Appendix: Sums Over Sets

You are already familiar with infinite sums of the form

$$\sum_{n=1}^{\infty} a_n$$

where $\{a_n\}$ is a sequence of real numbers. Here we extend the notion of an infinite sum to allow indexing sets other than the natural numbers.

> **Definition: Sums Over Sets**
> Let S be a set, and let $f\colon S \to [0,\infty]$ be a function. The **sum of f over S** is defined by the formula
>
> $$\sum_{s\in S} f(s) \;=\; \sup\bigl\{f(s_1)+\cdots+f(s_n) \;\big|\; \{s_1,\ldots,s_n\} \text{ is a finite subset of } S\bigr\}.$$

If $\{s_1,\ldots,s_n\}$ is a finite subset of S, the sum $f(s_1)+\cdots+f(s_n)$ is called a **finite partial sum** for $\sum_{s\in S} f(s)$. Thus $\sum_{s\in S} f(s)$ is the supremum of all of its finite partial sums.

Note that we are not allowing negative terms in our infinite sums. Since the terms of our sum are not ordered, it isn't possible to assign a sum to a conditionally convergent series. Of course, it would be possible to restrict ourselves to absolutely convergent sums with negative terms, using the definition

$$\sum_{s\in S} f(s) \;=\; \sum_{\substack{s\in S\\ f(s)\geq 0}} f(s) \;-\; \sum_{\substack{s\in S\\ f(s)<0}} |f(s)|$$

but we will not have a need for this.

The following proposition shows that the new notion of an infinite sum agrees with the old notion in the case of sums over the natural numbers.

Proposition 1

For any function $f\colon \mathbb{N} \to [0,\infty]$, *we have* $\displaystyle\sum_{n\in\mathbb{N}} f(n) = \sum_{n=1}^{\infty} f(n)$

PROOF Note first that each partial sum of the series $\sum_{n=1}^{\infty} f(n)$ is also a finite partial sum for $\sum_{n\in\mathbb{N}} f(n)$. Thus

$$\sum_{n\in\mathbb{N}} f(n) \;\geq\; \sum_{n=1}^{N} f(n)$$

for each $N \in \mathbb{N}$, and therefore

$$\sum_{n\in\mathbb{N}} f(n) \;\geq\; \sum_{n=1}^{\infty} f(n).$$

For the other direction, let $\{n_1, \ldots, n_k\}$ be any finite subset of $\mathbb{N}$, and let N be an integer upper bound for $\{n_1, \ldots, n_k\}$. Then

$$f(n_1) + \cdots + f(n_k) \;\leq\; \sum_{n=1}^{N} f(n) \;\leq\; \sum_{n=1}^{\infty} f(n).$$

This holds for any finite subset $\{n_1, \ldots, n_k\}$ of $\mathbb{N}$, so

$$\sum_{n\in\mathbb{N}} f(n) \;\leq\; \sum_{n=1}^{\infty} f(n). \qquad \blacksquare$$

Sums over sets have many other basic properties, which are listed in the exercises. Roughly speaking, sums over sets always behave exactly as one might expect.

Although sums over sets are a convenient notation, these aren't really a new kind of series. In particular, the following proposition shows that a sum over an uncountable set always reduces to a countable sum.

Proposition 2

Let S be an uncountable set, and let $f\colon S \to [0,\infty]$. If

$$\sum_{s\in S} f(s) \;<\; \infty,$$

then $f(s) = 0$ for all but countably many s.

PROOF Suppose that the given sum is less than infinity. Then for each $n \in \mathbb{N}$, the set

$$S_n = \{s \in S \mid f(s) \geq 1/n\}$$

must be finite, since otherwise the sum would be infinite. But

$$\bigcup_{n=1}^{\infty} S_n = \{s \in S \mid f(s) > 0\}$$

so this set must be countable, being a countable union of finite sets. ■

Exercises

1. Prove that if $f\colon S \to [0,\infty]$ and $\varphi\colon T \to S$ is a bijection, then
$$\sum_{s\in S} f(s) = \sum_{t\in T} f(\varphi(t)).$$

2. Let S be a set, and let $f, g\colon S \to [0,\infty]$ be functions.
   a) Prove that if $f \leq g$, then $\displaystyle\sum_{s\in S} f(s) \leq \sum_{s\in S} g(s)$.
   b) Prove that $\displaystyle\sum_{s\in S}\big(f(s)+g(s)\big) = \sum_{s\in S} f(s) + \sum_{s\in S} g(s)$.

3. Let S be a set, and let $f\colon S \to [0,\infty)$. Prove that
$$\sum_{s\in S} f(s) < \infty$$
if and only if for every $\epsilon > 0$, there exists a finite set $F \subseteq S$ so that
$$\sum_{s\in S-F} f(s) < \epsilon.$$

4. Prove that if S and T are sets and $f\colon S\times T \to [0,\infty]$, then
$$\sum_{(s,t)\in S\times T} f(s,t) = \sum_{s\in S}\sum_{t\in T} f(s,t).$$

5. a) Prove that if S and T are disjoint sets and $f\colon S \uplus T \to [0,\infty]$, then
$$\sum_{x\in S\uplus T} f(x) = \sum_{s\in S} f(s) + \sum_{t\in T} f(t).$$
   b) More generally, prove that if $\mathcal{C}$ is any collection of pairwise disjoint sets and $f\colon \bigcup\mathcal{C} \to [0,\infty]$, then
$$\sum_{x\in\bigcup\mathcal{C}} f(x) = \sum_{S\in\mathcal{C}}\sum_{x\in S} f(x).$$