Implementation of a Solution to the Conjugacy Problem in Thompson's Group F

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Abstract

We present an efficient implementation of the solution to the conjugacy problem in Thompson's group F. This algorithm checks for conjugacy by constructing and comparing directed graphs called strand diagrams. We provide a description of our solution algorithm, including the data structure that represents strand diagrams and supports simplifications.

1 Thompson's Group F and Strand Diagrams

The elements of Thompson's Group F [3] are piecewise, linear homeomorphisms of the interval $[0, 1]$ such that each piece has slope that is a power of 2 and, furthermore, the breakpoints between pieces take place at dyadic rational coordinates. The group operation is simply function composition. In a group, the **conjugacy problem** is the problem of determining whether any two elements are conjugate. The conjugacy problem is not solvable in general [5], but is solvable in certain cases.

A **strand diagram** [2] is a finite acyclic digraph embedded on the unit square. The digraph has a **source** along the top edge of the square and a **sink** along the bottom edge. Any internal vertex is either a **merge** or a **split** (Figure 1). Elements of Thompson's Group F can be translated to strand diagrams. Each element in a generating set corresponds to a particular strand diagram. A composition of such elements is represented by a concatenation of the associated strand diagrams.

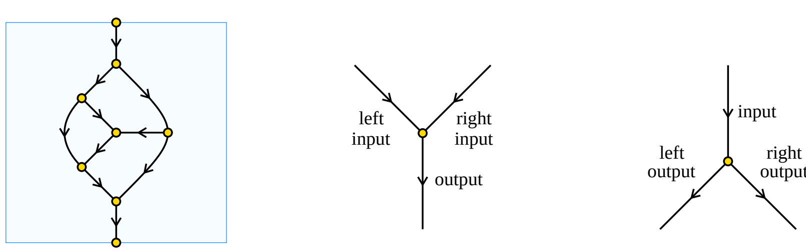

Figure 1: A strand diagram, a merge, and a split (image taken from [2]).

2 Algorithm for the Conjugacy Problem in F

The algorithm to determine whether two strand diagrams inhabit the same conjugacy class proceeds as follows. First, we convert the strand diagrams to **annular strand diagrams**. This is achieved by a process called **closing**, in which sources are identified with sinks. Next, the annular strand diagrams are reduced using a graphical rewriting system that is both confluent, terminating, and respects conjugacy [1]. Furthermore, any two connected and reduced annular strand diagrams s_1 and s_2 can be encoded into two planar graphs g_1 and g_2 respectively such that s_1 and s_2 represent conjugate elements if and only if g_1 and g_2 are isomorphic. Hence the problem reduces to checking whether two simplified planar graphs are isomorphic. Moreover, this enterprise can be carried out in linear time given a linear time planar-graph-isomorphism checker [4].

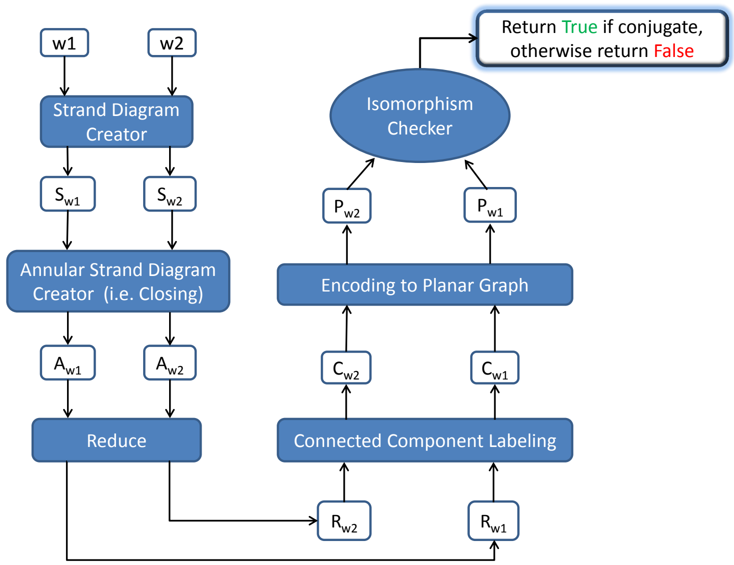

Figure 2: Algorithm Flowchart

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