Twisted Three-Eared Rabbits: Identifying Topological Quadratics Up To Thurston Equivalence

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Abstract

A topological quadratic is a two-sheeted branched covering map on the complex plane with one branch point. Such a map is called postcritically finite if the orbit of the branch point under iteration is finite. Two such maps have the same dynamics if there exists a self-homeomorphism of the complex plane conjugating the first to the second. The study of postcritically finite branched covers was initiated by Thurston, who characterized when such a map is homotopic to a conjugate of a polynomial map. The problem of which polynomial this would be, however, was left unsolved.

There are exactly three quadratic polynomials for which the branch point has period 3: the rabbit, the corabbit, and the airplane. In 2006, V. Nekrashevych and L. Bartholdi solved the *twisted rabbit problem*, which asked "given a topological quadratic whose branch point is periodic with period three, to which quadratic polynomial is it Thurston equivalent?"

Using braids and the mapping class group of a complex plane with punctures, we provide a new solution to the twisted rabbit problem. In addition, we solve the "twisted three-eared rabbit" problem, which is the analogous question for period-four quadratic polynomials.

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Dedication

To Mom, Dad, and Sam.

Acknowledgments

Ten years ago, even the prospect of finishing middle school seemed beyond my reach. The fact that I am here, writing this project and about to graduate from college, seems somewhat of a miracle when looked at from that perspective. I owe this triumph to a whole host of family, friends, and teachers who have helped me along my way.

First and foremost—this project would not exist at all without the seemingly endless support and guidance I received from my adviser and good friend Jim Belk. Going far above and beyond the standard duties of project adviser, he managed to make me feel less like a student and more like a full fledged mathematician. Without this environment, I could not possibly have made as much progress as I did, both as a mathematician and as a person.

I must also thank the entire Bard math department (students and faculty alike) for fostering an environment in which the primary objective of mathematics is to have fun. I will be sad to move on!

My parents have been one hundred percent awesome; who else would literally drive up here to deliver baked goods in the days before the due date? They have been steadfastly supportive through any crisis I've come upon—I cannot express how grateful I am to them for being everything that they are.

My brother Sam has been, as always, an invaluable friend. He sparked my interest in mathematics in the first place, and subsequently tolerated me droning on and on about my project. Although thousands of miles away, he still allowed me to complain endlessly at him.

There are countless others I am missing, but in the interest of brevity I must move on!

1
Background

1.1 Julia Sets and the Mandelbrot Set

Definition 1.1.1. Given a polynomial function $f\colon \mathbb{C} \to \mathbb{C}$ we define the **filled-in Julia set** for f to be the set

$$F = \{z \in \mathbb{C}\colon |f^n(z)| \text{ remains bounded as } n \to \infty\}.$$

We define the **Julia set** of f to be the boundary of F. △

The Julia set of a function is a picture of the dynamics of that function; everything outside of the Julia set diverges to infinity under iteration of f, and everything in the filled in Julia set stays inside.

Example 1.1.2. Take the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$. Then $f^n(z)$ diverges to infinity if $|z| > 1$, and converges to 0 otherwise. Hence the filled in Julia set for f is the closed unit disc, and the Julia set is the unit circle. Notice that f has a fixed point at 0, and so the interior of the filled in Julia set is the attracting basin for 0. ◊

Exploring further, however, shows that Julia sets become very interesting very quickly.

Example 1.1.3. Take the function $z \mapsto z^2 - 1$. Figure 1.1.1 shows the filled-in Julia set for this function. We can see immediately that this is not a simple object. Zooming in, we see that it exhibits a fractal structure. That is, it appears to have a pattern which repeats on all scales. Indeed, one could zoom in on its boundary forever and never encounter a smooth edge.

The dynamics of this function are different from those of $z \mapsto z^2$. Notice that $z \mapsto z^2 - 1$ contains a 2-cycle: $0 \mapsto -1 \mapsto 0 \mapsto -1 \cdots$. The interior of the filled-in Julia set is the attracting basin for this 2-cycle. $\diamondsuit$

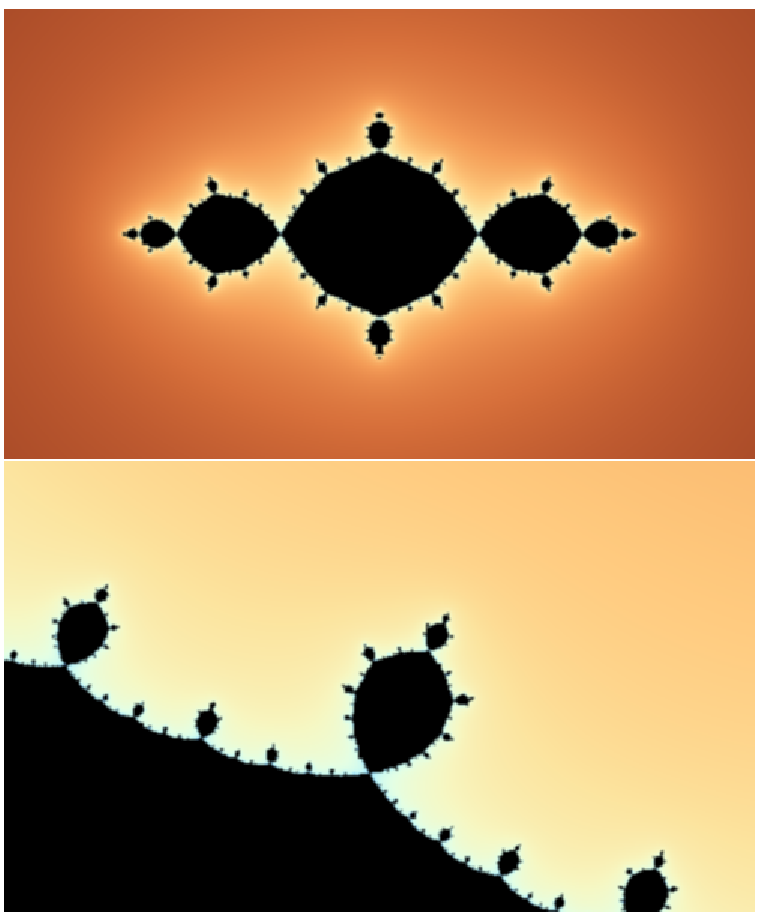


Figure 1.1.1: The Julia set for $f(z) = z^2 - 1$.

For now we will be restricting our attention to quadratic polynomials. In fact, if we are only looking at quadratics, it turns out that we need only consider maps of the form $z \mapsto z^2 + c$ where $c \in \mathbb{C}$. To see this, we recall the concept of topological conjugacy.

Definition 1.1.4. Let $f, g \colon \mathbb{C} \longrightarrow \mathbb{C}$. We say that f and g are **topologically conjugate** if there exists an orientation-preserving homeomorphism h so that $f = h \circ g \circ h^{-1}$. △

It follows immediately from this definition that topologically conjugate functions have the same dynamics, and so they are equivalent. This means that they will have the same Julia set up to homeomorphism, which motivates the following theorem.

Theorem 1.1.5. *Let $p \colon \mathbb{C} \longrightarrow \mathbb{C}$ be a quadratic polynomial. Then there exists $c \in \mathbb{C}$ so that p is topologically conjugate to $z \mapsto z^2 + c$.*

Proof. Let $p(z) = c_1 z^2 + c_2 z + c_3$, and let $w(z) = c_1 \left(z + \frac{c_2}{2c_1} \right)$. Then

$$(w \circ p \circ w^{-1})(z) = c_1 \left(c_1 \left(\frac{z}{c_1} + \frac{-c_2}{2c_1} \right)^2 + c_2 \left(\frac{z}{c_1} - \frac{c_2}{2c_1} \right) + c_3 + \frac{c_2}{2c_1} \right)$$

$$= z^2 + \left(-\frac{c_2^2}{4} + \frac{c_2}{2} + c_1 c_3 \right),$$

which is of the desired form. □

We now introduce the Mandelbrot set, which describes the behavior of Julia sets for the family of functions $z^2 + c$. The definition of the Mandelbrot set is based on the following Theorem.

Theorem 1.1.6. [4, Section 3.8] *Let $f(z) = z^2 + c$.*

1. *If the orbit of 0 is bounded, then the Julia set for f is connected.*

2. *If the orbit of 0 is unbounded, then the Julia set for f is homeomorphic to the Cantor set.*

Definition 1.1.7. Let $f_c\colon \mathbb{C} \longrightarrow \mathbb{C}$ be the function $z \mapsto z^2 + c$. The **Mandelbrot set** M is the set of all $c \in \mathbb{C}$ for which $\sup_{n \longrightarrow \infty} |f_c^n(0)| < \infty$. △

Figure 1.1.2 shows a picture of this set. Every c value in the black region yields a quadratic for which the orbit of 0 is bounded.

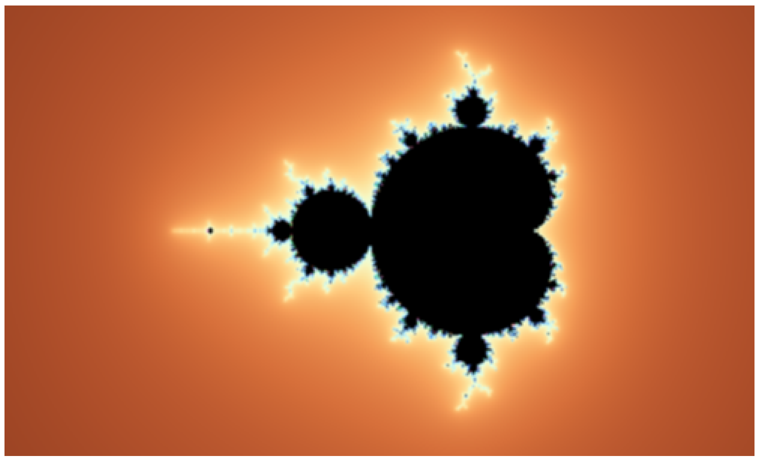


Figure 1.1.2: The Mandelbrot set.

Definition 1.1.8. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a function, and let $z \in \mathbb{C}$. The **forward orbit** of z is the set $O_f(z) = \{f^n(z) \mid n \in \mathbb{N}\}$. △

Definition 1.1.9. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function, and let P be the set of critical points of f. Let $C = \bigcup_{z \in P} O_f(z)$. We call C the **postcritical points** of f, and we say f is **postcritically finite** if C is finite. △

If we specify the kind of orbit we wish 0 to have, we can find some c value for which $z^2 + c$ has precisely those dynamics.

Example 1.1.10. Suppose we want a function $f(z) = z^2 + c$ for which 0 is part of a three cycle. That is, suppose 0 has the orbit $0 \longrightarrow c_1 \longrightarrow c_2 \longrightarrow 0 \longrightarrow \cdots$. Solving the equation

$$f^3(0) = (c^2 + c)^2 + c = 0$$

for c, we obtain the values

$$c_1 = 0$$

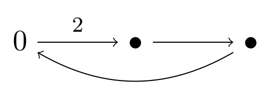
$$c_2 \approx -1.75488$$

$$c_3 \approx -0.12256 - 0.74486i$$

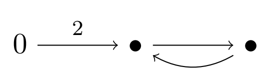
$$c_4 \approx -0.12256 + 0.74486i.$$

If we let $c = c_1 = 0$, we obtain the function $f_{c_1}(z) = z^2$, for which 0 is a fixed point. For all of the other solutions, however, we obtain maps f_{c_i} for which 0 belongs to a three cycle. We call $f_{c_2} = f_A$ the **airplane polynomial**, $f_{c_3} = f_{CR}$ the **corabbit polynomial**, and $f_{c_4} = f_R$ the **rabbit polynomial**. The Julia sets for these functions are shown in Figure 1.1.3, and the positions of the c values are shown in Figure 1.1.6. The orbit of 0 under f_R is shown in Figure 1.1.4. ◊

Remark 1.1.11. The description of the orbit of 0 in Example 1.1.10 can be described combinatorially by the following diagram:



The 2 above the first arrow means that an open ball around 0 maps 2-1 to an open ball around $f(0)$. We call this diagram a **critical portrait**. These make it easy to see which orbits for 0 are valid. Any point $p \in \mathbb{C}$ has at most two preimages under a map $z \mapsto z^2 + c$, and so the following diagram does not yield a valid orbit:



◊

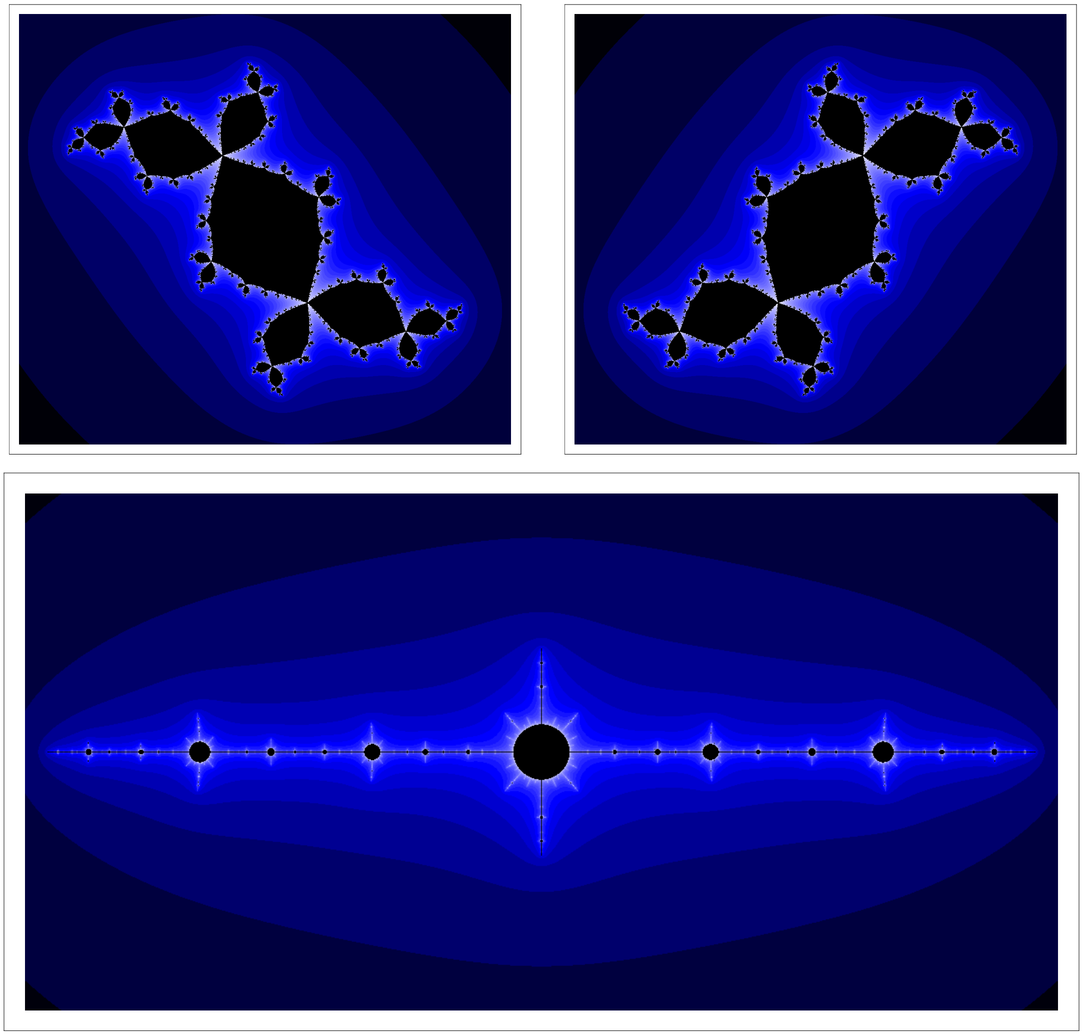
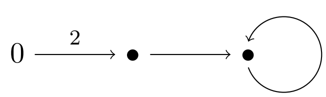


Figure 1.1.3: The Julia sets for f_R and f_{CR}, and f_A, respectively, from the top left.

Example 1.1.12. We will find the values of c for which $z \mapsto z^2 + c$ has the following critical orbit:



The polynomial we must solve is $f^2(0) = f^3(0)$. This is the equation

$$(c^2 + c)^2 + c = c^2 + c,$$

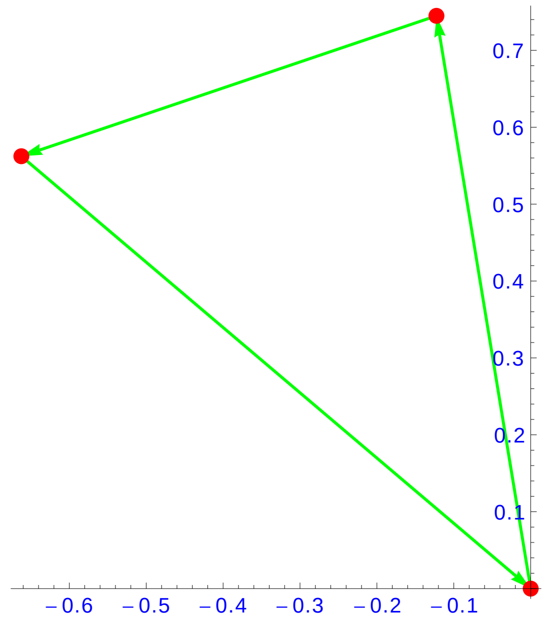


Figure 1.1.4: The critical orbit of f_R.

and the solutions are $c = 0$ and $c = -2$. We once again throw out the solution $c = 0$. The Julia set for $z \mapsto z^2 - 2$ is shown in Figure 1.1.5, and is in fact just the line segment from -2 to 2. ♢

Theorem 1.1.13 ([4, Theorem 4.6])**.** *Let $c \in \mathbb{C}$. Suppose $z \in \mathbb{C}$ is in an attracting periodic orbit under f_c. Then the critical point 0 lies in the basin of attraction of z.*

This theorem says that if f_c has an attracting cycle, then the orbit of 0 is bounded, so c lies in the Mandelbrot set. Furthermore, if f_c has an attracting cycle, then the same will be true for nearby values of c, and therefore c lies in the interior of the Mandelbrot set.

The connected components of the interior of the Mandelbrot set are called **bulbs**. The **hyperbolic components** are the bulbs whose corresponding functions each have an attracting cycle. It is conjectured that every bulb of the Mandelbrot set is a hyperbolic component.

The following Theorem helps to explain the importance of postcritically finite polynomials.

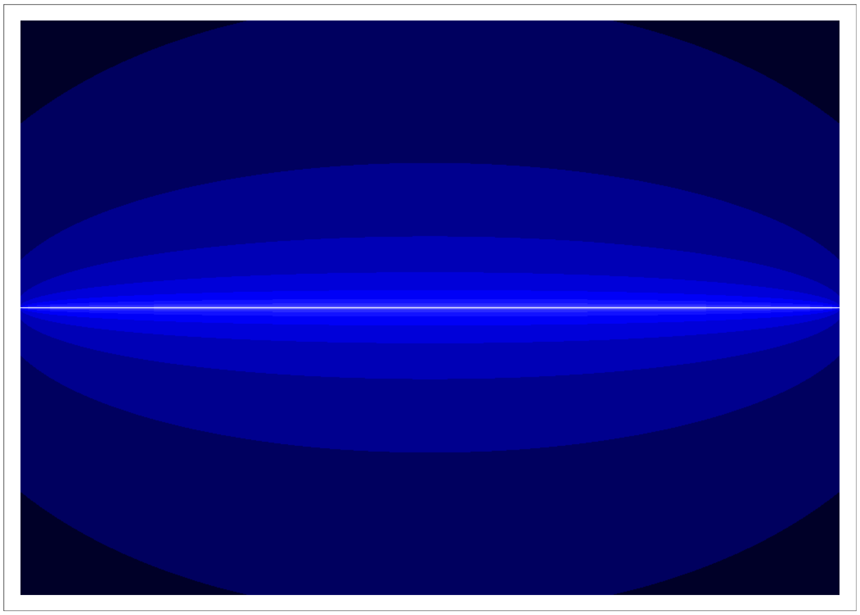


Figure 1.1.5: The Julia set for the map $z \mapsto z^2 - 2$.

Theorem 1.1.14. [3] *Every hyperbolic component of the Mandelbrot set contains a value of c for which 0 is periodic under f_c.*

As we saw in Example 1.1.12, 0 need not be periodic. In the case where 0 is not itself periodic, but where its orbit ends in a cycle, we call 0 **pre-periodic**. Values of c for which 0 is pre-periodic under f_c are called **Misiurewicz** points. The Julia sets for such maps are connected, but they have no interior.

1.2 Topological Quadratics

In this section we will construct a set of functions on the Riemann sphere which have all of the topological properties of quadratic polynomials.

Definition 1.2.1. Let S and T be surfaces, and let $f\colon S \to T$ be continuous. We say that f is a **branched cover** if there exists a finite set $D \subset T$ so that the restriction $f\colon S - f^{-1}(D) \to T - D$ is a cover. $\triangle$

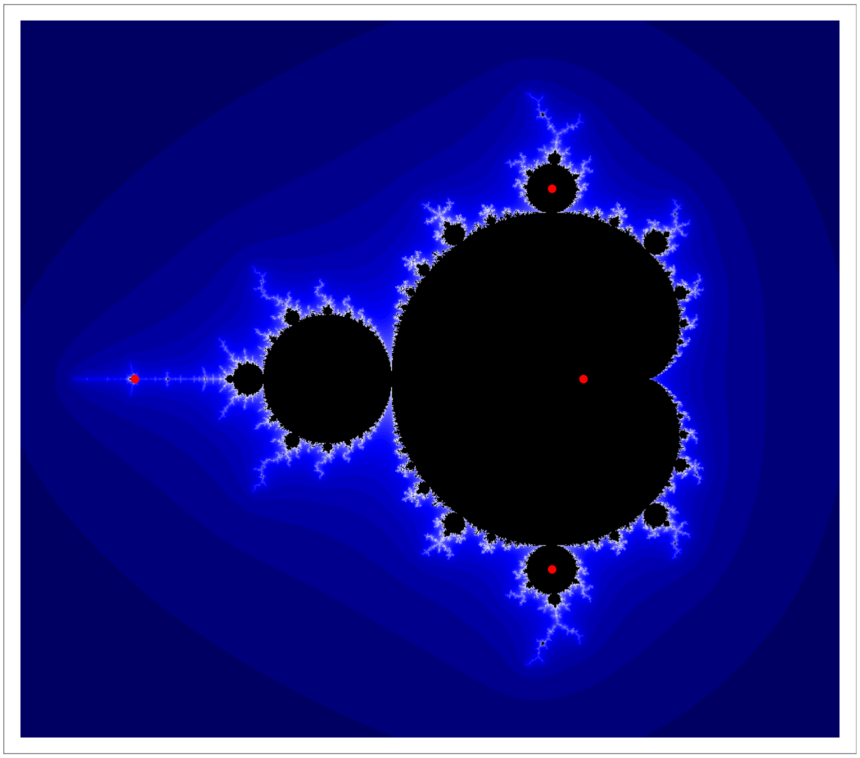


Figure 1.1.6: The locations of the points c_1, c_2, c_3, and c_4 from example 1.1.10.

The **degree** of a branched cover f is the degree of the associated covering map, and its **branch points** are the points in $f^{-1}(D)$.

Example 1.2.2. Let $\widehat{\mathbb{C}}$ be the Riemann sphere, and let $s\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the map $s(z) = z^2$. Then s is a branched cover of degree 2 with branch points 0 and ∞, each of which is a fixed point. In particular, any $z \in \widehat{\mathbb{C}} - \{0, \infty\}$ has two preimages under s, namely $\sqrt{z}$ and $-\sqrt{z}$. ◊

Example 1.2.3. Any quadratic polynomial $p\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched cover of degree 2. The branch points are ∞ and the critical point of p. ◊

Example 1.2.4. Any rational map $r\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched cover of $\widehat{\mathbb{C}}$, where the critical values of r (possibly including ∞) are the branch points. ◊

Definition 1.2.5. Let $\tau\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$. Then τ is a **topological quadratic** if it satisfies the following properties:

1. τ is a branched cover of degree 2 with two branch points, one of which is ∞,

2. τ is orientation-preserving, and

3. $\tau(\infty) = \infty$.

△

Remark 1.2.6. Since $\tau^{-1}(\infty) = \{\infty\}$ for any topological quadratic τ, we can restrict τ to $\mathbb{C}$ without penalty. It will sometimes be convenient to do this, and so we will occasionally be loose with the requirement that τ be a map on $\widehat{\mathbb{C}}$. ♢

Notice that if f is a quadratic polynomial and h is a homeomorphism, then $f \circ h$ is a topological quadratic. In fact, *any* topological quadratic can be written as $f \circ h$ for some quadratic polynomial f and some homeomorphism h.

Theorem 1.2.7. *Let $\tau\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a topological quadratic with branch point p. Let $s\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be the map $z \mapsto z^2$. Then $\tau = s \circ h$ for some homeomorphism $h\colon (\widehat{\mathbb{C}}, p, \infty) \longrightarrow (\widehat{\mathbb{C}}, 0, \infty)$.*

We can still talk about postcritical points for topological quadratics. We will have the branch points play the role of critical points, allowing us to generalize our previous definition.

Definition 1.2.8. Let $\tau\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a topological quadratic, and let p be the branch point of τ. Define the **postcritical set** of τ to be the set $O_\tau(p)$. Define τ to be **postcritically finite** if its postcritical set is finite. △

This definition agrees with our earlier definition since quadratic polynomials have precisely one critical point, which is the same as the branch point. Extending f_R, f_{CR}, and

f_A to $\widehat{\mathbb{C}}$, we see that they all are topological quadratics, and in particular, they are postcritically finite topological quadratics.

1.3 Thurston Equivalence

We have seen that topological quadratics may or may not have representations as quadratic polynomials. But what if we allow continuous deformation? That is, is every topological quadratic homotopic to a quadratic polynomial? The answer to this is "not necessarily." The clarification of when this is and is not the case is the subject of this project. We formally define Thurston equivalence as follows:

Definition 1.3.1. An **isotopy** is a map $\phi\colon I \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that ϕ_t is a homeomorphism for all $t \in I$. △

Definition 1.3.2. Let $\phi\colon I \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an isotopy, and let $A \subset \widehat{\mathbb{C}}$. We say that ϕ is an **isotopy relative to** A, abbreviated isotopy rel A, if $\phi_t|_A$ does not depend on t. △

Remark 1.3.3. Let B and C be subsets of X with the same cardinality. We will commonly use the notation $f\colon (X, B) \to (X, C)$ to mean "f is a function from X to X which takes B bijectively to C". ♢

Definition 1.3.4. Let $\tau\colon (\widehat{\mathbb{C}}, P_\tau) \to (\widehat{\mathbb{C}}, P_\tau)$ and $\upsilon\colon (\widehat{\mathbb{C}}, P_\upsilon) \to (\widehat{\mathbb{C}}, P_\upsilon)$ be topological quadratics with postcritical sets P_τ and P_υ, respectively. Then τ and υ are **Thurston equivalent**, denoted $\tau \sim \upsilon$, if there exist orientation-preserving homeomorphisms $h, k\colon (\widehat{\mathbb{C}}, P_\tau) \to (\widehat{\mathbb{C}}, P_\upsilon)$ where h and k are isotopic relative to P_τ, and so that the following diagram commutes:

$$\begin{array}{ccc}
(\widehat{\mathbb{C}}, P_\tau) & \xrightarrow{\ \tau\ } & (\widehat{\mathbb{C}}, P_\tau) \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\
(\widehat{\mathbb{C}}, P_\nu) & \xrightarrow[\ \upsilon\ ]{} & (\widehat{\mathbb{C}}, P_\nu)
\end{array}$$

△

In other words, τ and υ are equivalent if they are topologically conjugate, up to isotopy relative to their postcritical sets. If $h = k$, then τ and υ are topologically conjugate.

1.4 Mapping Class Groups

The problems we will be tackling will be of the following form:

> **Question**: Let τ be a postcritically finite topological quadratic, let P_τ be the postcritical set of τ, and let $h\colon (\widehat{\mathbb{C}}, P_\tau) \longrightarrow (\widehat{\mathbb{C}}, P_\tau)$ be a homeomorphism.
>
> 1. Is $\tau \circ h$ Thurston equivalent to some quadratic polynomial p?
>
> 2. If so, what is p?

Thurston solved the first of these questions using a criteria involving the cohomology of $\widehat{\mathbb{C}} - P_\tau$, but the second question remains unsolved.

The first step we will take in tackling this problem is to classify all homeomorphisms $h\colon (\widehat{\mathbb{C}}, M) \longrightarrow (\widehat{\mathbb{C}}, M)$ up to isotopy rel M.

Definition 1.4.1. Let $M \subset \widehat{\mathbb{C}}$ be finite, and let $h\colon (\widehat{\mathbb{C}}, M) \longrightarrow (\widehat{\mathbb{C}}, M)$ be a homeomorphism. The **mapping class** of h is the equivalence class $[h]$ of maps $g\colon (\widehat{\mathbb{C}}, M) \longrightarrow (\widehat{\mathbb{C}}, M)$ homotopic rel M to h. △

Definition 1.4.2. Let $M \subset \widehat{\mathbb{C}}$ be finite, and let $\mathrm{MCG}(\widehat{\mathbb{C}}, M)$ be the set of mapping classes on $(\widehat{\mathbb{C}}, M)$. Define the binary operation $*$ on $\mathrm{MCG}(\widehat{\mathbb{C}}, M)$ by $[h] * [g] := [h \circ g]$. △

Theorem 1.4.3. *The set* $\mathrm{MCG}(\widehat{\mathbb{C}}, M)$ *forms a group under the operation* $*$.

Definition 1.4.4. The group $(\mathrm{MCG}(\widehat{\mathbb{C}}, M), *)$ is called the **mapping class group** of $(\widehat{\mathbb{C}}, M)$. △

Remark 1.4.5. Since the elements of $\mathrm{MCG}(\widehat{\mathbb{C}}, M)$ are equivalence classes, we abuse notation by saying that $h \in \mathrm{MCG}(\widehat{\mathbb{C}}, M)$ means any representative element h of $[h]$. ◊

Let p be a quadratic polynomial, and let $h\colon (\widehat{\mathbb{C}}, P_p) \longrightarrow (\widehat{\mathbb{C}}, P_p)$. Notice that this definition of h requires only that h permute the elements of P_p, and so $p \circ h$ will not necessarily have the same critical portrait as p.

Definition 1.4.6. The **pure mapping class group** of $(\widehat{\mathbb{C}}, M)$, denoted $\mathrm{PMCG}(\widehat{\mathbb{C}}, M)$, is the subgroup of $\mathrm{MCG}(\widehat{\mathbb{C}}, P_p)$ for which each element fixes P_p pointwise. △

Remark 1.4.7. Let $M \subset \widehat{\mathbb{C}}$ be finite. Let $\phi\colon \mathrm{MCG}(\widehat{\mathbb{C}}, M) \longrightarrow S_n$ be the homomorphism which takes $[h] \in \mathrm{MCG}(\widehat{\mathbb{C}}, M)$ to its corresponding permutation of M. The kernel of ϕ is then subgroup $\mathrm{PMCG}(\widehat{\mathbb{C}}, M)$ of $\mathrm{MCG}(\widehat{\mathbb{C}}, M)$. ♢

The generating elements of $\mathrm{PMCG}(\widehat{\mathbb{C}}, M)$ are homeomorphisms called Dehn twists, which we now define.

Definition 1.4.8. Let V be the annulus $\{z \in \widehat{\mathbb{C}} \colon 1 \leq |z| \leq 2\}$. Let $D\colon V \longrightarrow V$ be defined by

$$D(z) = ze^{2\pi i(2-|z|)} \text{if } |z| \in [1, 2]$$

Then D is a homeomorphism from V to V which is the identity on the boundary of V. We call D the **canonical Dehn twist**. △

Example 1.4.9. Let $p\colon I \longrightarrow \widehat{\mathbb{C}}$ be the path $p(t) = (-1-i)t + (1-t)(1+i)$. Figure 1.4.1 shows $(D \circ p)(t)$ and $(D \circ (-p))(t)$. ♢

Definition 1.4.10. Let α be a simple closed curve in $(\widehat{\mathbb{C}}, M)$, which encloses precisely two elements of M. Let D be the canonical Dehn twist. Let $h\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism which takes the annulus $\{z \in \mathbb{C} \mid |z| \in [1, 2]\}$ to an annulus with inner boundary α and outer boundary a small enough distance away from α so that it does not intersect M. We define D_α to be the homeomorphism such that the following

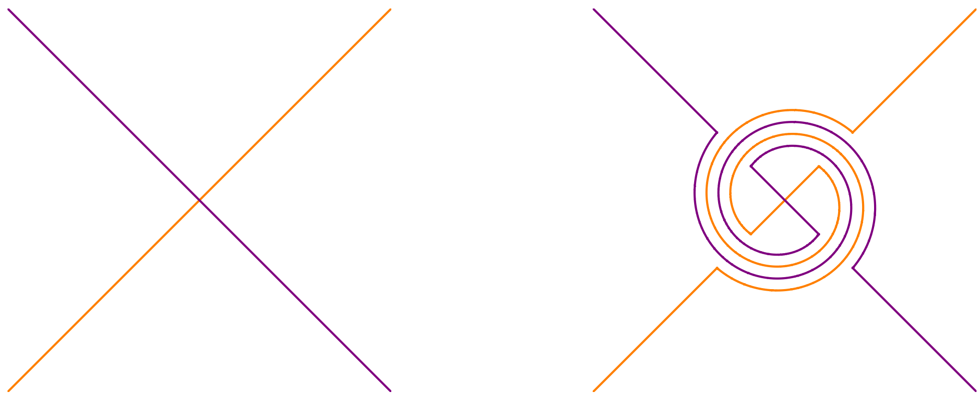


Figure 1.4.1: The result of the canonical Dehn twist on two lines through the origin.

diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{h} & \mathbb{C} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D_\alpha} \\ \mathbb{C} & \xrightarrow[h]{} & \mathbb{C} \end{array}$$

We call D_α a **Dehn twist around** α. △

Example 1.4.11. Let α, β, and γ be the loops in the first image in Figure 1.4.2. Then D_α, D_β, and D_γ occur in the annuli shown in the second image. ◊

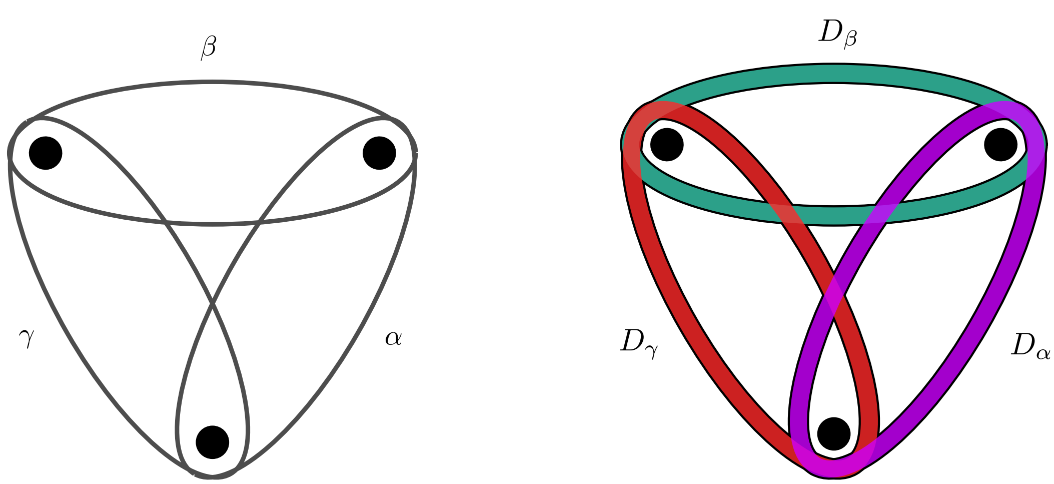


Figure 1.4.2: Three curves α, β, and γ, and the annuli in which their Dehn twists D_α, D_β, and D_γ take place.

For a given set M of marked points, the pure mapping class group is generated by a set of Dehn twists, one around each pair of points [10]. It is not in general an easy task, however, to define a presentation for $\mathrm{PMCG}(\widehat{\mathbb{C}}, M)$ using Dehn twists as generators. Later on, we will use braid groups to find presentations for $\mathrm{MCG}(\widehat{\mathbb{C}}, M)$ for arbitrary M. For now, though, we will restrict to the case where $|M| \leq 4$.

Remark 1.4.12. It must be noted that Dehn twists around closed curves containing a single point of M are trivial. Dehn twists around null-homotopic loops are trivial as well. $\diamondsuit$

Theorem 1.4.13 (The Lantern Relation)**.** *Let p_1, p_2, and p_3 be three marked points in a disc. Let α, β, γ, a, b, c, and d be the curves shown in Figure 1.4.3. Then*

$$D_\alpha D_\beta D_\gamma = D_\gamma D_\alpha D_\beta = D_\beta D_\gamma D_\alpha = D_a D_b D_c D_d.$$

For now we present this theorem without proof, and we refer the reader to Chapter 3 for a treatment of mapping class groups involving braids.

Remark 1.4.14. Notice that the twists D_a, D_b, D_c, and D_d all commute with each other, since the curves a, b, c, and d are all disjoint. $\diamondsuit$

Remark 1.4.15. The most important aspect of a Dehn twist is the points around which it occurs. Where it is clear, we will refer to a Dehn twist around a set of marked points M as D_M. We mean by this a Dehn twist around a curve separating M from the rest of $\widehat{\mathbb{C}}$. $\diamondsuit$

Lemma 1.4.16. *Let $M \subset \widehat{\mathbb{C}}$ be finite, let $p \in M$, and let $D_{\{p\}} \in \mathrm{PMCG}(\widehat{\mathbb{C}}, M)$ be a Dehn twist around $\{p\}$. Then $D_{\{p\}} = id$.*

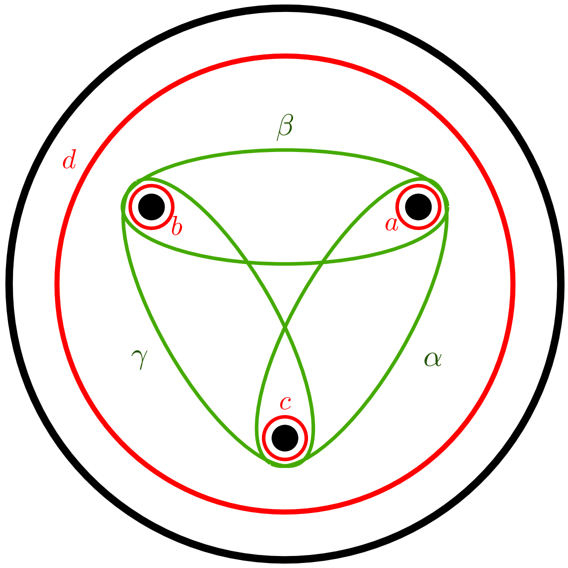


Figure 1.4.3

Proof. Notice that the canonical Dehn twist D can be written as the result of the isotopy $\phi\colon I \times \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ defined by

$$\phi_t(z) = \begin{cases} ze^{2\pi i t(2-|z|)} & \text{if } |z| \in [1,2] \\ ze^{2\pi i t} & \text{otherwise} \end{cases} .$$

Notice that $\phi_t(0) = 0$ for all $t \in I$. Choose a homeomorphism $h\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ so that h takes the annulus $\{z \mid |z| \in [1,2]\}$ to the proper annulus around $\{p\}$ and so that $h(0) = p$. Then $D_{\{p\}} = h \circ D \circ h^{-1}$, and so it is the result of the isotopy $h \circ \phi_t \circ h^{-1}$, which fixes p for each $t \in I$. Hence $D_{\{p\}} = id$. □

Theorem 1.4.17. *Let* $M \subset \widehat{\mathbb{C}}$.

1. *If* $|M| \leq 3$, *then* $\mathrm{PMCG}(\widehat{\mathbb{C}}, M) \cong 1$.
2. *If* $|M| = 4$, *then* $\mathrm{PMCG}(\widehat{\mathbb{C}}, M) \cong \langle A, B, C \mid ABC = CAB = BCA = id \rangle$.

Proof. We will only prove this for the first case, leaving the second to chapter 3.

Let $|M| = 1$. Notice that $\widehat{\mathbb{C}} - M$ is homeomorphic to $\mathbb{C}$. Since all orientation-preserving homeomorphisms of $\mathbb{C}$ are isotopic to the identity, we are done.

Let $|M| = 2$. Notice that any curve around the two points in M is contractible, and so the Dehn twist around the two points is trivial.

Let $|M| = 3$. Any curve separating two points in M is homotopic to a curve separating just one, and so any Dehn twist around two points is trivial.

□

Remark 1.4.18. The basic idea for the proof in the case where $|M| = 4$ is as follows, although the details here are fuzzy since we do not yet have the machinery to deal with it. Let $p, x \in M$. Notice that $D_{\{p,x\}} = D_{\{y,z\}}$ where y and z are the other two points in M. Hence any Dehn twist involving p can be written as a Dehn twist involving the other three points, yielding three generating elements A, B, and C. Supposing, without loss of generality, that these twists are arranged counterclockwise starting with A (as the curves α, β, and γ in Figure 1.4.3), we obtain the relation $ABC = BCA = CAB = D_p = id$. The fact that there are no other relations will for now be taken on faith (we will tackle this problem in Chapter 3). This leaves us the desired presentation. ◊

2

Lifting and Rewriting

2.1 Dehn Twist Lifting Theorem

The rest of this project hinges on the fact that we can say things about the Thurston classes of postcritically finite topological quadratics, if we know what generators of the mapping class group they come from. The following Theorem is how we derive these relationships:

Theorem 2.1.1 (Dehn Twist Lifting Theorem)**.** *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a topological quadratic. Let α be a simple closed curve in $\mathbb{C}$.*

1. *If $f^{-1}(\alpha) = \beta \cup \gamma$ where β and γ are disjoint simple closed curves in $\mathbb{C}$, then*

$$D_\alpha \circ f = f \circ D_\beta \circ D_\gamma.$$

2. *If $f^{-1}(\alpha) = \beta$ where β is a simple closed curve in $\mathbb{C}$, then*

$$D_\alpha^2 \circ f = f \circ D_\beta.$$

It follows from Thurston's theorem that if $\tau\colon \mathbb{C} \longrightarrow \mathbb{C}$ is a topological quadratic with the period three critical portrait, then τ is Thurston equivalent to either f_R, f_{CR}, or f_A. Hubbard and Nekrashevych, in [9] used monodromy groups to show that it is possible to determine which of these three τ is equivalent to, if you are given the sequence of Dehn twists ω so that $\tau \simeq \omega f_R$. The following sections present an alternative proof of that second fact, using Theorem 2.1.1 and the mapping class group of the 3-punctured complex plane.

2.2 Rewriting Systems

Let A be the Dehn twist around 0 and $f_R(0)$, let B be the Dehn twist around $f_R(0)$ and $f_R^2(0)$, and let C be the Dehn twist around $f_R^2(0)$ and 0. Nekrashevych showed that $Cf_R \sim f_A$ and that $C^{-1}f_R \sim f_{CR}$. We wish to know, given a sequence ω of Dehn twists, which class ωf_R belongs to. The way we go about figuring this out is by rewriting ω until it reduces to one of C, C^{-1}, or the identity (henceforth denoted 1). To be more formal, we wish to find a sequence $\omega_1, \omega_2, \ldots, \omega_n$ so that for each ω_i, we have $\omega_i f \sim \omega_{i+1} f$, and so that $\omega_n \in \{C, C^{-1}, 1\}$. Showing the existence of such a sequence for any ω is equivalent to showing that it is always possible to discover which equivalence class ωf belongs to, and to do so in a finite amount of time.

We need a formal system for this sort of computation, and we turn to rewriting systems for that structure. Much of the following notation comes from [5, Chapter 2].

Definition 2.2.1. We define a **rewriting system** as a pair $(S, \to)$, where S is a set and $\to$ is a binary relation on S. That is, $\to \subseteq S \times S$, and we write $a \to b$ if $(a, b) \in \to$. We write $a \xrightarrow{*} b$ if there exist $a_1, a_2, \ldots, a_n$ so that $a \to a_1 \to a_2 \to \cdots \to a_n \to b$. If there is no $b \in S$ such that $a \to b$, then we say that a is **irreducible**. △

Definition 2.2.2. We say that a rewriting system $(S, \to)$ is **confluent** if whenever $a \xrightarrow{*} x_1$ and $a \xrightarrow{*} x_2$, there is some b so that $x_1 \xrightarrow{*} b$ and $x_2 \xrightarrow{*} b$. We say that $(S, \to)$ is **terminating** if there does not exist an infinite chain $a_1 \to a_2 \to a_3 \to \cdots$. If $(S, \to)$ is both confluent and terminating, we say that it is **convergent**. △

Theorem 2.2.3. *Let $(S, \to)$ be a convergent rewriting system. Let $R \subseteq S$ be the set of irreducible elements of S. Let $s \in S - R$. Then $s \xrightarrow{*} r$ for a unique $r \in R$.*

Proof. Let $s \in S - R$. Since $(S, \to)$ is terminating, there is some element of R so that $s \xrightarrow{*} r$. To prove uniqueness, assume there exist distinct $r_1, r_2 \in R$ so that $s \xrightarrow{*} r_1$ and $s \xrightarrow{*} r_2$. Since $(S, \to)$ is confluent, there must be some element $b \in S$ so that $r_1 \xrightarrow{*} b$ and $r_2 \xrightarrow{*} b$. But r_1 and r_2 are irreducible, so this is a contradiction. Hence $s \xrightarrow{*} r$ for a unique $r \in R$. □

We can now rephrase our question. Let $\mathcal{L}$ be the set of reduced words in $\langle B, C \mid \rangle$, and let 1 denote the empty string in $\langle B, C \mid \rangle$. We wish to find a relation $\to$ on $\mathcal{L}$ with the following properties:

1. If $\omega \to \rho$, then $\omega f \sim \rho f$.

2. C, C^{-1}, and 1 are the only irreducible elements of $\mathcal{L}$.

3. $(\mathcal{L}, \to)$ is convergent.

This is precisely what we will do next.

2.3 Relations

Figure 2.3.1 shows the lifts of the curves used for the Dehn twists A, B, and C. We use these in combination with Theorem 2.1.1 to obtain homotopy relations for A, B, and C. Since a lifts to a single curve homotopic to c, we obtain the relation $A^2 f_r \simeq f_r C$. Since b lifts to a single curve homotopic to a, we obtain the relation $B^2 f_r \simeq f_r A$. Since c lifts to two disjoint curves, one of them homotopic to the identity, and the other homotopic to b, we obtain the relation $C f_r \simeq f_r B$.

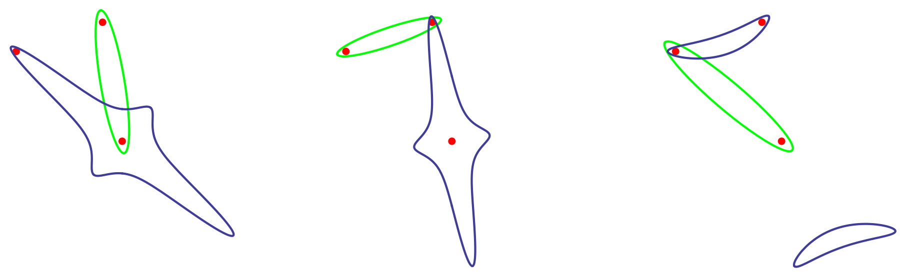


Figure 2.3.1: The lifts of a, b, and c, respectively. The original curves (a, b, and c) are shown in green, and their lifts are shown in blue. Note that each curve is counterclockwise.

Since we have the lantern relation, we can rewrite these rules in terms of only two of the generators. Since $ABC \simeq I$, we have $A \simeq C^{-1}B^{-1}$. Hence our rules and their inverses can be rewritten as

$$C^{-1}B^{-1}C^{-1}B^{-1}f \simeq fC \tag{2.3.1}$$

$$BCBCf \simeq fC^{-1} \tag{2.3.2}$$

$$B^2 f \simeq fC^{-1}B^{-1} \tag{2.3.3}$$

$$B^{-2} f \simeq fBC \tag{2.3.4}$$

$$Cf \simeq fB \tag{2.3.5}$$

$$C^{-1}f \simeq fB^{-1} \tag{2.3.6}$$

2.4 Liftable Subgroup

It will be useful in our proof to identify exactly which elements of $\mathrm{MCG}(\mathbb{C}^{*M})$ can be "moved across" f, and in what way. That is, given a sequence of Dehn twists ω, we wish to know when there exists another sequence of Dehn twists ρ so that $\omega f \simeq f\rho$. Additionally, we would like to identify what ρ is, given ω. This prompts the following definition.

Definition 2.4.1. Let $f_R \colon \mathbb{C} \longrightarrow \mathbb{C}$ be the rabbit polynomial. The **liftable subgroup** of $\mathrm{MCG}(\mathbb{C}, M)$ with respect to f_R is the set

$$L = \{S \in \mathrm{MCG}(\mathbb{C}, M) \mid \exists T \in \mathrm{MCG}(\mathbb{C}, M) \text{ so that } Sf_r \simeq f_r T\}.$$

△

To find this subgroup, we begin with the fact that $Cf_R \simeq f_R B$, and that $B^2 f_R \simeq f_R C^{-1}B^{-1}$. We then ask if $BCB^{-1}f_R \simeq f_R\omega$ for some word ω. Observe that

$$BCB^{-1}f_R = BCBB^{-2}f_R \simeq BFBf_R BC \simeq BCBCf_R C \simeq f_R C^{-1}C = f_R.$$

Hence $BCB^{-1}f_R \simeq f_R$. Then we know that anything that can be written as a combination of C, B^2, BCB^{-1}, and their inverses, can be moved across f_R, and in what way.

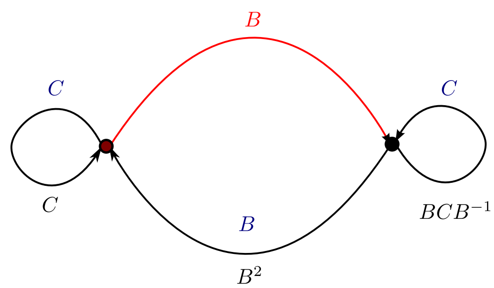


Figure 2.4.1

Observe that $\langle B^2, C, BCB^{-1} \mid\rangle$ is precisely the subgroup of $\langle B, C \mid\rangle$ generated by paths in the directed graph in Figure 2.4.1 beginning and ending at the base point (denoted as a red dot). Each vertex has one C edge going in, one B edge going in, one C edge going out, and one B edge going out. We choose a spanning tree (the red B edge). Then, for every edge not in the spanning tree, we choose a path along the spanning tree, then along that edge, and then back along the spanning tree. For example, the C edge on the right becomes BCB^{-1}.

Then for any path beginning and ending at the base point, we record the edges it traverses (eg. $BCCBC$), remove each red letter, and replace each blue letter with its corresponding spanning tree path. For example, $BCCBC$ becomes $BCB^{-1}BCB^{-1}B^2C$.

Since each path beginning and ending at the base point represents a word with an even number of Bs, we see that $\langle B, C, BCB^{-1} \mid \rangle$ generates the subgroup of $\langle B, C \mid \rangle$ with an even number of Bs. Hence words with an even number of Bs are liftable. Since we have no relation for Bf_R, it follows that words with an odd number of Bs are not liftable, and this prompts the following proposition:

Proposition 2.4.2. *Let f_R be the rabbit polynomial, and let ω be a word in $\langle B, C \mid \rangle$. If ω contains an even number of Bs, then $\omega f_R \simeq f_R\rho$ for some word ρ in $\langle B, C \mid \rangle$. If ω contains an odd number of Bs, then $\omega f_R \simeq Bf_R\rho$ for some word ρ in $\langle B, C \mid \rangle$.*

Proof. Let ω contain an even number of Bs. Then it can be written as a product of the generators B, C, BCB^{-1}, and their inverses. Each of those is liftable, so $\omega f_R \simeq f_R\rho$ where ρ is a product of B, $C^{-1}B^{-1}$, 1, and their inverses. Now assume ω has an odd number of Bs. Then form the word $BB^{-1}\omega$. Since $BB^{-1} = 1$, it follows that $BB^{-1}\omega f_R \simeq \omega f_R$. Observe that $B^{-1}\omega$ has an even number of Bs, so $BB^{-1}\omega f_R \simeq Bf_R\rho$ where rho is a product of B, $C^{-1}B^{-1}$, 1, and their inverses. This completes the proof. □

We can now begin to construct the reduction we will put on $\mathcal{L}$. Using Proposition 2.4.2 along with the relations we have on B, C, and BCB^{-1}, we construct the following Thurston equivalences:

Proposition 2.4.3. *Let ω be a reduced word in $\langle B, C \mid \rangle$. Let f_r be the rabbit polynomial. Then the following equivalences hold.*

$$\omega C f_R \sim B\omega f_R \qquad \omega C^{-1} f_R \sim B^{-1}\omega f_r \tag{2.4.1}$$

$$\omega B^2 f_R \sim C^{-1}B^{-1}\omega f_R \qquad \omega B^{-2} f_R \sim BC\omega f_r \tag{2.4.2}$$

$$\omega CB f_R \sim C^{-1}B^{-1}\omega B^{-1} f_R \qquad \omega C^{-1}B f_R \sim C^{-1}B^{-1}\omega B^{-1} f_R \tag{2.4.3}$$

$$\omega CB^{-1} f_R \sim \omega B^{-1} f_R \qquad \omega C^{-1}B^{-1} f_R \sim \omega B^{-1} f_R \tag{2.4.4}$$

Proof. Observe that equation 2.4.1 follows directly from the original relations. Since $\omega C f_R \simeq \omega f_R B$, we can move the B around to obtain $\omega C f_R \sim B\omega f_R$. The inverse case is the same. Equation 2.4.2 also follows directly from the original relations.

For equation 2.4.3, observe that $\omega CBf_R \simeq \omega B^{-1}BCBf_R \simeq \omega B^{-1}BCB^{-1}B^2 f_R \simeq \omega B^{-1} f_R C^{-1}B^{-1}$. Since $\omega B^{-1} f_R C^{-1} B^{-1} \sim C^{-1}B^{-1}\omega B^{-1}$, we have $\omega CBf_R \sim C^{-1}B^{-1}\omega B^{-1}$. The case for $\omega C^{-1}Bf_R$ is similar.

For equation 2.4.4, observe that $\omega CB^{-1} f_R \simeq \omega B^{-1}BCB^{-1} f_R \simeq \omega B^{-1} f_R$, so $\omega CB^{-1} f_R \sim \omega B^{-1} f_R$. The case for $\omega C^{-1}B^{-1} f_R$ is similar.

□

2.4.1 Proof

Observe that the relations in Proposition 2.4.3 are *all* of the possible endings for a given word. Therefore, for any given word, one and only one of these rules will apply. We are now ready to define our reduction on $\mathcal{L}$. Let $\mathcal{A}\colon \mathcal{L} - \{B, B^{-1}, 1\} \longrightarrow \mathcal{L}$ be defined by $\mathcal{A}(\omega) = \rho$ where ρ is obtained by the appropriate relation in Proposition 2.4.3, and then reducing it. Since we have no relation if $\omega = B$, if $\omega = B^{-1}$, or if $\omega = 1$, we do not define $\mathcal{A}$ on those elements. For example, if $\omega = \alpha C$ for some reduced word α, then $\mathcal{A}(\omega) = B\alpha$ if α does not begin with B^{-1}, and $\mathcal{A}(\omega) = \alpha$ if it does. Observe that this is a well-defined function: each $\omega \in \mathcal{L}$ is a reduced word, so it has only one representative in $\mathcal{L}$, and $\mathcal{A}$ is defined on all of its domain.

Define $\rightarrow \subset \mathcal{L} \times \mathcal{L}$ to be the set $\{(\omega, \rho) \in \mathcal{L} \times \mathcal{L} \mid \mathcal{A}(\omega) = \rho\}$. Observe that, by construction, $\omega f_R \sim \mathcal{A}(\omega) f_R$. Hence if $\omega \rightarrow \rho$, then $\omega f_R \sim \rho f_R$. Also by construction, the set of irreducible elements in $(\mathcal{L}, \rightarrow)$ is $\{B, B^{-1}, 1\}$. Hence the only thing we have left to prove is that $(\mathcal{L}, \rightarrow)$ is convergent.

Since $\mathcal{A}(\omega)$ is only a single element for each $\omega \in \mathcal{L} - \{B, B^{-1}, 1\}$, it follows that, for any ω, we have $\omega \to \rho$ for a unique $\rho \in \mathcal{L}$. Hence $(\mathcal{L}, \to)$ is confluent.

To prove that $(\mathcal{L}, \to)$ is terminating, we define an order on $\mathcal{L}$ by looking at the "length" of each $\omega \in \mathcal{L}$. That is, we define $|\omega|$ to be the number of letters in ω. We will show that for any $\omega \in \mathcal{L} - \{B, B^{-1}, 1\}$ such that $|\omega| > 1$, there exists some $\rho \in \mathcal{L}$ so that $\omega \xrightarrow{*} \rho$ and $|\rho| < |\omega|$. We use the notation $(\omega - n)$ to denote a word with n fewer letters than ω.

Proposition 2.4.4. *Let $\omega \in \mathcal{L} - \{B, B^{-1}, 1\}$. Then there exists $\rho \in \mathcal{L}$ such that $|\rho| < |\omega|$ and $\omega \xrightarrow{*} \rho$.*

Proof. If $\omega = \alpha C^n$ for some $n \in \mathbb{Z} - \{0\}$, we know that $\omega f_R \sim B^n(\alpha - 1)B^i f_R$ for some $i \in \mathbb{Z} - \{0\}$. So we can assume, without loss of generality, that $\omega = \alpha B^n$ for some $n \in \mathbb{Z} - \{0\}$. We have four cases.

Case 1: Assume $\omega = \alpha CB$. Then

$$\alpha CB \to C^{-1}B^{-1}\alpha B^{-1}.$$

If $\alpha = (\alpha - n)B^{-n}$, then

$$C^{-1}B^{-1}\alpha B^{-1} \xrightarrow{*} (C^{-1}B^{-1})^{\frac{n}{2}+1}(\alpha - n - 1)C^{\pm 1}B^{-1} \to (C^{-1}B^{-1})^{\frac{n}{2}+1}(\alpha - n - 1)B^{-1}.$$

Repeating this process one more time gives us

$$(C^{-1}B^{-1})^{\frac{n}{2}+1}(\alpha - n - 1)B^{-1} \xrightarrow{*} (C^{-1}B^{-1})^{\frac{n+p}{2}+1}(\alpha - n - p - 2)B^{-1}$$

for some $p \in \mathbb{N} \cup \{0\}$. Observe that $|(C^{-1}B^{-1})^{\frac{n+p}{2}+1}(\alpha - n - p - 2)B^{-1}| = |\alpha| + 1 < |\alpha| + 2 = |\omega|$. Hence $\omega \xrightarrow{*} \rho$ for some ρ with $|\rho| < |\omega|$.

Case 2: The proof for $\omega = \alpha C^{-1}B$ is exactly the same as in case 1.

Case 3: Assume $\omega = (\omega - 2)C^{\pm 1}B^{-1}$. Then $\omega \to (\omega - 2)B^{-1}$, so we are done.

Case 4: Assume $\omega = (\omega - 2)B^{\pm 2}$. Then $\omega \to (CB)^{\pm 1}(\omega - 2)$. Since $(CB)^{\pm 1}(\omega - 2)$ has the same number of twists as ω but one fewer B terms, we are done.

□

The previous proposition shows that $(\mathcal{L}, \to)$ is terminating. Hence $(\mathcal{L}, \to)$ is convergent. Since the set of irreducible words in $\mathcal{L}$ is precisely $\{C, C^{-1}, 1\}$, it follows that ωf_r is Thurston equivalent to precisely one of $C f_r$, $C^{-1} f_r$, or f_r, and that we can determine which one it is in a finite amount of time.

3
Braid Groups

Braids are very versatile objects: they have an intuitive geometric interpretation, a nice algebraic structure, and are used widely throughout mathematics. We will be using them because of their intimate relationship with homeomorphisms of $\mathbb{C}$—their algebraic structure can be used as a powerful tool to attack the problem of Thurston equivalence. In this chapter we will first develop some of the general theory behind braids, and then introduce their connection to the mapping class group.

3.1 The Braid Group B_n

A braid, in essence, is a geometrically intuitive object. Think of the standard hair braid: we take three bundles of hair, do a sequence of twists, and then tie it off. The "endpoints" of the bundles are then fixed (i.e. attached to the head or tied off), and the bundles certainly cannot pass through each other. The notion of being unable to undo a braid gives us a natural idea of equivalence—two braids are the same if, when one pulls the strands tight, they have the same crossings.

These are essentially all of the properties we want a mathematical braid to have—two or more strands twisted together and then fixed, so that the strands cannot untwist. Two braids will be equivalent if one can be deformed into the other without crossing the strands.

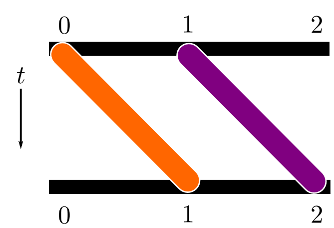
Definition 3.1.1. A **braid** $\beta\colon I \longrightarrow \mathbb{C}$ is a set of continuous paths

$$\beta(t) = \{p_1(t), p_2(t), \dots, p_n(t)\}$$

so that at any time t we have $p_i(t) \neq p_j(t)$ for all $i \neq j$. △

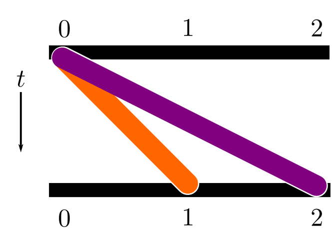
Remark 3.1.2. Each path p_i in a braid β is called a **strand**. It will occasionally be useful to refer to a specific strand of a braid β, and for this we use the notation $\beta_t(i) := p_i(t)$. ◊

Example 3.1.3. The path $t \mapsto \{t, 1+t\}$ is a braid, since $t \neq 1+t$ for all $t \in I$. The following is a graph of this path, looking at it from the perspective of the negative imaginary axis.



◊

Example 3.1.4. The path $t \mapsto \{t, 2t\}$, shown below, is *not* a braid, since at $t = 0$, we have $t = 2t = 0$.



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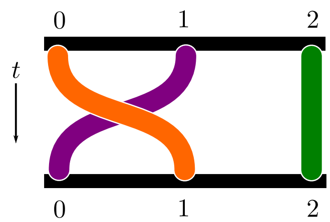
Example 3.1.5. The path

$$t \mapsto \left\{ \left(-\frac{1}{2}e^{i\pi t} + \frac{1}{2} \right), \left(\frac{1}{2}e^{i\pi t} + \frac{1}{2} \right), t \right\}$$

is a braid with a crossing of the first and second strands:



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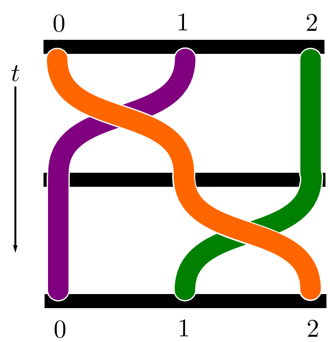
Example 3.1.6. Let β be the path from Example 3.1.5, and let γ be the equivalent crossing of 1 and 2. We can form a new braid α by concatenating β and γ. We denote this as $\alpha = \beta\gamma$:



Note that we can only do this when $\beta_1 = \alpha_0$. ◊

Since we can only concatenate braids when they have endpoints in common, we will define some notation to make keeping track of said endpoints easier.

Definition 3.1.7. Let β be a braid. We call the set of points in β_0 the **begin points** of β, and we call the set of points in β_1 the **end points** of β. △

Definition 3.1.8. Let $A, B \subset \mathbb{C}$ be finite sets. We will let $\mathrm{Braid}(A, B)$ denote the set of braids with begin points A and end points B. △

We can also think of a braid on n strands as a path in what is called "configuration space." The configuration space of n points in $\mathbb{C}$ is simply the space of n-element subsets $\mathbb{C}$.

Definition 3.1.9. The space

$$\mathcal{C}_n(\mathbb{C}) = \big\{\{x_1, x_2, \ldots, x_n\} \in X^n \mid x_i \neq x_j \text{ for } i \neq j\big\}$$

is called the **configuration space** of n points in $\mathbb{C}$. △

The configuration space inherits a topology as a quotient of a subspace of $\mathbb{C}^n$. Using this topology, it is easy to see that any continuous path $\beta\colon I \longrightarrow \mathcal{C}_n(\mathbb{C})$ in the configuration space is in fact a braid, and so we can use this as an alternate but equivalent definition.

We wish to consider two braids the same if one can be deformed into the other, keeping the endpoints fixed, and without crossing the strands. The natural way to do this is to look at homotopy classes of paths in $\mathcal{C}_n(X)$.

Definition 3.1.10. Let $\alpha, \beta \in \mathrm{Braid}(A, B)$. We say that α and β are **braid homotopic**, denoted $\alpha \simeq \beta$, if there exists a homotopy of paths $\psi\colon I \times I \longrightarrow \mathcal{C}_n(\mathbb{C})$ so that $\psi_0 = \alpha$, and $\psi_1 = \beta$. △

This is precisely what we want; at any two times t and r, $\phi_t(r)$ is still an element of $\mathcal{C}_n(\mathbb{C})$, and so the strands of a braid never cross during homotopy.

We can now define a group structure on braids. If we choose a base point $N \in \mathcal{C}_n(\widehat{\mathbb{C}})$, then we can form the fundamental group $\pi_1(\mathcal{C}_n(\widehat{\mathbb{C}}), B)$ of homotopy classes of braids starting and ending at N.

Definition 3.1.11. Let $n \in \mathbb{N}$, and let $N = \{0, 1, 2, \ldots, n-1\} \in \mathcal{C}_n(\widehat{\mathbb{C}})$. The **braid group on n strands**, denoted B_n, is the group $\pi_1(\mathcal{C}_n(\widehat{\mathbb{C}}), N)$. △

Taking two elements β and α of B_n, we see that the group product $\beta\alpha$ is the same as the concatenation defined in Example 3.1.6

Remark 3.1.12. At this point we have several different sets of notation for the same thing. Part of the reason braids are so useful is that they have so many equivalent representations; this same property, however, makes it difficult do define a consistent notation for them.

For this reason, we will in general use whatever notation is convenient, without further discussion. So a braid β on a finite set A is either a path in $\mathcal{C}_n(\mathbb{C})$, a tuple of paths, or a family of functions $\beta_t \colon A \longrightarrow \mathbb{C}$.

We will also, in general, not care about the distinction between braids and homotopy classes of braids. Therefore the set $\mathrm{Braid}(A, B)$ refers to homotopy classes of braids, and so we can use $\mathrm{Braid}(A, A)$ to denote the group $\pi_1(\mathcal{C}_n(\mathbb{C}), A)$. The standard braid group B_n is still specifically the group of braids on $\{0, 1, \ldots, n\}$. $\Diamond$

The group B_n is generated by $n-1$ crossings $X_1, X_2, \ldots, X_{n-1}$, where X_j is the braid in which the strand j crosses "in front" (that is, via the lower half-plane) of $j+1$.

Definition 3.1.13. Let $n \in \mathbb{N}$ and $j \in \{1, \ldots, n-1\}$. The braid $X_j \in B_n$ is defined to be the homotopy class of the braid

$$t \mapsto \left\{1, 2, \ldots, j-1, \left(-\frac{1}{2}e^{i\pi t} + \frac{2j+1}{2}\right), \left(\frac{1}{2}e^{i\pi t} + \frac{2j+1}{2}\right), j+2, \ldots, n-1\right\}.$$

$\triangle$

One can check that this is indeed an element of $\mathcal{C}_n(\widehat{\mathbb{C}})$ for all $t \in I$. Notice that the inverse element of X_j is the class of braids generated by

$$t \mapsto \left\{1, 2, \ldots, j-1, \left(-\frac{1}{2}e^{-i\pi t} + \frac{2j+1}{2}\right), \left(\frac{1}{2}e^{-i\pi t} + \frac{2j+1}{2}\right), j+2, \ldots, n-1\right\}.$$

We use $\overline{X}_j$ to refer to this element.

The identity element of B_n is the braid $\mathrm{id}(t) = \{0, 1, \ldots, n-1\}$. Figure 3.1.1 shows $X_1\overline{X}_1$, which is the same as the identity.

Theorem 3.1.14. *The group B_n is generated by $n-1$ elements $X_1, \ldots, X_{n-1}$, with the following relations:*

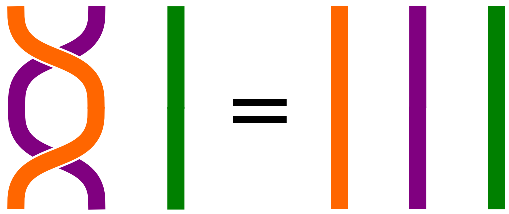


Figure 3.1.1: The braid $X_1\overline{X}_1$, which is equivalent to *id*.

1. *If* $|i-j| \geq 2$, *then* $X_iX_j = X_jX_i$.

2. $X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}$ *for all* $i < n-1$.

We refer the reader to [8] for a proof of this fact. Figure 3.1.2 shows the geometric idea behind the relation $X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}$.

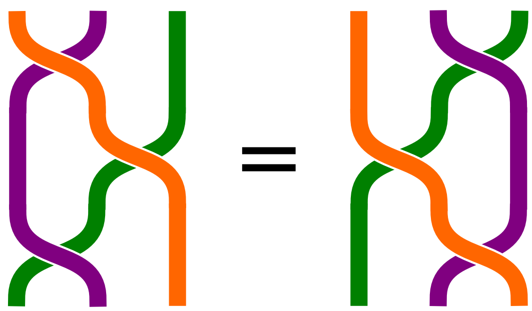


Figure 3.1.2: The relation $X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}$.

Remark 3.1.15. We will mostly be working with B_3 and B_4, and so instead of using the general notation defined above, I make the following notational conventions: $A = X_1$ and $a = \overline{X}_1$, $B = X_2$ and $b = \overline{X}_2$, etc. So our relations are nicely written as $AC = CA$, $ABA = BAB$, and so on. Figure 3.1.3 shows the generators A and B for B_3, along with their inverses. ◊

A a B b

Figure 3.1.3: The two generators A and B of B_3, and their inverses.

The following is a relation that we will use often in Chapter 4.

Lemma 3.1.16. *Let $n \in \mathbb{N}$, and let $i < n-1$, and let $X_i \in B_n$. Then $\overline{X}_i X_{i+1}^q X_i = X_{i+1} X_i^q \overline{X}_{i+1}$.*

Proof. Observe that $\overline{X}_i X_{i+1}^q X_i = \overline{X}_i X_{i+1}^q X_i X_{i+1} \overline{X}_{i+1} = \overline{X}_i X_{i+1}^{q-1} X_i X_{i+1} X_i \overline{X}_{i+1} = \overline{X}_i X_i X_{i+1} X_i^q \overline{X}_{i+1} = X_{i+1} X_i^q \overline{X}_{i+1}$. □

3.2 The Mapping Class Group and Braids

Braids in $\mathbb{C}$ and homeomorphisms of $\mathbb{C}$ are intimately related. In particular, if M is a set of marked points, any homeomorphism $h\colon (\mathbb{C}, M) \longrightarrow (\mathbb{C}, M)$ has unique braid representative in $\text{Braid}(M, M)$. In addition, any such braid is a representative for a unique class of homeomorphisms in the mapping class group of $(\mathbb{C}, M)$. This section will illuminate some of the theory behind this connection.

Definition 3.2.1. Let $M = \{m_1, m_2, \ldots, m_n\}$. Given a homeomorphism $h\colon (\widehat{\mathbb{C}}, M) \longrightarrow (\widehat{\mathbb{C}}, N)$, let ϕ_t be an isotopy with $\phi_0 = \text{id}$ and $\phi_1 = h$. The braid β defined by

$$\beta(t) = \{\phi_t(m_1), \ldots, \phi_t(m_n)\}$$

is called a **braid representative** for h. △

Example 3.2.2. Let $n \geq 1$, and let $B = \{1, 2, \ldots, n\} \subset \mathbb{C}$. Consider an isotopy $\phi\colon I \times \mathbb{C} \longrightarrow \mathbb{C}$. Since ϕ is an isotopy, it follows that $\phi_t(j) \neq \phi_t(k)$ for any $j \neq k$ and all $t \in I$. Hence for any isotopy ϕ and any set of marked points $\{m_1, \ldots, m_n\}$, we get a braid

$$t \mapsto \{\phi_t(m_1), \ldots, \phi_t(m_n)\}\,.$$

Figure 3.2.1 shows a braid induced by some isotopy ϕ.

◊

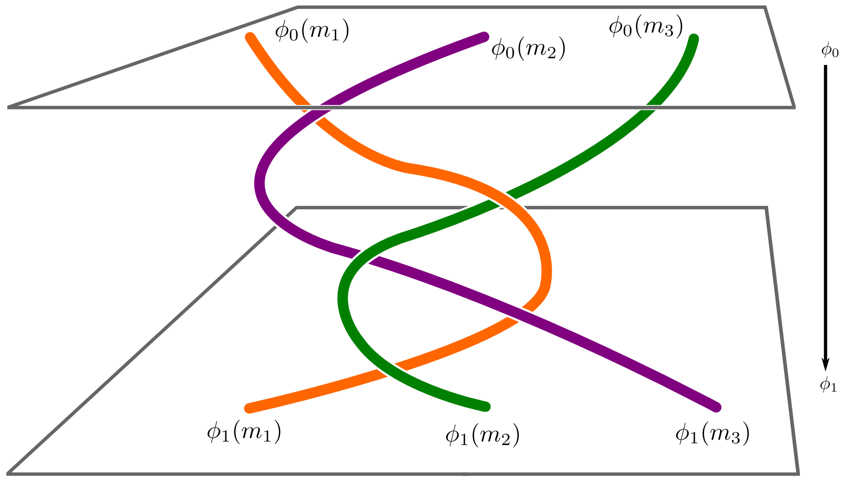
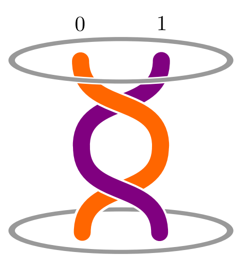


Figure 3.2.1: A braid in $\mathbb{C}$ induced by some isotopy ϕ.

Example 3.2.3. The representative braid for a Dehn twist around two points is a double twist of the two points. The figure below shows the representative braid for a Dehn twist around 0 and 1.



◊

We know that any homeomorphism gives rise to a braid, but we also want the other direction. That is, we want a class of braids to be a representative for a unique class of homeomorphisms. Proofs of the following theorems can be found in [8, Chapter 1].

Theorem 3.2.4. *Every braid is a representative for some homeomorphism.*

Theorem 3.2.4 essentially says that any braid beginning and ending at a set of marked points M is a picture of some homeomorphism $h\colon (\mathbb{C}, M) \to (\mathbb{C}, M)$.

Theorem 3.2.5. *Let $h, k\colon (\widehat{\mathbb{C}}, M) \to (\widehat{\mathbb{C}}, N)$ be homeomorphisms. If h and k have a common braid representative, then h and k are isotopic rel M.*

Theorem 3.2.5 in combination with Theorem 3.2.4 says that to every $\beta \in \mathrm{Braid}(M)$ we can assign a unique (up to isotopy rel M) homeomorphism. That is, we have a function $\mathrm{Braid}(M) \to \mathrm{MCG}(\mathbb{C}, M)$. This function is not, however, one to one. That is, it is not true that every homeomorphism class in $\mathrm{MCG}(\mathbb{C}, M)$ has a unique braid representative in $\mathrm{Braid}(M)$—a full twist around all points in M is isotopic rel M to the identity. Any such twist is represented in the braid group as a $360°$ twist around all of the strands. The following sequence of theorems expresses this idea formally.

Definition 3.2.6. Let $n \in \mathbb{N}$. The element

$$\Delta_n = (X_1 X_2 \cdots X_n)(X_1 X_2 \cdots X_{n-2}) \cdots (X_1 X_2) X_1 \in B_n$$

is the $180°$ twist in B_n. The element $\theta_n = \Delta_n^2$ is the $360°$ twist in B_n. △

Figure 3.2.2 shows Δ_5 (the 180 degree twist on 5 strands) and θ_5 (the 360 degree twist on 5 strands).

Theorem 3.2.7 ([8, Theorem 1.24])**.** *Let $n \geq 3$. Then the center of B_n, is the infinite cyclic subgroup generated by the $360°$ twist θ_n*

Remark 3.2.8. We can talk about the group $\mathrm{Braid}(M)$ for any arbitrary finite set M. In fact, if $|M| = n$, this group is always isomorphic to B_n. To see this, pick any element γ of $\mathrm{Braid}(\{0, 1, \ldots, n\}, M)$. Then conjugating elements of $\mathrm{Braid}(M)$ by γ gives us the desired isomorphism. ♢

Lemma 3.2.9. *Let $h \in \mathrm{MCG}(\widehat{\mathbb{C}}, M)$, and let β be in the center of $\mathrm{Braid}(M)$. If h has braid representative β, then h is isotopic rel M to the identity.*

Proof. We will prove this only for the case where $M = \{1, \ldots, n\}$. By Theorem 3.2.7, β is either the identity, or it is some power of θ_n. Let ϕ be the isotopy which twists $\mathbb{C}$ by

Figure 3.2.2: The braids Δ_5 and θ_5, respectively.

$360°$ around the origin. Then the braid representative for ϕ is the element $\theta_n \in \text{Braid}(M)$. By Theorem 3.2.5, any homeomorphism $h\colon (\widehat{\mathbb{C}}, M) \to (\widehat{\mathbb{C}}, M)$ with braid representative θ_n is isotopic rel M to ϕ_1, which is isotopic rel M to the identity. □

Armed with Lemma 3.2.9, we can finally construct our isomorphism. Modding out $\text{Braid}(M)$ by its center gives us precisely the equivalence on braids that we want: any braid with a 360 degree twist is equivalent to itself without the 360 degree twist.

Theorem 3.2.10. *Let $M \subset \widehat{\mathbb{C}}$. Let Z be the center of* $\text{Braid}(M)$*. The function*

$$\text{Braid}(M)/Z \longrightarrow \text{MCG}(\widehat{\mathbb{C}}, M)$$

taking a braid to its homeomorphism class is an isomorphism.

Proof. This follows directly from Theorem 3.2.5 and Lemma 3.2.9. □

Remark 3.2.11. Now that the isomorphism between $\text{MCG}(\mathbb{C}, M)$ and $\text{Braid}(M)/Z$ is established, we will make no distinction between homeomorphisms and braids. That is, we will write things like $\beta \circ h$ where β is a braid and h is a homeomorphism. In such a case we will mean "the homeomorphism to which β is associated, composed with h." $\Diamond$

Example 3.2.12. Recall the lantern relation from Section 1.4. We can now prove this relation using braids. Let $M = \{0, 1, 2\}$. Then $\mathrm{MCG}(\mathbb{C}, M)$ is generated by the three Dehn twists $D_{0,1}$, $D_{1,2}$ and $D_{0,2}$. As braids, these are represented as the $D_{0,1} = A^2$, $D_{1,2} = B^2$, and $D_{0,2} = BA^2\bar{B}$. The inverse 360 degree twist in B_3 is the braid *ababab*, shown in the following figure:

The lantern relation says that the composition of Dehn twists $D_{0,1}D_{1,2}D_{0,2}$ is equivalent to a 360° twist around the set $\{0, 1, 2\}$. In the braid group, the relation is as follows:

$$BA^2bB^2A^2 = \theta_3.$$

A proof of this fact via braid diagrams is shown inf Figure 3.2.3 (note the introduction of *ababab* in step 3).

Additionally, we can write down a purely algebraic proof using the relations in the braid group:

$$BA^2bB^2A^2 = BA^2BA^2 = BABABA = ABAABA = \theta_3.$$

$\Diamond$

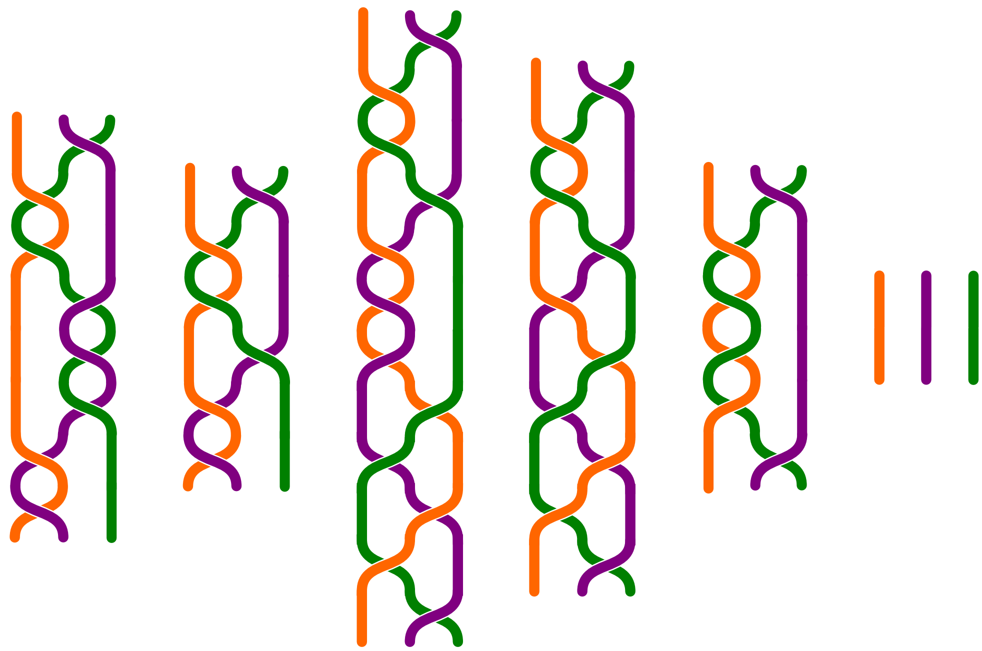


Figure 3.2.3

4

Braids and Postcritically Finite Polynomials

Recall that any topological quadratic can be written as $\tau = f \circ h$ for some quadratic polynomial f and some homeomorphism h taking the postcritical set to itself. This fact, in combination with the correspondence between homeomorphisms and braids, allows us to use braids to shed some light on the problem of Thurston equivalence between topological quadratics.

The basic idea of this chapter is that any given topological quadratic τ is Thurston equivalent to $s \circ h$ where s is the map $z \mapsto z^2$. Then h will give us a unique braid representing τ. To determine the Thurston class of τ, we then simplify h as much as possible by lifting it across s and conjugating, much as in Chapter 2. We begin by showing that one can lift braids across quadratics in a meaningful way.

4.1 Squaring And Lifting

Recall that a braid can be thought of as an injective function $\beta\colon I \times N \to \mathbb{C}$, where N is a finite subset of $\mathbb{C}$. In this section we will simply use N to denote the set $\{0, 1, 2, \ldots, n\}$. Instead of using the map $z \mapsto z^2$, we will use the angle-doubling map:

Definition 4.1.1. The **angle-doubling map** is the function $\varsigma\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\varsigma(z) = |z|e^{2i\arg(z)}.$$

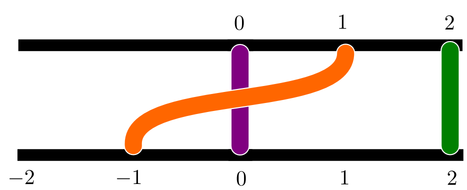
△

Remark 4.1.2. We will refer to the set of braids in $\mathrm{Braid}(N, M)$ which fix 0 as $\mathrm{Braid}_0(N, M)$. ◊

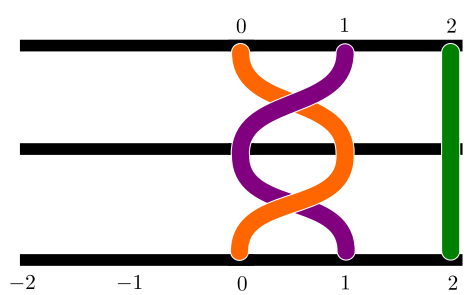
Definition 4.1.3. A **lift of** N is a set $\tilde{N}$ with $|\tilde{N}| = |N|$ and $\varsigma(\tilde{N}) = N$. △

Definition 4.1.4. Let $\tilde{N}_1$ and $\tilde{N}_2$ be lifts of N. Let $\beta \in \mathrm{Braid}_0(\tilde{N}_1, \tilde{N}_2)$ with $\beta_t(j) \neq -\beta_t(k)$ for $j, k \in N - \{0\}$ for all $t \in I$. The **square** of β is the braid $\alpha_t = \varsigma \circ \beta_t$. △

Example 4.1.5. Let $\beta \in \mathrm{Braid}(\{0, 1, 2\}, \{-1, 0, 2\})$ be a single crossing of the first two strands:



Then β squares to the following braid in $\mathrm{Braid}(\{0, 1, 2\}, \{0, 1, 2\})$:



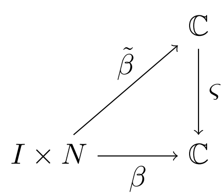
◊

Theorem 4.1.6. *Let* $\gamma \in \mathrm{Braid}_0(\tilde{N}_1, \tilde{N}_2)$. *Then there exists a unique (up to isotopy rel* N*) braid* $\beta \in \mathrm{Braid}_0(N, N)$ *so that* $\beta = \varsigma \circ \gamma$.

Proof. First, observe that there exists an $\alpha \in \mathrm{Braid}(N, N)$ so that $\gamma \simeq_N \alpha$, where $\varsigma \circ \alpha_t$ injective for all $t \in I$. Then the braid $\beta \in \mathrm{Braid}(N, N)$ defined by $\beta_t = \varsigma \circ \alpha_t$ satisfies the requirements. □

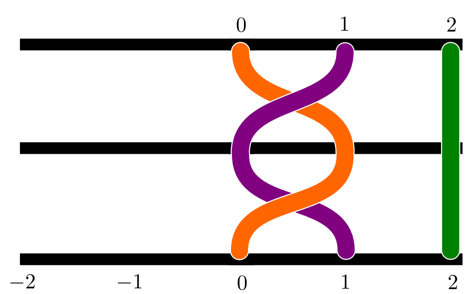
Now that we have defined the square of a braid, we wish to understand which braids are liftable—i.e. which braids are squares of other braids. We define this concept as follows.

Definition 4.1.7. Let $\beta \in \mathrm{Braid}_0(N, N)$, and let $\widetilde{N}_1$ be a lift of N. We say β is **liftable** starting at $\tilde{N}_1$ if there exists a lift $\tilde{N}_2$ of N and a braid $\tilde{\beta} \in \mathrm{Braid}(\tilde{N}_1, \tilde{N}_2)$ so that the following diagram commutes.

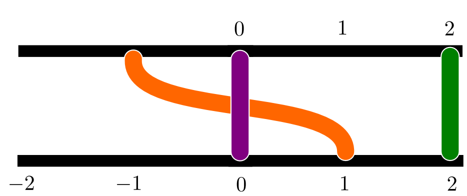


We call $\tilde{\beta}$ a **lift** of β starting at $\tilde{N}_1$. △

Example 4.1.8. Let $\beta \in \mathrm{Braid}(N, N)$ be the braid a^2:



Then the following braid $\tilde{\beta}$ in $\mathrm{Braid}(\{-1, 0, 2\}, \{0, 1, 2\})$ is a lift of β.



◊

The homotopy lifting property from algebraic topology tells us that, given a braid $\beta \in \mathrm{Braid}_0(N, N)$ and a lift of β_0, we get a unique braid lifting β:

Proposition 4.1.9 ([6, p. 60]). *Given a braid $\beta_t \colon N \longrightarrow \mathbb{C}$ fixing 0 and a map $\tilde{\beta}_0 \colon N \longrightarrow \mathbb{C}$ so that $\varsigma \circ \tilde{\beta}_0 = \beta_0$, there exists a unique braid $\tilde{\beta}_t \colon N \longrightarrow \mathbb{C}$ so that*

$$\varsigma \circ \tilde{\beta} = \beta.$$

Proposition 4.1.10. *Let $\beta \in \mathrm{Braid}_0(N, N)$, and let $\tilde{N}_1$ be a lift of N. Then there exists a unique lift $\tilde{N}_2$ of N and a unique (up to isotopy rel N) lift $\tilde{\beta} \in \mathrm{Braid}(\tilde{N}_1, \tilde{N}_2)$ of β.*

4.2 Representatives

Now that we know how to lift braids across ς, we can set about finding braid representatives for topological quadratics.

Theorem 4.2.1. *Let τ be a topological quadratic with postcritical set P_τ and branch points $\{p, \infty\}$. There exists an orientation-preserving homeomorphism $h \colon (\mathbb{C}, p, \infty) \longrightarrow (\mathbb{C}, 0, \infty)$ so that*

$$\tau \sim \varsigma \circ h$$

and so that $h(N) \subset N \cup -N$.

Proof. Let $h \colon (\mathbb{C}, N, 0) \longrightarrow (\mathbb{C}, P_\tau, \tau(p))$ be a homeomorphism. Let k be a homeomorphism such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C}, \pm N, 0) & \xrightarrow{\;k\;} & (\mathbb{C}, \tau^{-1}(P_\tau), p) \\ \big\downarrow \varsigma & & \big\downarrow \tau \\ (\mathbb{C}, N, 0) & \xrightarrow[\;h\;]{} & (\mathbb{C}, P_\tau, \tau(p)) \end{array}$$

Then we have $\tau \circ k = h \circ \varsigma$, and so

$$\begin{aligned} \tau &\sim k^{-1} \circ \tau \circ k \\ &\sim k^{-1} \circ h \circ \varsigma \\ &\sim \varsigma \circ k^{-1} \circ h. \end{aligned}$$

Then $k^{-1} \circ h$ is the desired homeomorphism. □

Definition 4.2.2. Let $\tau\colon (\mathbb{C}, P_\tau) \longrightarrow (\mathbb{C}, P_\tau)$ be a topological quadratic, and let $h\colon (\mathbb{C}, N, 0) \longrightarrow (\mathbb{C}, \pm N, 0)$ be a homeomorphism so that $\tau \sim \varsigma \circ h$. Let ϕ_t be an isotopy taking the identity to h. The braid $\phi_t(N)$ is called a **braid representative** of τ. △

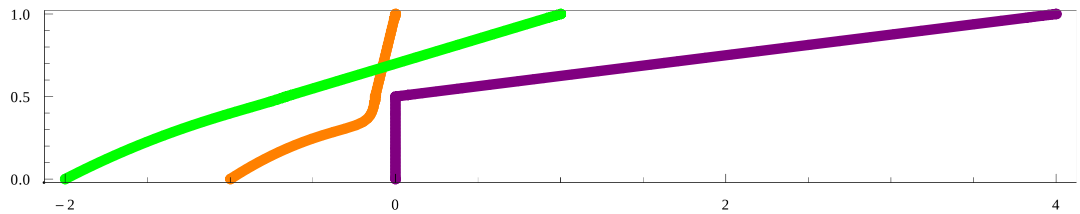
Theorem 4.2.3. *Let τ and υ be topological quadratics with $|P_\tau| = |P_\upsilon|$, and suppose that β is a braid representative for both τ and υ. Then $\tau \sim \upsilon$.*

Proof. This follows directly from Theorem 3.2.5. □

Example 4.2.4. Let f_R be the rabbit polynomial. Let $c = f_R(0)$, and let $d = f_R^2(0)$. Let $k\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a homeomorphism that sends $-c$ to 0, 0 to 0, and c^2 to 1. Let $q\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be the function $z \mapsto z - c$. Let h be a homeomorphism so that the following diagram commutes:

$$\begin{CD} (\widehat{\mathbb{C}}, d, 0, c) @>{id}>> (\widehat{\mathbb{C}}, d, 0, c) @>{h}>> (\widehat{\mathbb{C}}, -2, 0, -1) \\ @V{f_R}VV @VV{s}V @VV{s}V \\ (\widehat{\mathbb{C}}, 0, c, d) @>>{z-c}> (\widehat{\mathbb{C}}, -c, 0, c^2) @>>{k}> (\widehat{\mathbb{C}}, 4, 0, 1) \end{CD}$$

The following is a graph of a braid representative for the homeomorphism $k^{-1} \circ h$, and hence a braid representation for f_R.



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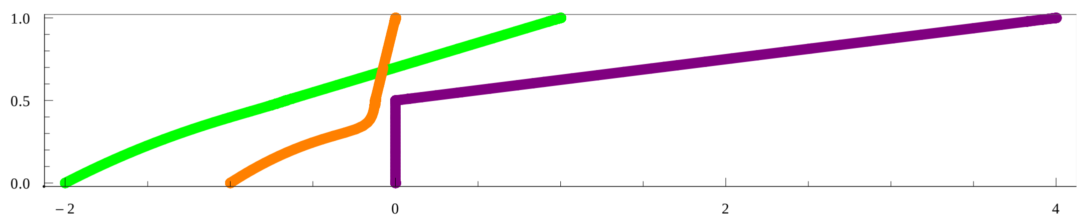
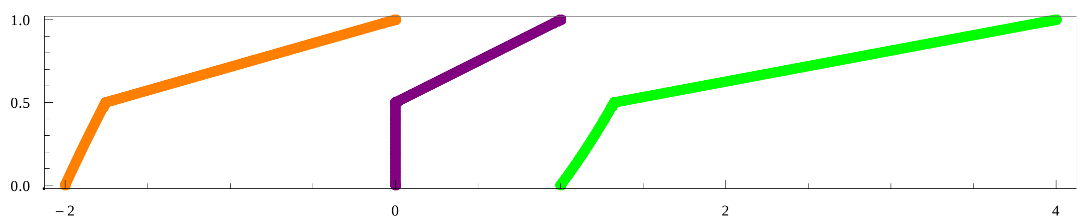
Example 4.2.5. We obtain braid representatives for f_{CR} and f_A in a similar fashion to Example 4.2.4:



A braid representative for the corabbit polynomial f_{CR}.



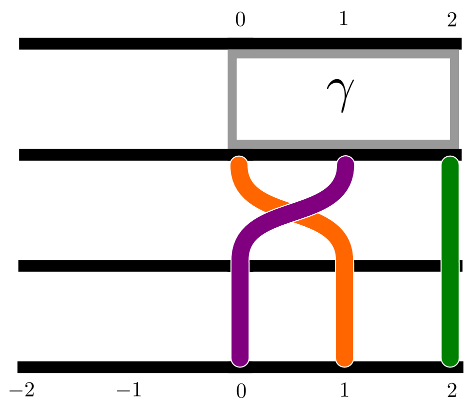
A braid representative for the airplane polynomial f_A.

◊

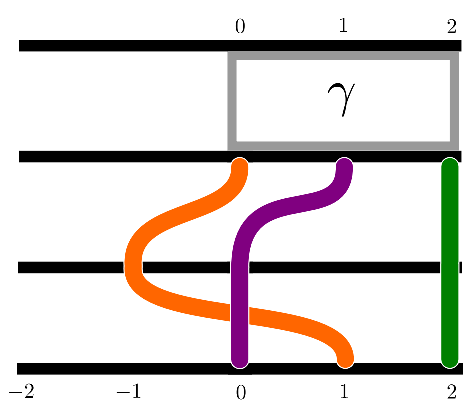
4.3 Lifting Relations

Remark 4.3.1. A braid in $\text{Braid}(N, \tilde{N})$ may be thought of as a braid $\beta \in \text{Braid}(N, N)$ followed by a braid $\gamma \in \text{Braid}(N, \pm N)$. Notice that γ can be described entirely by $\tilde{N}$. We will then describe braid representatives of topological quadratics as objects of the form $\beta\tilde{N}$. ◊

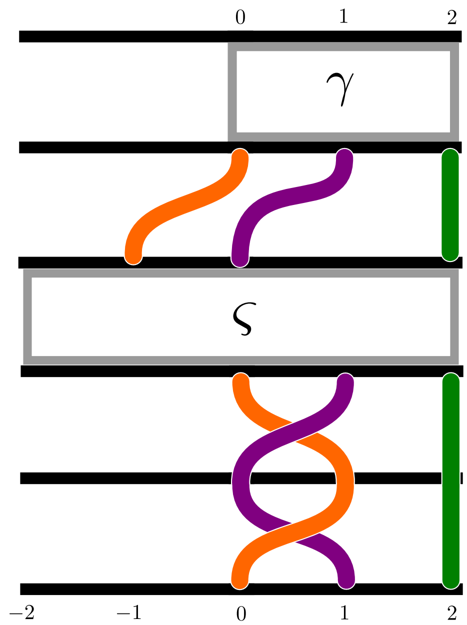
Example 4.3.2. Let $\gamma \in \mathrm{Braid}(\{0,1,2\},\{0,1,2\})$, and let $\beta \in \mathrm{Braid}(\{0,1,2\},\{0,1,2\})$ be a crossing of the first two strands, as shown below:



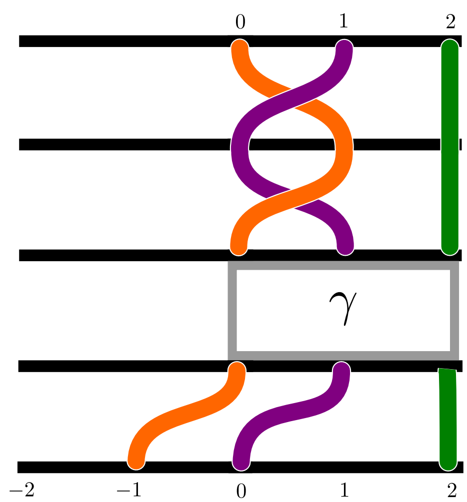
We move the first strand over:



Squaring the bottom-most braid yields a double crossing:



We move the bottom braid up to the top via Thurston equivalence:



We can write out the above process as follows:

$$\varsigma\,\{0,1,2\}\,a\,\gamma \sim$$

$$\varsigma\,a^2\,\{-1,0,2\}\,\gamma \sim$$

$$\varsigma\,\{-1,0,2\}\,\gamma\,a^2$$

◊

The algebraic version of Example 4.3.2 is seriously cumbersome. In order to do any sorts of calculations using that method, we need to streamline the notation. We make the following conventions:

1. We will use a more compact notation for the sets $\tilde{N}$: we write $\{-1, 0, 2\}$ as (102), we write $\{-2, -1, 0\}$ as (210), and so on.

2. Since we are always squaring via ς, we entirely drop it from the calculation.

3. We reverse the order of composition to better reflect the "top down" nature of the graphs.

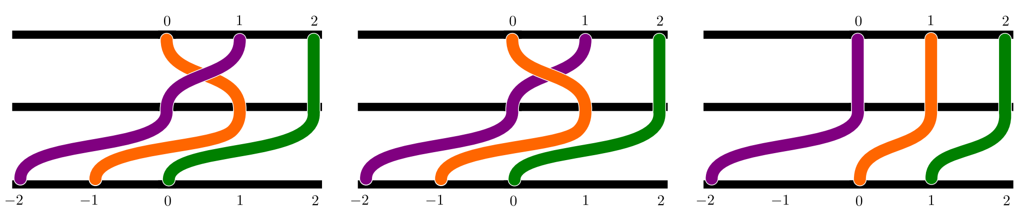
Example 4.3.3. The calculation in Example 4.3.2 becomes

$$\gamma a(012) \sim$$

$$a^2\gamma(102)$$

$\diamondsuit$

Example 4.3.4. The braids representatives for the rabbit, corabbit and airplane polynomials can be written as follows:



$f_R \sim\ a(210)$ $\quad f_{CR} \sim\ A(210)$ $\quad f_A \sim\ (201)$

$\diamondsuit$

The following lifting relations can be easily verified using the method detialed in Example 4.3.2

Theorem 4.3.5. *Let* $\omega \in \mathrm{Braid}(N, N)$*. Let* $\tilde{N}$ *be a lift of* N*, and let* $m_i, m_{i+1} \in N$*. Suppose* $|m_i - m_{i+1}| = 1$*.*

1. *If neither of* m_i, m_{i+1} *is* 0*, and if* $m_i < m_{i+1}$*, then* $X_{m_i}\omega\tilde{N} \sim \omega X_i \tilde{N}$*.*
2. *If neither of* m_i, m_{i+1} *is* 0*, and if* $m_{i+1} < m_i$*, then* $X_{m_{i+1}}\omega\tilde{N} \sim \omega X_i \tilde{N}$*.*
3. *If* $m_i = 0$ *or* $m_{i+1} = 0$*, then* $X_0^2\omega\tilde{N} \sim \omega X_i \tilde{N}_2$*, where* $\tilde{N}_2$ *is the same as* $\tilde{N}$ *with the* 0 *and* 1 *strands swapped.*

Theorem 4.3.6. *Let* $\omega \in \mathrm{Braid}(N, N)$*. Let* $\tilde{N}$ *be a lift of* N*, and let* $m_i, m_j \in \tilde{N}$*. Suppose* $|m_i - m_j| = 1$ *and* $m_i < 0 < m_j$*.*

1. *If* $|m_i| < |m_j|$*, then*

$$X_i^{\pm 1}\omega\tilde{N} \sim \omega\tilde{N}_2,$$

where $\tilde{N}_2$ *is* $\tilde{N}$ *with the* m_i *and* m_j *strands swapped.*

2. *If* $|m_j| < |m_i|$*, then*

$$X_j^{\pm 1}\omega\tilde{N} \sim \omega\tilde{N}_2,$$

where $\tilde{N}_2$ *is* $\tilde{N}$ *with the* m_i *and* m_j *strands swapped.*

Definition 4.3.7. Let $\omega \in \mathrm{Braid}(N, N)$. The **transpose** of ω, denoted ω^T, is the braid obtained by conjugating ω by Δ_n. △

Example 4.3.8. Let $\omega \in \mathrm{Braid}(\{0,1,2,3\}, \{0,1,2,3\})$ be the braid A^2BCA. Then

$$\omega^T = ABCABA\,A^2BCA\,abcaba = C^2BAC.$$

◊

Remark 4.3.9. The transpose of a braid essentially flips all of the crossings about the the midpoint of N. That is, if we are in B_4 then $A^T = C$ and $B^T = B$. If we are in B_3, however, $A^T = B$ and $B^T = A$.

Additionally, we will abuse notation by defining $\tilde{N}^T$ to be $\tilde{N}$ with the order of the strands reversed. ◊

Theorem 4.3.10. *Let* $\omega \in \mathrm{Braid}(N, N)$, *and let* $\tilde{N}$ *be a lift of* N. *Then*

$$\omega \Delta_n \tilde{N} \sim \omega \tilde{N}^T.$$

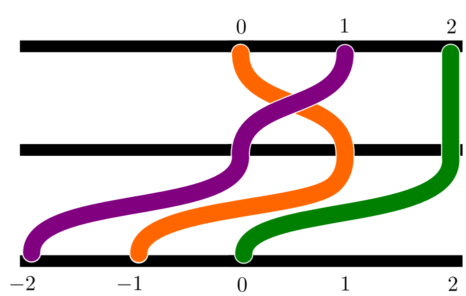
Proof. Observe that the square of a 180° twist is a 360° twist, which is equivalent to the identity. □

Corollary 4.3.11. *Let* $\omega \in \mathrm{Braid}(N, N)$ *and let* $\tilde{N}$ *be a lift of* N. *Then*

$$\omega \tilde{N} \sim \Delta_n \omega^T \tilde{N}^T$$

4.4 The Rabbit Revisited

We are now ready to begin twisting the rabbit. Recall that the rabbit polynomial f_R has the following braid representative:



We can write this down algebraically as $a(210)$. A Dehn twist around 0 and $f_R^2(0)$ then has braid representative A^2. We begin by showing that twisting f_R once yields the corabbit, as expected. We will use the relations defined in Theorems 4.3.5, 4.3.6, and 4.3.10, as well as all of the relevant braid relations from Chapter 3.

$$A^2a(210) \sim$$

$$A(210) \sim$$

$$f_{CR}$$

Things get slightly more interesting as we add another twist:

$$A^4a(210) \sim$$

$$A^3(210) \sim$$

$$AB(201) \sim$$

$$bBAB(201) \sim$$

$$b(102) \sim$$

$$(201) \sim$$

$$f_A$$

We will now prove that given any period-three topological quadratic τ and any orientation-preserving homeomorphism h, we can determine the Thurston equivalence class of $\tau \circ h$. We begin with a Lemma.

Lemma 4.4.1. *Let $\omega \in B_3$ be the braid B^m for some $m \in \mathbb{N}$ with $m > 1$. Let $N = \{0, 1, 2\}$, and let $\tilde{N}$ be a lift of N. Then $\omega\tilde{N} \sim \gamma\tilde{N}_2$ where $\tilde{N}_2$ is a lift of N and where γ has strictly fewer crossings than ω.*

Proof. If $\tilde{N} = (102)$ or (201), we are done, since $B\omega(102) \sim \omega(201)$ and vice-versa. Suppose $\tilde{N} = (012)$. Then we have

$$
\begin{aligned}
B^m(012) &\sim \\
A^2a^2B^m(012) &\sim \\
a^2B^mA(102) &\sim \\
B^mAb^2(102) &\sim \\
BAb^2(102) \text{ or } (201)
\end{aligned}
$$

Observe that

$$BAb^2(102) \sim a^2BA(102) \sim BAa(012) \sim B(012),$$

and

$$BAb^2(201) \sim Ab^2(102) \sim ab^2A(012) \sim Ba^2b(012) \sim a^2bB(012) \sim A(102).$$

This completes the proof. □

Theorem 4.4.2. *Let $\omega \in B_3$. Let $N = \{0, 1, 2\}$ and let $\tilde{N}$ be a lift of N. Suppose ω has m crossings, where $m > 1$. Then $\omega\tilde{N} \sim \gamma\tilde{N}_2$ where γ has strictly fewer crossings than ω.*

Proof. Lemma 4.4.1 shows that if ω is a power of B, then we are done. Suppose then that ω contains at least one of A or a. Since B and A^2 are always liftable, we can assume without loss of generality that $\omega = AB\gamma$ for some $\gamma \in B_3$. Observe that

$$
\begin{aligned}
&AB\gamma\tilde{N} \sim \\
&ABABA\gamma^T\tilde{N}^T \sim bBABABA\gamma^T\tilde{N}^T \qquad \sim b\gamma\tilde{N}^T,
\end{aligned}
$$

and so we are done. Now assume that $\omega = aB\gamma$ for some $\gamma \in B_3$. Then we have

$$\begin{aligned} aB\gamma\tilde{N} &\sim \\ ABAbA\gamma^T\tilde{N}^T &\sim \\ BABbA\gamma^T\tilde{N}^T &\sim \\ BA^2\gamma^T\tilde{N}^T&. \end{aligned}$$

In the worst case, $BA^2\gamma^T\tilde{N}^T$ lifts to either $\gamma^T AB(201)$ or $\gamma^T BA(102)$. Observe that

$$\begin{aligned} \gamma^T AB(201) &\sim \\ \gamma^T bBAB(201) &\sim \\ \gamma^T b(102)&, \end{aligned}$$

and similarly

$$\begin{aligned} \gamma^T BA(102) &\sim \\ \gamma^T aABA(102) &\sim \\ \gamma^T a(201)&, \end{aligned}$$

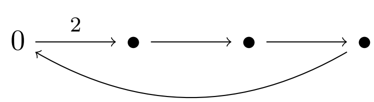
so we are done. □

Theorem 4.4.2 shows that any topological quadratic is Thurston equivalent to $\varsigma \circ \beta$ where β has at most one crossing. It is not hard to show that any such braid is equivalent to one of $a(210)$, $A(210)$, or (201), which solves the twisted rabbit problem.

5
The 3-eared Rabbit

5.1 Period-4 Quadratics and Their Braids

There are 6 quadratic polynomials with the following critical portrait:

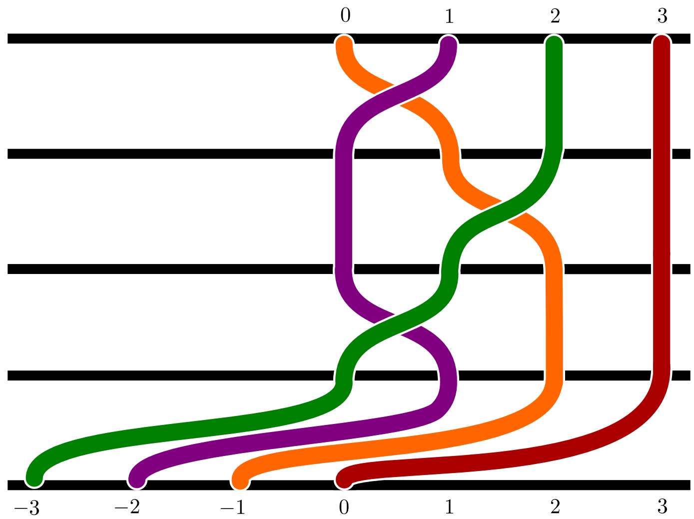


Solving the equation $((c^2+c)^2+c)^2+c=0$ for c, we obtain the following solutions:

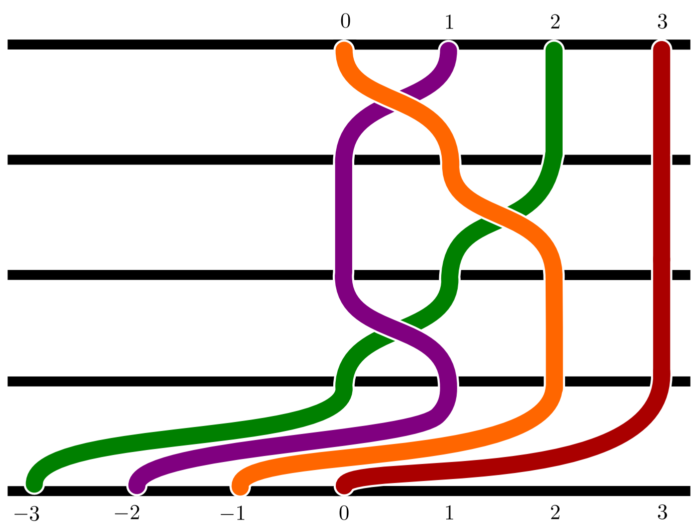
$$\begin{aligned}
c_1 &\approx -1.9408\\
c_2 &\approx -1.3107\\
c_3 &\approx -0.15652-1.03225i\\
c_4 &\approx -0.15652+1.03225i\\
c_5 &\approx 0.282271-0.530061i\\
c_6 &\approx 0.282271+0.530061i\\
c_7 &= -1\\
c_8 &= 0
\end{aligned}$$

We throw out c_7 and c_8, since they yield maps which are period 0 and period 2, respectively. The values c_1 through c_6, however, are all period 4 maps. We name the corresponding polynomials as follows:

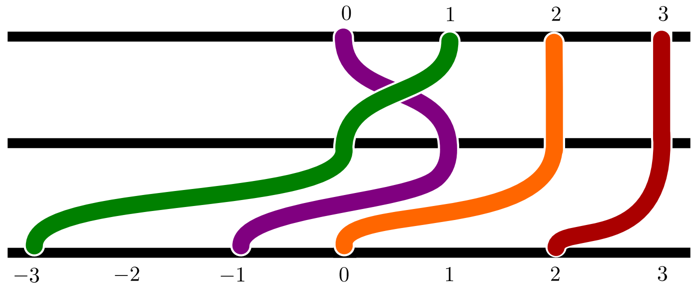
- We call f_{c_6} the **three-eared rabbit**. This has braid representative $aba(3210)$, shown below.



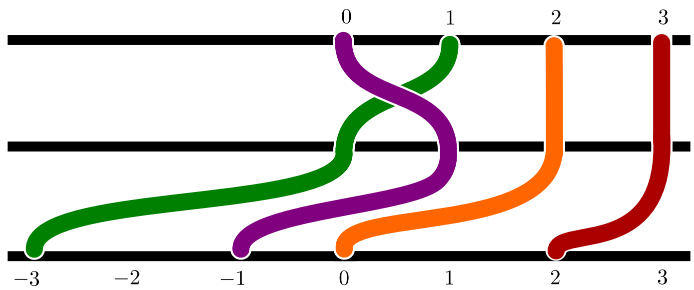
- We call f_{c_5} the **co-three-eared rabbit**. This has braid representative $ABA(3210)$, shown below.



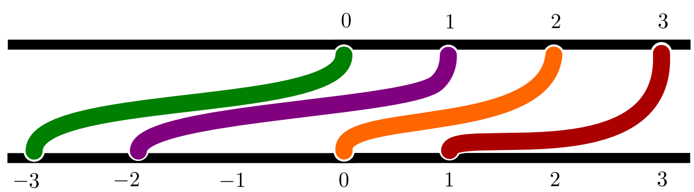
- We call f_{c_4} the **circledrite**. This has braid representative $a(3102)$, shown below.



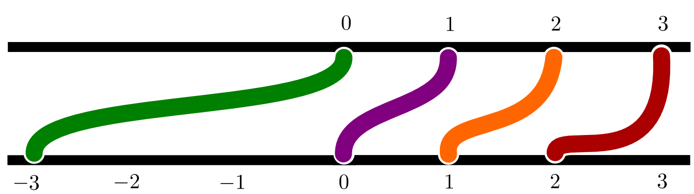
- We call f_{c_3} the **co-circledrite**. This has braid representative $A(3102)$, shown below.



- We call f_{c_2} the **2-basilica**. This has braid representative (3201), shown below.



- We call f_{c_1} the **long plane**. This has braid representative (3012), shown below.



The Julia sets for these functions are shown in Appendix A.

Example 5.1.1. Let f_{3R} be the three-eared rabbit polynomial. We will show that f_{3R} composed with a Dehn twist around 0 and $f^3(0)$ yields a topological quadratic which is Thurston equivalent to the 2-basilica.

$$
\begin{aligned}
A^2aba(3210) &\sim \\
Aba(3210) &\sim \\
baB(3210) &\sim \\
a(3210) &\sim \\
a^2A(3210) &\sim \\
Ac(3201) &\sim \\
cA(3201) &\sim \\
Aa(3201) &\sim \\
(3201)
\end{aligned}
$$

Hence $f_{3R} \circ A^2 \sim f_{2B}$, where f_{2B} is the 2-basilica polynomial. ◊

Example 5.1.2. Similarly, twisting f_{3R} twice yields the long plane:

$$
\begin{aligned}
A^4aba(3210) &\sim \\
A^3aba(3210) &\sim \\
baB^3(3210) &\sim \\
aB^2(3210) &\sim \\
AB^2c(3201) &\sim \\
baB^2c(3102) &\sim \\
A^2Bac(3102) &\sim \\
BacB(3012) &\sim
\end{aligned}
$$

$$
\begin{aligned}
acBC(3012) &\sim \\
aBC(2013) &\sim \\
ABCb(2103) &\sim \\
AcBC(2103) &\sim \\
ABC(3102) &\sim \\
bcb(2013) &\sim \\
cb(1023) &\sim \\
bc(1023) &\sim \\
c(2013) &\sim \\
(3012)
\end{aligned}
$$

◊

Based on a very preliminary inspection, it appears to be the case that the polynomial we get by twisting f_{3R} by $(A^2)^n$ depends on the base-4 expansion of n, as is the case for the rabbit polynomial.

Appendix A

Julia Sets for the Period 4 Maps

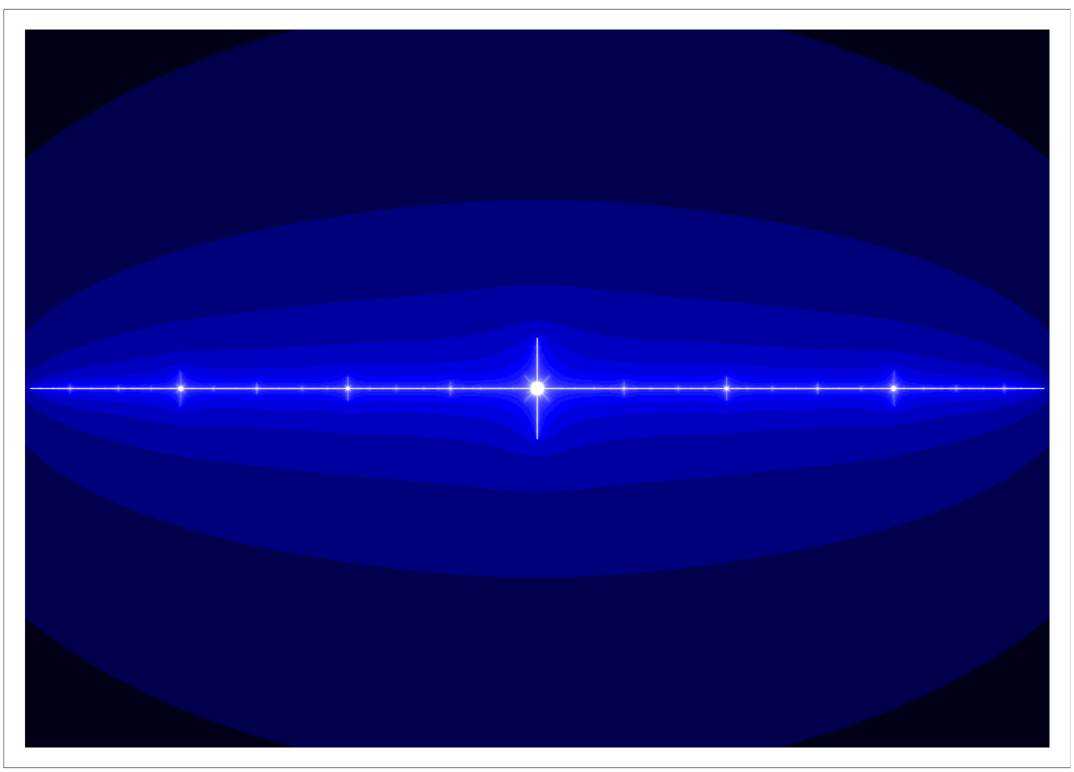


Figure A.0.1: The Julia set for $f_{c_1} \approx z^2 - 1.9408$

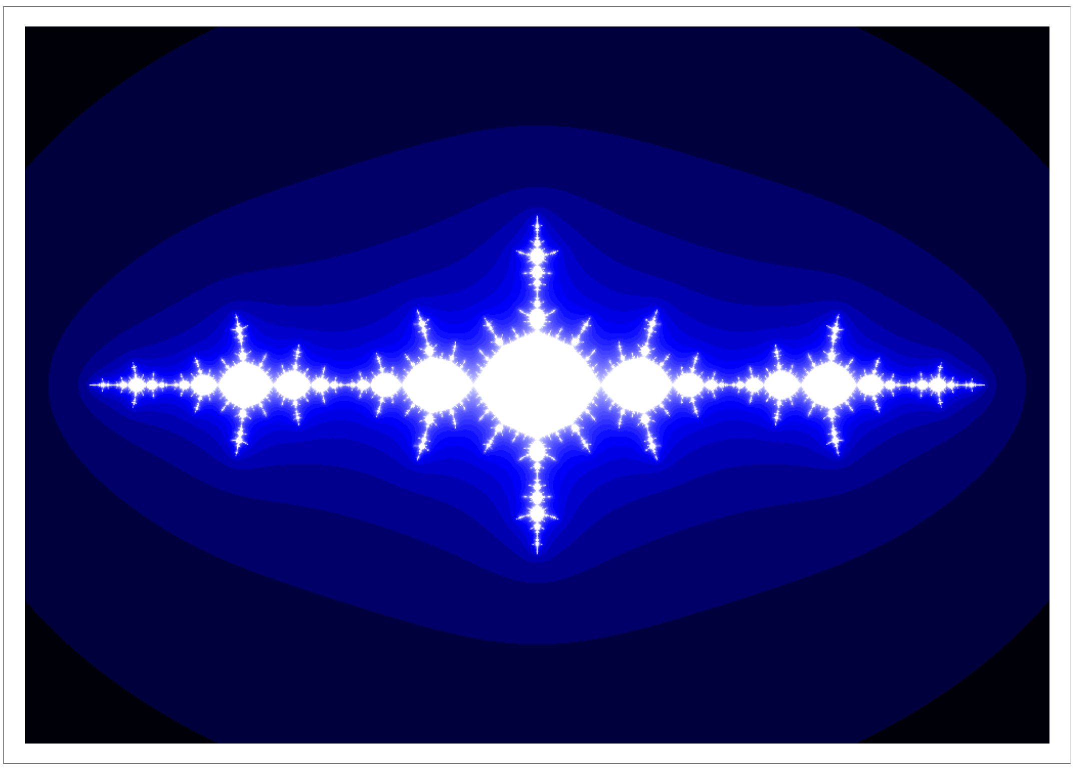


Figure A.0.2: The Julia set for $f_{c_2} \approx z^2 - 1.3107$

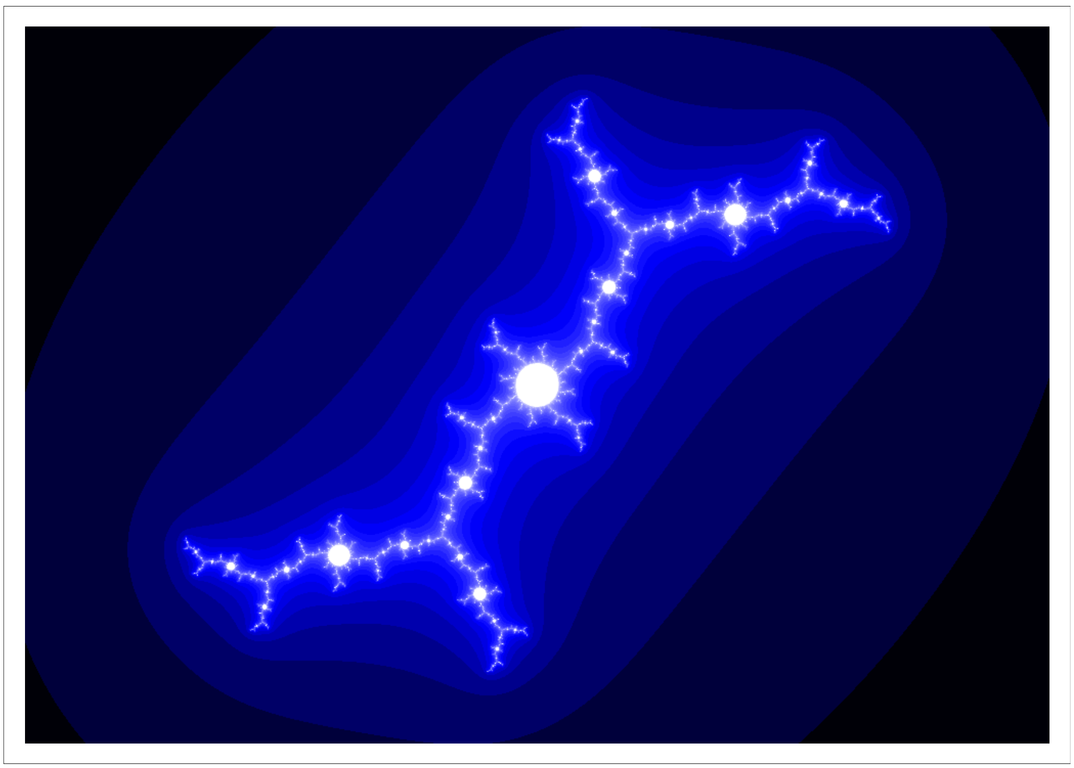


Figure A.0.3: The Julia set for $f_{c_3} \approx z^2 - 0.15652 - 1.03225i$

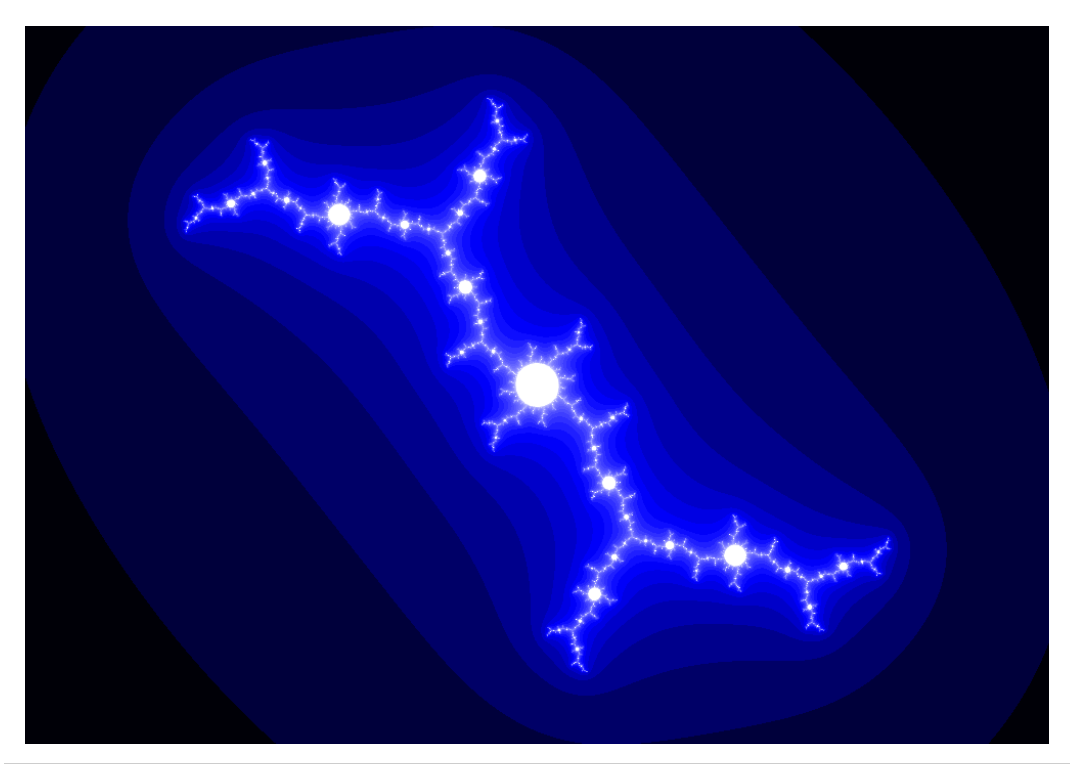


Figure A.0.4: The Julia set for $f_{c_4} \approx z^2 - 0.15652 + 1.03225i$

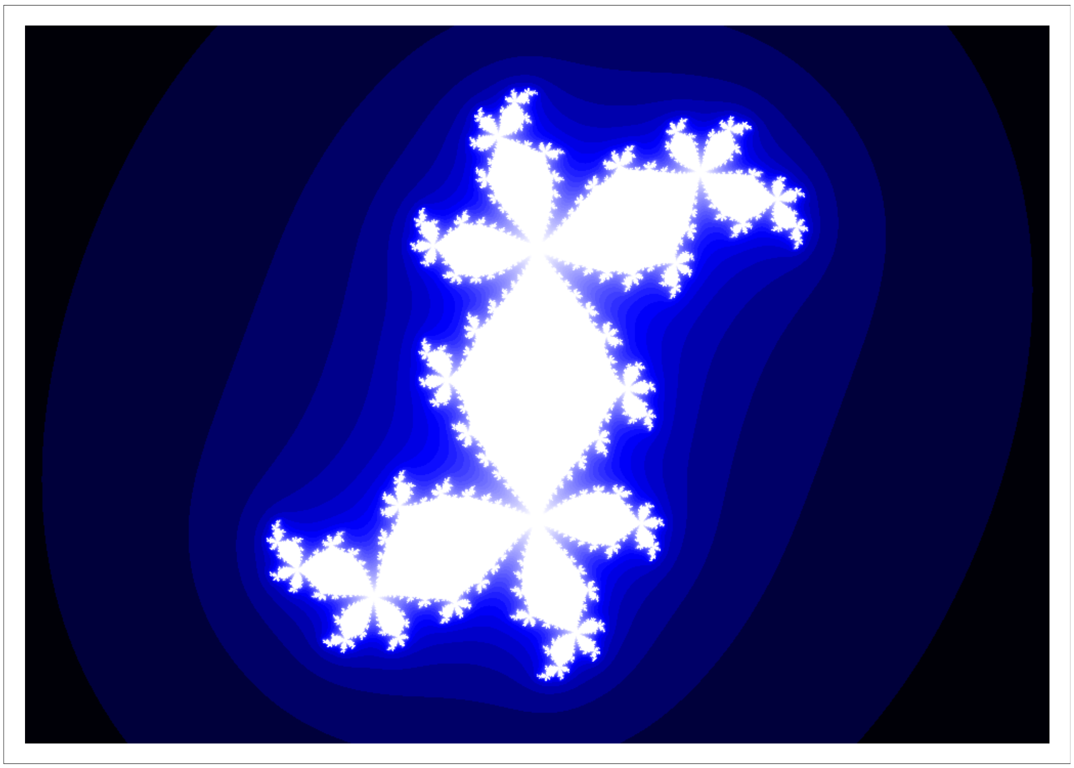


Figure A.0.5: The Julia set for $f_{c_5} \approx z^2 + 0.282271 - 0.530061i$

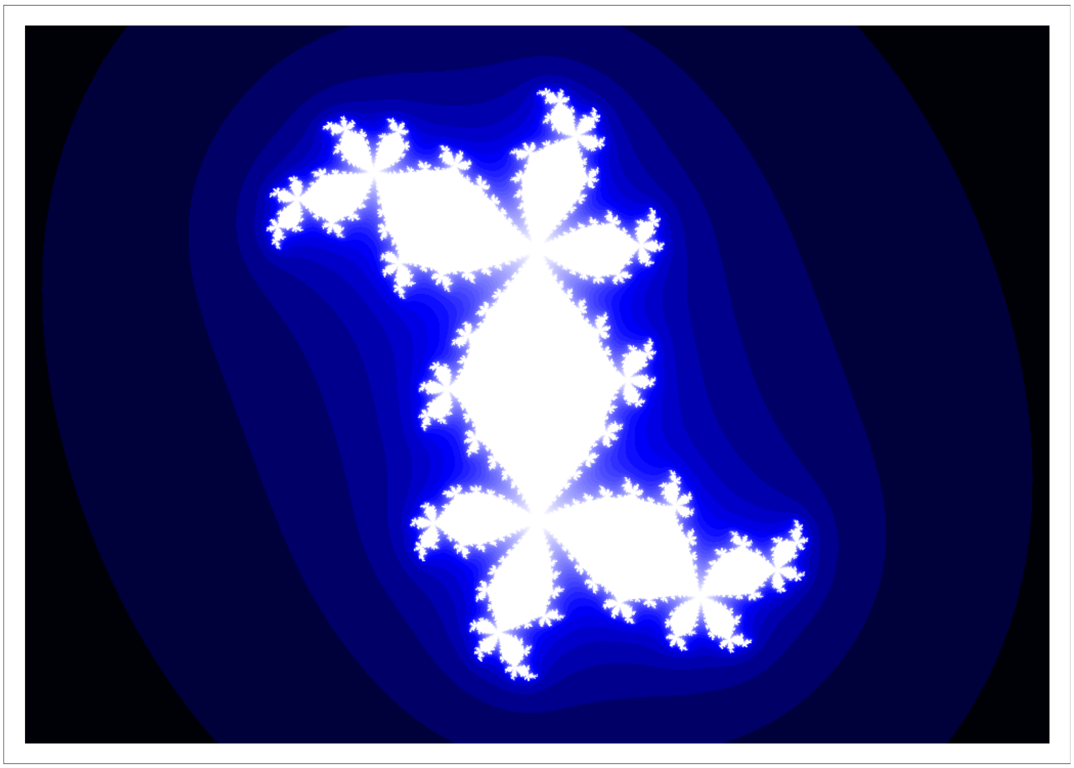


Figure A.0.6: The Julia set for $f_{c_6} \approx z^2 + 0.282271 + 0.530061i$

Bibliography

[1] James R. Munkres, *Topology*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.

[2] John Meier, *Groups, Graphs and Trees*, 1st ed., Cambridge University Press, New York, NY, 2008.

[3] M. Rees, *Components of Degree Two Hyperbolic Rational Maps*, Inventiones Mathematicae **100** (1990), 357-382.

[4] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, Boulder, CO, 2003.

[5] Franz Baader and Tobias Nipkow, *Term Rewriting and All That*, 2nd ed., Cambridge University Press, Cambridge, UK, 1998.

[6] Allen Hatcher, *Algebraic Topology*, 1st ed., Cambridge University Press, New York, NY, 2001.

[7] John Stillwell, *Classical Topology and Combinatorial Group Theory*, 2nd ed., Springer, New York, NY, 1993.

[8] Christian Kassel and Vladimir Turaev, *Braid Groups*, 1st ed., Springer, New York, NY, 2008.

[9] Laurent Bartholdi and Volodomyr Nekrashevych, *Thurston equivalence of topological polynomials*, Acta Mathematica **197** (2006), 1-51. 10.1007/s11511-006-0007-3.

[10] Dan Margalit and Jon McCammond, *Geometric Presentations for the Pure Braid Group*, Journal of Knot Theory and Its Ramifications **18** (2009), 1-20.