattice square $[j, j+1) \times [k, k+1)$, with values shown in the following picture:

$$
\begin{array}{cccccc}
\vdots & & & & & \iddots \\
\hline
\multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{-1} & \\
\hline
\multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} & \\
\hline
\multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \\
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \cdots \\
\hline
\end{array}
$$

The function f is zero in the second, third, and fourth quadrants.

Now, since each 1 cancels horizontally with a -1, we see that

$$\int_{\mathbb{R}} f(x,y)\, dm(x) \;=\; 0$$

for all $y \in \mathbb{R}$, and therefore

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dm(x)\, dm(y) \;=\; 0.$$

In the vertical direction, each 1 cancels with a -1 except for the first, so

$$\int_{\mathbb{R}} f(x,y)\, dm(y) \;=\; \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dm(y)\, dm(x) \;=\; 1,$$

so the iterated integrals are different. In this case, the Lebesgue integral

$$\int_{\mathbb{R}^2} f\, d\mu$$

does not exist, since f^+ and f^- each have infinite integral. ■

Note that this example was really about iterated *sums*. In particular, if

$$a_{ij} \;=\; \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} a_{ij} \;=\; 0 \qquad \text{and} \qquad \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} a_{ij} \;=\; 1.$$

Thus some hypotheses will be necessary if we want to understand integrals with respect to product measures.

Theorem 2 Fubini's Theorem

Let $(X, \mathcal{E}, \mu)$ and $(Y, \mathcal{F}, \nu)$ be measure spaces, and let $\xi = \mu \times \nu$. Let f be a measurable function on $X \times Y$, and suppose that either $f \geq 0$ or $f \in L^1(\mathbb{R}^2)$. Then the integrals

$$\int_X f(x,y)\, d\mu(x) \qquad \textit{and} \qquad \int_Y f(x,y)\, d\nu(y)$$

are defined for almost all $x \in X$ and $y \in Y$, the functions

$$x \mapsto \int_Y f(x,y)\, d\nu(y) \qquad \textit{and} \qquad y \mapsto \int_X f(x,y)\, d\mu(x)$$

are measurable, and

$$\int_{X\times Y} f\, d\xi \;=\; \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) \;=\; \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$

In some textbooks, the name **Fubini's theorem** refers only the case where f is L^1, whereas the case where $f \geq 0$ is known as **Tonelli's theorem**.

Radon-Nikodym Derivatives

Recall the following definition.

Definition: Weighted Measures
Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a non-negative measurable function on X. The resulting **weighted measure** $\nu\colon \mathcal{M} \to [0, \infty]$, denoted

$$d\nu \;=\; f\,d\mu$$

is the measure defined by

$$\nu(E) \;=\; \int_E f\,d\mu$$

for all $E \in \mathcal{M}$.

This definition takes the point of view that we start with f and use it to construct ν. If instead we start with ν, it is common to use slightly different terminology.

Definition: Absolute Continuity, Radon-Nikodym Derivative
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\nu\colon \mathcal{M} \to [0, \infty]$ be a measure. We say that ν is **absolutely continuous** with respect to μ if there exists a non-negative measurable function f on X such that

$$d\nu \;=\; f\,d\mu.$$

In this case, the function f is called the **Radon-Nikodym derivative** of ν with respect to μ, denoted

$$f \;=\; \frac{d\nu}{d\mu}.$$

The most important case is when μ is Lebesgue measure on $\mathbb{R}$. For this case, ν is absolutely continuous with respect to μ if the "distribution of mass" on $\mathbb{R}$ corresponding to ν can be described by a density function. The following example describes a measure on $\mathbb{R}$ that is not absolutely continuous with respect to Lebesgue measure.

EXAMPLE 3 δ-Measure
Let $\delta\colon \mathcal{M} \to [0, \infty]$ be the function defined by

$$\delta(E) \;=\; \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E. \end{cases}$$

It is easy to check that this is a measure on $\mathbb{R}$. Intuitively, this corresponds to a point mass at the origin with a measure of 1, with no mass anywhere else on the real line.

Since $\delta(\{0\}) = 1$, this measure cannot be absolutely continuous with respect to Lebesgue measure m. In particular, if f is any non-negative measurable function, then

$$\int_{\{0\}} f\,dm \;=\; 0 \;\neq\; 1$$

and hence $f\,dm \neq d\delta$.

Incidentally, the measure δ has the property that

$$\int_{\mathbb{R}} f\,d\delta \;=\; f(0)$$

for any measurable function f. ■

The following theorem characterizes absolute continuity.

Theorem 3 Radon-Nikodym Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\nu\colon \mathcal{M} \to [0,\infty]$ be a measure. Then ν is absolutely continuous with respect to μ if and only if for all $E \in \mathcal{M}$,

$$\mu(E) = 0 \qquad \Rightarrow \qquad \nu(E) = 0.$$

In the case where μ is Lebesgue measure, there is a simple formula for the Radon-Nikodym derivative.

Theorem 4 Lebesgue Differentiation Theorem

Let f be a non-negative measurable function on $\mathbb{R}$, and let $d\nu = f\,dm$. Then for almost all $x \in \mathbb{R}$,

$$f(x) \;=\; \lim_{h\to 0^+} \frac{\nu\big([x-h, x+h]\big)}{2h}$$

That is, if ν is a measure on $\mathbb{R}$ and ν is absolutely continuous with respect to Lebesgue measure, then

$$\frac{d\nu}{dm}(x) \;=\; \lim_{h\to 0^+} \frac{\nu\big([x-h, x+h]\big)}{2h}.$$

EXAMPLE 4 Let I denote the interval $[0,1]$, and let $d\nu = \chi_I\, dm$. Note then that

$$\nu(E) \;=\; m(E \cap I)$$

for any measurable set $E \subseteq \mathbb{R}$. It is easy to check that

$$\lim_{h\to 0^+} \frac{\nu\big([x-h,x+h]\big)}{2h} \;=\; \begin{cases} 1 & \text{if } x \in (0,1) \\ 1/2 & \text{if } x = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this limit is equal to $\chi_I(x)$ for almost all $x \in \mathbb{R}$. ■

Pushforward Measures

There is one more measure-theoretic construction you should be aware of.

Definition: Pushforward of a Measure
Let $(X, \mathcal{M}, \mu)$ be a measure space, let Y be a set, and let $f \colon X \to Y$ be a function.

1. The **pushforward σ-algebra** $f(\mathcal{M})$ on Y is the collection

$$\{S \subseteq Y \mid f^{-1}(S) \in \mathcal{M}\}.$$

2. The **pushforward** of μ by f is the measure $\nu \colon f(\mathcal{M}) \to [0,\infty]$ defined by

$$\nu(S) \;=\; \mu\big(f^{-1}(S)\big).$$

The pushforward of μ by f is sometimes denoted $f_*\mu$.

EXAMPLE 5 Arc Length on the Circle
Let S^1 be the unit circle in $\mathbb{R}^2$, and let $f \colon [-\pi,\pi] \to S^1$ be the function

$$f(\theta) \;=\; (\cos\theta, \sin\theta).$$

Then the pushforward f_*m of Lebesgue measure on $[-\pi,\pi]$ is a measure on the circle with the property that the measure of any arc is equal to the length of the arc. That is, f_*m is length measure on the circle. ■

EXAMPLE 6 Area on the Sphere
Let S^2 be the unit sphere in $\mathbb{R}^3$, let R be the rectangle $[-\pi,\pi] \times [0,\pi]$, and let

$f\colon R \to S^2$ be the function

$$f(\theta,\phi) \;=\; (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

Let μ denote Lebesgue measure on R. Then the pushforward $f_*\mu$ is *not* a good area measure on S^2, for it weighs area near the north and south poles more heavily than area near the equator. However, if we let ν be the weighted measure

$$d\nu \;=\; \sin\phi\, d\mu$$

on R, then the pushforward $f_*\nu$ is a good measure of area on the sphere. ■

The following proposition is quite helpful for evaluating integrals with respect to a pushforward measure. It can be used, for example, to integrate a real-valued measurable function on the sphere.

Proposition 5 Integration Using Pushforward Measures

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f\colon X \to Y$ be a function, and let ν be the pushforward of μ by f. Then

$$\int_Y g\, d\nu \;=\; \int_X (g\circ f)\, d\mu.$$

for every measurable function g on Y.