Switching the Shannon Switching Game

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
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Abstract

The Shannon switching game is a combinatorial game for two players, which we refer to as the cop and the robber. In this project, we explore a few variations of the original rules that make the game more interesting. One of these variations is a game involving multiple cops and one robber. We present a formal recursive definition of this game which we use to prove several basic theoretical results. Next, we consider this game on complete graphs and complete bipartite graphs. On each family of graphs, we investigate the winning conditions for the players depending on who goes first. We describe these conditions as functions and prove several asymptotic results. Finally, after looking at the variation with multiple cops and one robber, we also study the game with multiple cops and multiple robbers and compare the two variations.

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Dedication

I would like to dedicate this project to Bard College as without this institution I would not have been given the opportunity to write a senior project.

Acknowledgments

I would like to thank my senior project adviser Jim Belk for his guidance throughout my project. He has been tremendously helpful and I am very grateful. Jim has been the best adviser a student could ask for and has made this year-long journey educational and worth while. I would also like to acknowledge Maria Belk and Sam Hsiao for agreeing to be on my senior project board. I would like to thank Sam for his suggestions in our midway board meeting to include basic definitions and propositions about the game. This truly contributed to my understanding of the project and served to be very important. All three professors have taught me in various classes and I would like to thank them for their dedication and contribution to my academic life. I have the utmost respect for them.

1
What is the Shannon Switching Game?

1.1 Introduction

Combinatorial game theory is an important branch of mathematics which combines the study of combinatorics, game theory and graph theory. A combinatorial game is a game with two players each having separate turns and making different moves to achieve a winning positive. The players have alternate turns with perfect information such that the moves of each player are known by both players. Mathematicians study the pure strategies for each player involved in a combinatorial game. They also study how these strategies affect the outcome of the game. They aim to explore situations such as: given the two players play perfectly, who wins the game and how does the situation define the result. Combinatorial games are so interesting as they provide the opportunity to explore several cases and questions within the respective game being studied. One of the most notable books on combinatorial game theory is *Winning Ways for your Mathematical Plays* [1]. The book includes mathematical strategies and proofs for several combinatorial games therefore serving as a resource for scholars to explore these games in greater depth.

The Shannon switching game is a combinatorial game invented by Claude Shannon and its solution found in 1964 by Alfred Lehman [3]. This game has been previously studied by many mathematicians for its intrinsic interest and relationship to graph theory and matroid theory. As the game is played on a graph, we can study the positions of the two players while investigating the game on different graphs. The game has also been generalized to a game on matroids and this was initially done by Alfred Lehman in his breakthrough with the game [5]. Matroid theory and graph theory are also connected as the winning condition for one of the players involves the presence of disjoint spanning trees within a graph which Lehman also proved using matroids in his solution [5]. The Shannon switching game is also closely related to the game of Hex such that instead of using edges, vertices are used in each move instead [2]. In 1976 [2], it was proven that this version of the Shannon switching game related to Hex is PSPACE-complete and the same was concluded in 1981 [2] for Hex as well. Therefore there is a vast pool of knowledge and mastery from mathematicians of this game.

The game involves two players that we will call the cop and the robber and is played on a graph. The Shannon switching game has been classified in a set of games called Maker-Breaker games [4]; which are games where the goal of one player, the robber, is to create a specific subgraph and the goal of the other player, the cop, is to prevent this from happening. The robber wants to mark a path while the cop wants to destroy it. In this project, we will be investigating variations of the original rules which make the game more exploratory. In Section 1.2 we will provide a description of the game and its rules as well as the types of games that exist. We will also show examples for the types of games and among these is a winning strategy for the robber mentioned earlier involving disjoint spanning trees [6].

Though the game is traditionally played with only two players, we will study a game with multiple cops and one robber. We will refer to this game as the n-cop game. Section 2.1

of Chapter 2 begins with an intuitive definition of this variation. We will provide a formal definition using recursion of the n-cop game which we will use to prove several fundamental properties. The game has been formally described with the method that the robber claims edges by marking them on his turn and the cop deletes edges on his turn. We will define this game as the Deletion-Marking Game. However, we will also define a method involving the robber contracting edges and the cop deleting edges. We will refer to this game as the Deletion-Contraction game. We will include in this chapter an illustration showing that these two methods result in the same game.

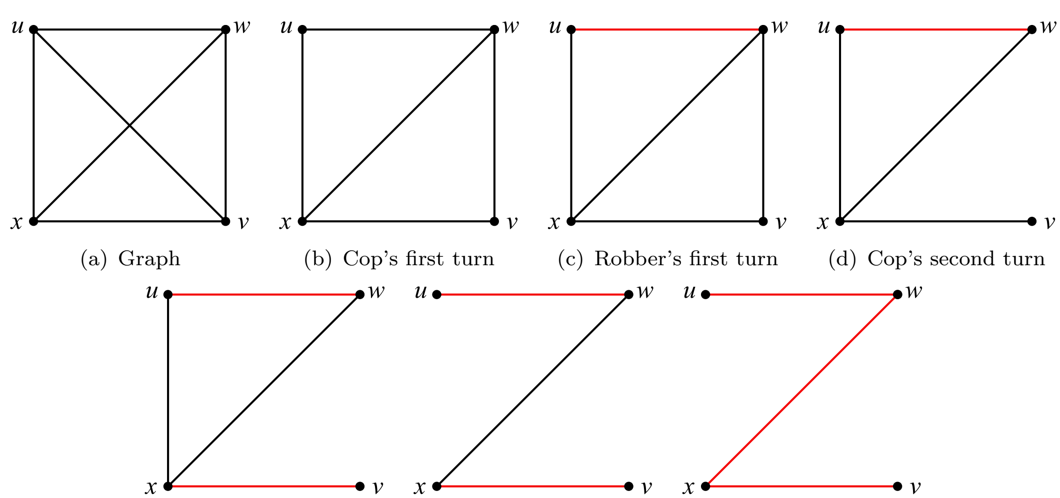
As the game can be played on any graph, we will restrict our attention to specific families of graphs. As the n-cop game is a new variation to the Shannon switching game, before it is possible to prove any generalization, we must first focus on one class of graphs. This is because the task of discovering winning strategies for the n-cop game on an arbitrary graph would be very difficult if no test studies were done. Therefore in Chapter 3 we will explore the n-cop game on *complete graphs* in Section 3.1. Our aim will be to research the winning conditions for the players and the relationship between different starting positions. We will describe these conditions using functions and we will prove different results. After exploring the game on complete graphs, we will further investigate the n-cop game played on *complete bipartite graphs* in Section 3.2. Our goal here is similar as in the case of complete graphs. However, our results will differ as complete bipartite graphs are also unique. At the end of each section in Chapter 3, we will conclude it with observations and questions related to the results obtained and conjectures that we believe to be true.

Chapter 4 investigates a game played on complete bipartite graphs where there are multiple cops and multiple robbers of the same number. We will define this game as the n-cop, n-robber game and aim to find results for this as well. We will also compare the results found in this chapter to those found when playing the n-cop game. How advantageous is it to have more robbers?

1.2 Rules of the Game

The Shannon switching game has two players, the cop and the robber, and is played on a graph. Given an undirected graph $G = (V, E)$ with two distinguished vertices u and v, the objective of the robber is to mark a path from vertex u to vertex v. The objective of the cop is to prevent the robber from escaping by deleting edges to disconnect the robber's paths, prohibiting the robber from getting from u to v.

The two players alternate turns and either player may begin the game. The cop can delete any edge as long as the robber has not previously claimed it. On the first turn, suppose that the cop plays first. He may delete any edge of his choosing. The robber plays next, marking his edge with red. The game ends once the robber has marked a complete path from u to v and therefore escapes or the cop successfully blocks all paths between u and v.



(a) Graph (b) Cop's first turn (c) Robber's first turn (d) Cop's second turn

(e) Robber's second turn (f) Cop's third turn (g) Robber's third turn

Figure 1.2.1. Example of a Game

Figure 1.2.1 is an example of a game where the cop has played first. It shows the sequence of turns. Let us now go through these plays from the figure to further explain how the game works.

(a) The original graph we are playing on.

(b) The cop has the first turn and chooses to delete the edge between vertices u and v. This move is necessary for the cop to make in order to prevent the robber from winning.

(c) The robber now plays second and chooses to mark the edge between vertices u and w.

(d) The cop now continues on his turn and chooses to delete the edge between vertices w and v. The cop is forced to delete this edge as otherwise the robber can win on his next turn.

(e) The robber now proceeds and chooses to mark the edge between vertices x and v.

(f) The cop is again forced to delete the edge between vertices u and x to prevent the robber from winning.

(g) The robber now makes the final turn of the game and marks the edge between vertices w and x. Now that the robber has successfully marked a path from vertex u to vertex v, he has won the game.

1.3 Types of Games

On certain graphs, the robber will always be able to make an escape regardless of who plays first. Shown in Figure 1.3.1 is a graph such as this where the robber will always win even when the cop has the first turn. This is called a *positive game* [6].

Figure 1.3.1 is an example of a positive game. There are two dashed edges marked in red shown in the graph. These two edges are between u and v so therefore there are two direct paths between u and v. If the cop goes first, he may only delete one of these paths therefore leaving the second path for the robber to mark on his turn. This is an example

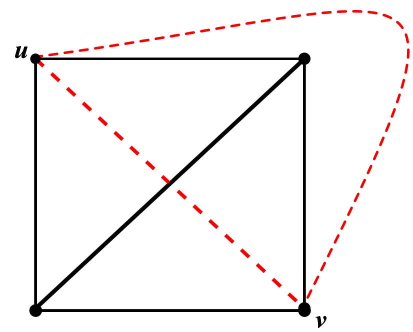


Figure 1.3.1. Example of a graph that is a Positive Game

of a positive game as it shows that the robber can win even when the cop has the first turn.

Definition 1.3.1. A spanning tree in a graph G is a a selection of edges in G where there are no cycles and the edges are all connected. We say two spanning trees are disjoint when they share no common edges. △

Theorem 1.3.2. *Disjoint Spanning Tree Theorem*

A game is positive if and only if there is a subgraph containing vertices u and v that has two disjoint spanning trees [3].

The Disjoint Spanning Tree Theorem is basically a strategy for the robber to use to guarantee his escape when playing on a positive game. Let us now prove this theorem and therefore show the robber's strategy. We will show this by using the following figure as our graph where the blue and orange edges each represent the two disjoint spanning trees.

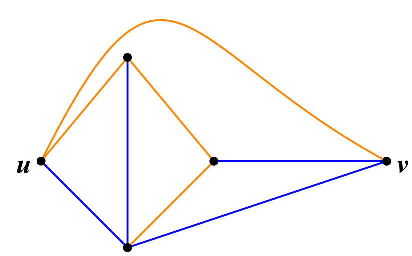
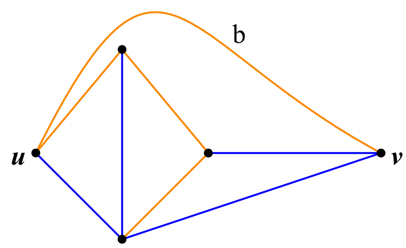


Figure 1.3.2. Positive Game

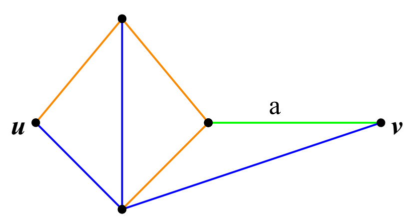
Example 1.3.3. The following illustrates the Spanning Tree Theorem

We will show using these two spanning trees and the following strategy that this is an example of a positive game and that the strategy works for positive games. If the robber can win playing second, he can also win playing first, (later proven in Proposition 2.3.4) so it is only necessary to show that the robber can win when the cop goes first. Therefore, the cop will always make the first move in this strategy.

Cop's First Turn: From Figure 1.3.2, the cop should obviously delete the direct path between u and v, edge b.

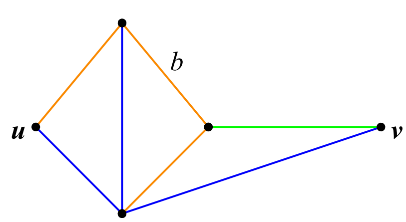


Robber's First Turn: The robber may now choose any edge a, except the edge that the cop has previously deleted:

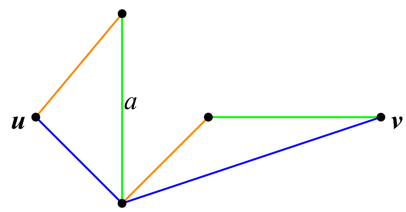


Now the two spanning trees have one edge in common, a. We have made this edge green to make it clear that this edge is now part of both spanning trees.

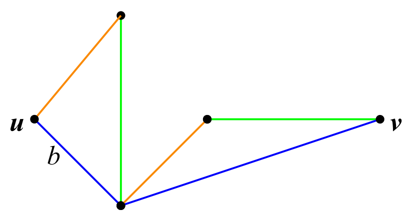
Cop's Second Turn: Now the cop may choose any edge b available to delete:

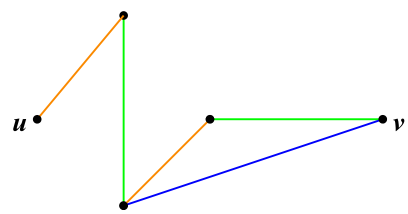


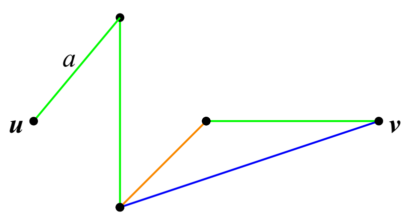
Robber's Second Turn: Now the robber must choose an edge a. The edge he chooses must complete both spanning trees, which are now altered by the deletion of the edge b, so that a is now part of both spanning trees. We have again changed its color to green to show this:

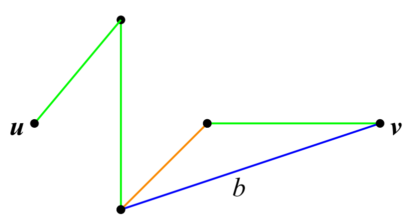


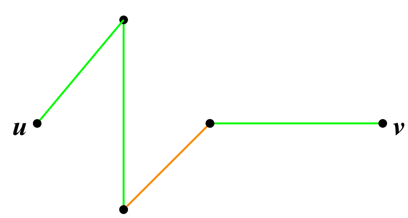
This strategy continues through the end of the game, guaranteeing that there is always a path between u and v. Below are the rest of the steps, shown through the graphs:

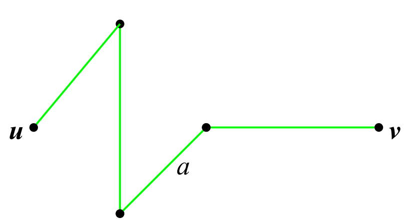












Thus we have shown the algorithm that guarantees the robber a successful escape in a positive game. ◊

On other graphs, the cop will always be able to prevent the robber from escaping regardless of who plays first, shown in Figure 1.3.3. A graph such as this where the cop wins regardless of who plays first is called a *negative game* [6].

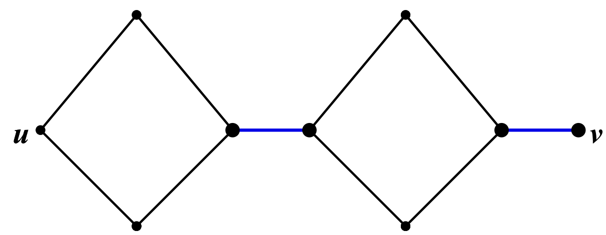


Figure 1.3.3. Example of a graph that is a Negative Game

Figure 1.3.3 is an example of a negative game. Notice that there are two edges in the graph highlighted in blue. If the cop deletes any of these edges the graph is disconnected and therefore the robber cannot escape. The cop's objective is to disconnect the graph, which would destroy all paths between u and v. Therefore if the robber goes first, the cop will disconnect the graph on his second turn. If the cop plays first, he will immediately disconnect the graph thus making this a negative game.

On other graphs, the winner of the game is determined by who plays first. A graph such as this where either the cop or the robber wins contingently is called a *neutral game* [6].

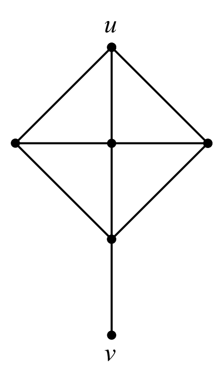


Figure 1.3.4. Example of a graph that is a Neutral Game

Figure 1.3.4 illustrates an example for a neutral game where the robber is trying to mark a path between u and v. Let us now show why this is a neutral game by playing it with the cop starting and then playing it with the robber starting.

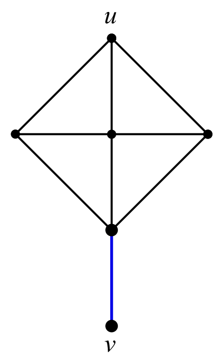


Figure 1.3.5. Neutral Game where cop plays first

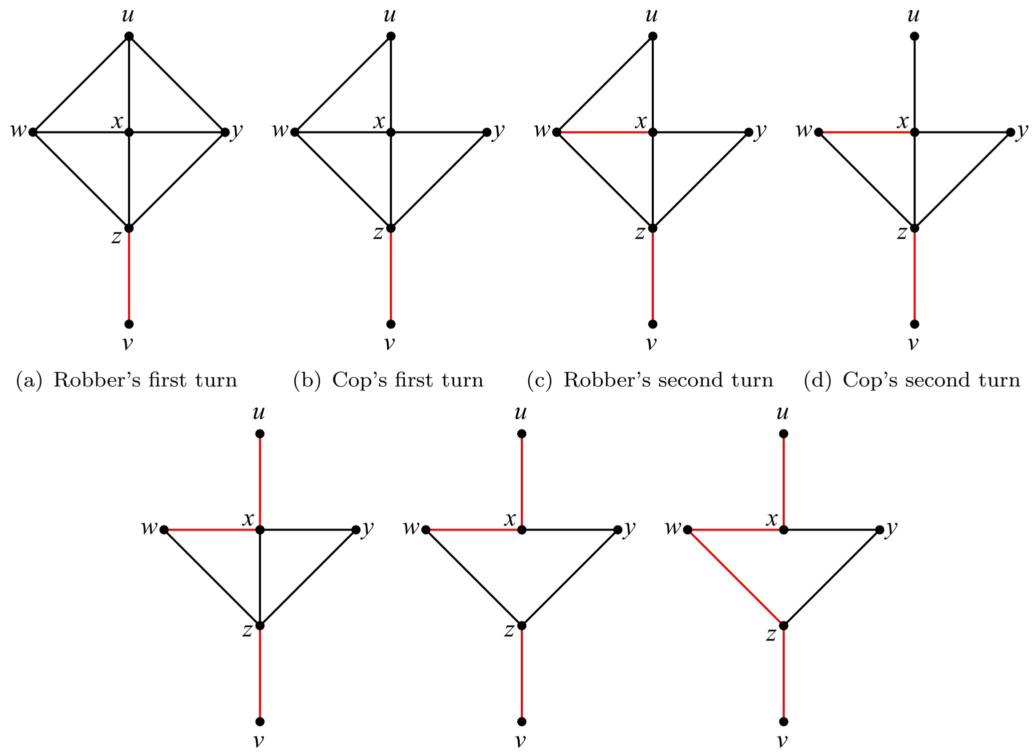
Figure 1.3.5 shows the graph with a highlighted blue edge. If the cop plays first, he should delete this edge and therefore disconnect the graph. Thus the cop wins playing first.

Figure 1.3.6 now shows the sequence of turns made by each player with the robber starting. The edges have been labeled $u, \ldots, z$ and the robber is trying to get from u to v. Here we are asserting that this is a neutral game. Other cases to consider for the cop's moves in each turn are not explicitly shown. Let us now go through these turns from the figure to further explain.

(a) The robber on his first turn marks the edge between vertices v and z. This is a necessary move as otherwise the cop can disconnect the graph on his following turn.

(b) The cop then chooses to delete the edge between vertices u and y. There are other moves to consider however they do not increase his chances of winning.

(c) The robber now plays second and chooses to mark the edge between vertices w and x.



(a) Robber's first turn (b) Cop's first turn (c) Robber's second turn (d) Cop's second turn

(e) Robber's third turn (f) Cop's third turn (g) Robber's final turn

Figure 1.3.6. Neutral game where robber plays first

(d) The cop now continues on his turn and chooses to delete the edge between vertices w and u.

(e) The robber now proceeds and chooses to mark the edge between vertices x and u.

(f) The cop is now forced to delete two edges but only has one move so he deletes edge between vertices x and z.

(g) The robber now makes the final turn of the game and marks the edge between vertices w and z.

Now that the robber has successfully marked a path from vertex u to vertex v, the robber has won the game.

Formal definitions of the types of game follow in the next chapter.

2

N-Cop Game

2.1 Introduction

The Shannon switching game is traditionally played with only two distinguished players: one robber and one cop. In this chapter, we introduce a variation of the game involving multiples cops and one robber. We refer to the multiple cops as n cops. Having n cops intuitively means that one of the players has n moves per turn. Therefore when the n cops play, it is equivalent to one cop having n moves and therefore deleting n edges on each turn. Therefore in the n-cop game, the two players alternate turns, however on the cop's turn, he has n cops and therefore n moves. The n cops in this sense therefore move collectively together to delete a total of n edges. The robber proceeds as before having only one move per turn. Therefore the variation is consistent with the framework of combinatorial games of having two players.

With this variation, a *positive game* is now a game where the robber still wins regardless of whether the n cops play first. In a *negative game*, the n cops are able to win regardless of if the robber starts. *Neutral games* remain the same such that the winner is determined by whether the n cops start or the robber starts. Let us show an example of a game with two cops and one robber. We will play the game on the complete graph K_5 with the 2 cops playing first. For the purposes of explaining the game, the vertices have been labelled u through y where the robber is trying to get from vertex u to vertex v.

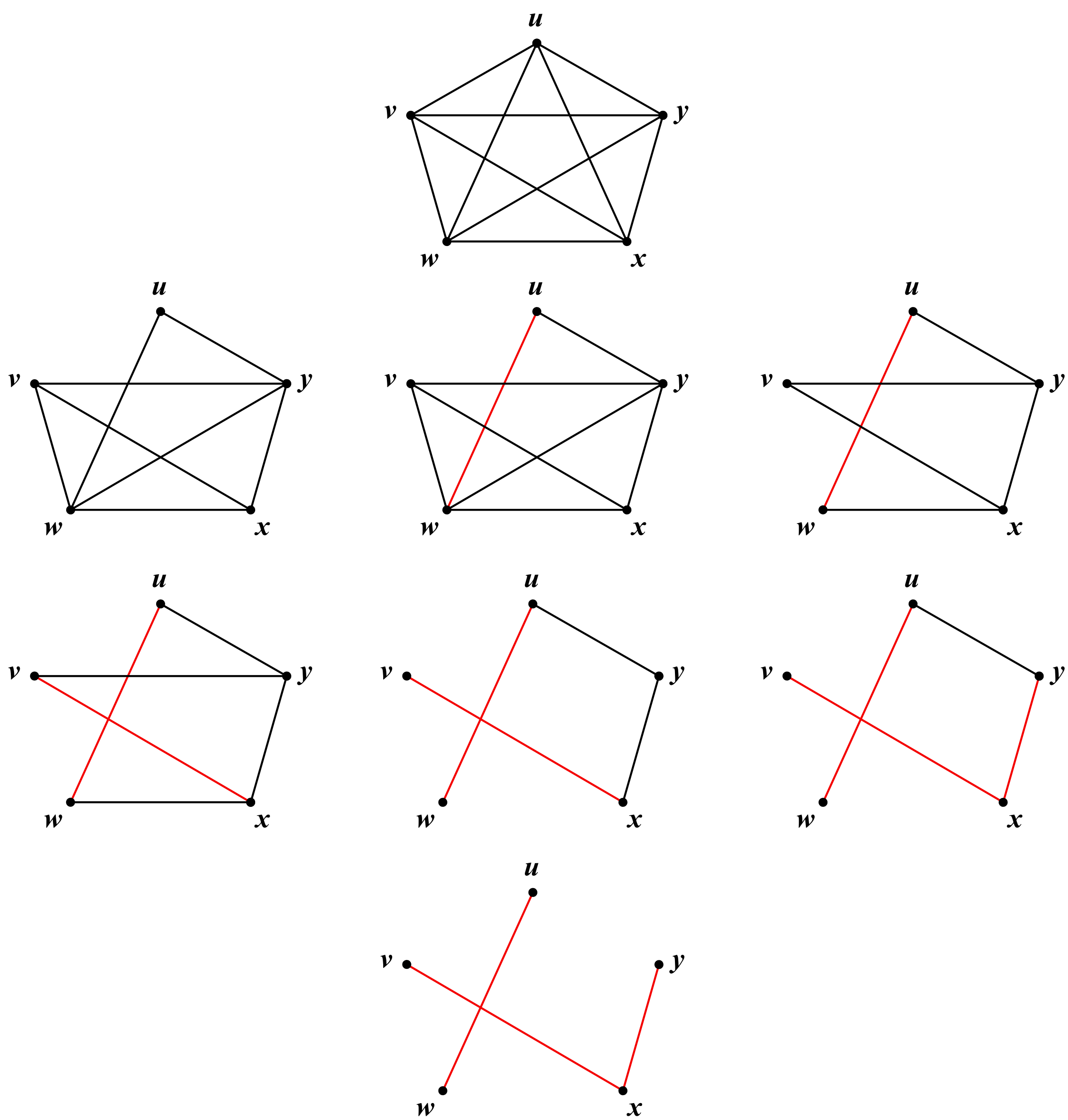


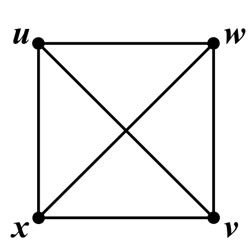
Figure 2.1.1. 2-Cops

Figure 2.1.1 shows the game played on K_5. On the cops' first move they delete the edge between vertices u and v as otherwise the robber wins. They also delete the edge between vertices u and y. Next, the robber marks with red the edge between vertices u and w. The turns by the players continue until the final move shown in the last graph where the cops have deleted the edge between vertices u and x. Here the cops did not even have to use their second move in order to disconnect the graph and therefore win the game.

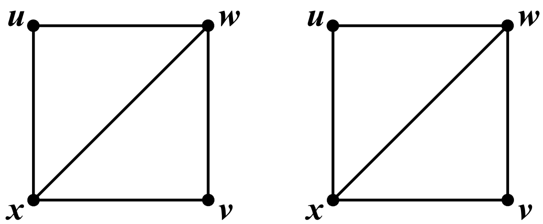
2.2 Deletion-Marking vs. Deletion-Contraction Game

The Shannon switching game is normally defined with the robber's moves as marking edges and the cop's moves as deleting edges. When playing the game however, deleting and marking edges for each player is equivalent to deleting and contracting edges also. Therefore, when referring to the play of each player, either game can be used. The deletion-marking game is a game where the robber's moves are marking edges and the cop's moves are deleting edges. The deletion-contraction game is a game where the robber's moves involve contracting edges and the cop's moves involve deleting edges. We will show in this section why the two ways are the same.

Let us look at the complete graph K_4. We will use this graph to illustrate that the two games result in the same outcome. The first figure shows the complete graph K_4. The following figures show graph 1 and graph 2 where the deletion-marking game is played on graph 1 (g1) while simultaneously the deletion-contraction game is played on graph 2 (g2).

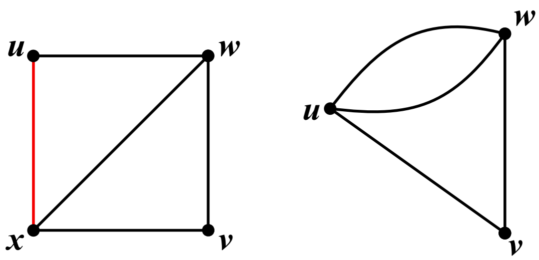


Complete Graph K_4: We will show the two ways by playing on the complete graph K_4



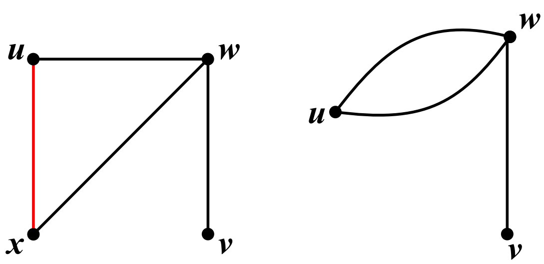
g1 g2

Cop's first move: On g1 and g2 the cop has chosen to delete the edge between vertices u and v as otherwise the robber will win.



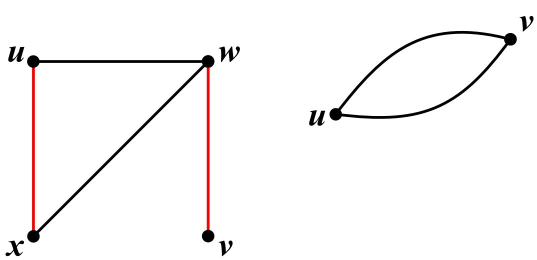
g1 g2

Robber's first move: On g1, the robber has chosen to mark with red the edges between vertices u and x. On g2, the robber has chosen to contract the same edge between vertices u and x resulting in the contracted graph.



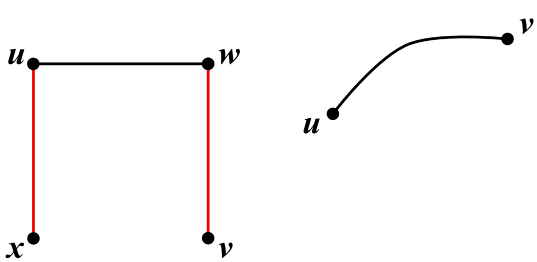
g1 g2

Cop's second move: On g1, the cop is forced to delete the edge between vertices x and v otherwise the robber will win. On g2, the cop deletes the edge between vertices u and v as otherwise the robber will win too.



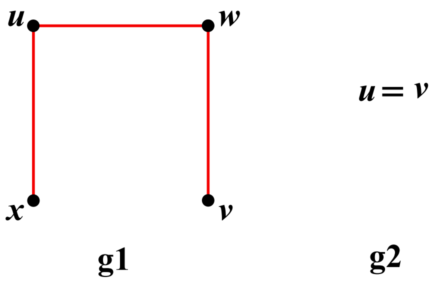
g1 g2

Robber's second move: On g1 the robber has chosen to mark with red the edge between vertices w and v. On g2 the robber has chosen to contact the same edge resulting in the contracted graph.



g1 g2

Cop's third move: On g1 in order to prevent the robber from creating a path, the cop needs to delete two edges. The edge between vertices x and w and the edge between vertices u and w however he only has one move and so chooses to delete the edge between vertices x and w. The same goes for g2 where two edges between vertices u and v need to be deleted however the cop can only delete one.



Robber's third move: On g2 the robber has chosen to mark with red the last remaining edge between vertices u and w creating a direct path between vertices u and v therefore winning the game. On g2 the robber chose to contract the last remaining edge resulting in $u = v$. Therefore the robber has also won go g2.

2.3 Formal Definitions and Propositions

In this section we provide a formal recursive definition of the n-cop game and prove several basic theoretical propositions. In order for us to prove our results obtained in chapters to come, we must first have a structured definition to reference. We use a recursive definition as we are defining the game for a fixed number of cop size n. Therefore it is useful as it can be applied repeatedly to all terms of this sequence of n for all $n \in \mathbb{N}$.

Earlier in our introduction we intuitively discussed how the game works. Here we will define the game using ordered triples. Formal definitions are very useful here because in order for us to prove these propositions, we first need to have definitions that can provide clarity and structure. This formality will also help us to understand other theorems in Chapter 3. The types of games have been originally described as positive, negative and neutral games. In this section however, we will formally define positive and non-negative games. Intuitively the types of games are as follows:

- A non-negative game is a game where the robber can win when the robber starts.
- A positive game is a game where the robber can win when the cops start.
- A negative game is a game where the cop can win when the robber starts.
- A non-positive game is a game where the cop can win when the cop starts.

- A neutral game is a game that is both non-negative and non-positive.

NOTE: In the following definitions and proofs, we will consider a fixed number of n cops and one robber for all games mentioned. Therefore it can be assumed that we are referring to n cops and one robber.

Definition 2.3.1. A *game* is an ordered triple (G, u, v) where G is a graph with edge set $E(G)$ and vertex set $V(G)$ such that $u, v \in V(G)$. By convention the game (G, u, v) is equal to the game (G, v, u). △

In our definition that follows, the notation $(G - \{e_1, e_2...e_n\})$ means that a distinct set of n edges are deleted from G. The notation G/e means that the edge e is contracted in G.

Definition 2.3.2. Let (G, u, v) be a game. If $u = v$ then the game is always positive and non-negative.

If $u \neq v$ then we define positive and non-negative games recursively as follows:

- (G, u, v) is a *positive game* if G has at least n edges and $(G - \{e_1, e_2...e_n\}, u, v)$ is a non-negative game for all distinct edges $e_1, e_2 \ldots e_n \in E(G)$.

- (G, u, v) is a *non-negative game* if there exists an edge $e \in E(G)$ such that the graph $(G/e, u, v)$ is a positive game.

Note that if G has zero edges and $u \neq v$ then the game is neither positive nor non-negative. Also, if G has fewer than n edges and $u \neq v$ then the game cannot be positive. △

Proposition 2.3.3. *Let (G, u, v) be a game. Let H be a subgraph of G containing vertices u and v.*

1. *If (H, u, v) is a non-negative game, then (G, u, v) is a non-negative game.*

2. *If (H, u, v) is a positive game, then (G, u, v) is a positive game.*

Proof. We will prove this by induction on the number of edges. Suppose it holds true for G with fewer than m edges. We need to show it holds true for m edges. If $u = v$ then it follows that G is both positive and non-negative by definition. Therefore we can assume $u \neq v$ for a total of m edges.

For the first statement in the proposition, suppose (H, u, v) is a non-negative game. Then by definition there exists an edge $e \in E(H)$ such that $(H/e, u, v)$ is a positive game. However we know that the graph G/e has $m - 1$ edges and H/e is a subgraph of G/e. So therefore by our induction hypothesis $(G/e, u, v)$ is a positive game. Therefore by definition since $(G/e, u, v)$ is a positive game then (G, u, v) is a non-negative game.

For the second statement in the proposition, suppose (H, u, v) is a positive game. Then by definition $(H - \{e_1, e_2, \ldots, e_n\}, u, v)$ is non-negative for all $e_1, e_2, \ldots, e_n \in E(H)$. Now let $e_1, e_2, \ldots, e_n \in E(G)$. Therefore $(G - \{e_1, e_2, \ldots, e_n\})$ has $m - n$ edges. Suppose first that $e_1, e_2, \ldots, e_n \in E(H)$. Since we know $(H - \{e_1, e_2, \ldots, e_n\})$ is a subgraph of $(G - \{e_1, e_2, \ldots, e_n\})$ then by our induction hypothesis $(G - \{e_1, e_2, \ldots, e_n\}, u, v)$ is a non-negative game. Therefore by definition since $(G - \{e_1, e_2, \ldots, e_n\}, u, v)$ is a non-negative game then (G, u, v) is a positive game.

Suppose now that $\{e_1, e_2, \ldots, e_n\}$ are not all in $E(H)$. This means that the edges chosen by the cop to delete are not all in the subgraph H. Without loss of generality, we may assume that $e_1, \ldots, e_p \in E(H)$ and $e_{p+1}, \ldots, e_n \notin E(H)$ for some p such that $0 < p \leq n$. Let $d_1, \ldots, d_{n-p} \in E(H)$ be different from $e_1, \ldots, e_p \in E(H)$. These $d_1 \ldots d_{n-p}$ edges represent dummy moves made by the cop. Since (H, u, v) is positive then $(H - \{e_1, \ldots e_p, d_1, \ldots d_{n-p}\}, u, v)$ is a non-negative game. NOTE: We know that (H, u, v) is positive therefore by definition, $u \neq v$ so H has at least n edges. Since $(H - \{e_1, \ldots e_p, d_1, \ldots, d_{n-p}\} \subseteq (G - \{e_1 \ldots, e_n\}$ then this implies that $(G - \{e_1, \ldots, e_n\}, u, v)$ is also non-negative and therefore (G, u, v) is a positive game. $\square$

Proposition 2.3.4. *Let (G, u, v) be a game with $n > 1$ cops and one robber. If (G, u, v) is a positive game, then (G, u, v) is a non-negative game.*

This proposition is saying that if the robber can win playing second, then the robber can win playing first.

Proof. Suppose (G, u, v) is a positive game. If $u = v$ then by definition, (G, u, v) is a non-negative game. Suppose $u \neq v$. Assuming (G, u, v) is positive then we know it has at least n edges. Let $d_1, d_2, \ldots, d_n$ be dummy moves such that $d_1, d_2 \ldots d_n \in E(G)$. We also know that $(G - \{d_1, d_2, \ldots, d_n\})$ is a subgraph of G. Since (G, u, v) is positive then this implies $(G - \{d_1, d_2, \ldots, d_n\}, u, v)$ is non-negative. Therefore by the first statement of Proposition 2.3.3, (G, u, v) is a non-negative game. □

Proposition 2.3.5. *Let (G, u, v) be a game with $n > 1$ cops and one robber.*

1. *If (G, u, v) is a positive game with $n + 1$ cops, then (G, u, v) is a positive game with n cops.*

2. *If (G, u, v) is a non-negative game with $n + 1$ cops, then (G, u, v) is a non-negative game with n cops.*

Proof. We will prove this by induction on the number of edges. Suppose it holds true for G with fewer than m edges. We need to show it holds true for m edges. If $u = v$ then it follows that G is both positive and non-negative by definition. Therefore we can assume $u \neq v$ for a total of m edges.

For the first statement of the proposition, suppose (G, u, v) is positive with $n + 1$ cops then by definition $(G - \{e_1, e_2, \ldots, e_{n+1}\}, u, v)$ is a non-negative game for all $e_1, e_2, \ldots, e_{n+1} \in E(G)$. Let $e_1, e_2, \ldots, e_n \in E(G)$. Since (G, u, v) is positive with $n + 1$ cops then we know it has at least $n + 1$ edges. Suppose there exists d_1 such that d_1 is a dummy move for $d_1 \in E(G)$. Let the robber choose d_1 and $e_1, e_2, \ldots, e_n$. Therefore

$(G-\{e_1,e_2,\ldots,e_{n+1}+d_1\}$ is a subgraph of G. Therefore $(G-\{e_1,e_2,\ldots,e_{n+1}+d_1\},u,v)$ is a non-negative game. As the dummy move is a fake move, this is equivalent to saying that $(G-\{e_1,e_2,\ldots,e_n\},u,v)$ is non-negative for n cops. Therefore (G,u,v) is a positive game fore n cops.

For the second statement of the proposition, suppose (G,u,v) is non-negative with $n+1$ cops then by definition there exists an edge e such that the graph $(G/e,u,v)$ is a positive game with $n+1$ cops. Therefore by our induction hypothesis, since $(G/e,u,v)$ is positive with $n+1$ cops then it follows that $(G/e,u,v)$ is also positive for n cops. Therefore (G,u,v) is a non-negative game with n cops. □

Proposition 2.3.6. *For a given graph (G,u,v), if there exists a direct edge between u and v then the game is non-negative.*

Proof. If a game is non-negative then this means the robber can win when the robber starts. Thus on the robber's first turn if there is a direct edge between u and v, the robber can mark this edge and therefore win the game. □

3

Families of Graphs

3.1 Complete Graphs

The Shannon switching game can be played on any graph G. There are many families of graphs. Thus in order for us to learn how the n-cop game is related to graphs, we must start with applying it to a specific class of graphs. In this section we therefore explore the n-cop game played on *complete graphs*. The reason we need to specialize to complete graphs is due to the fact that we cannot yet make any conjectures regarding the n-cop game on an arbitrary graph. If we were to choose any arbitrary graph to study this game, it would be very challenging to prove anything about it. Therefore in order to generalize this variation to all graphs if possible, we must first start out small with test cases and prove what we can find about the game on complete graphs before investigating a different class of graphs. Studying the n-cop game on different types of graphs will contribute towards understanding how the n-cop game relates to *all* graphs.

Definition 3.1.1. Let $n \in \mathbb{N}$. A *complete graph* K_n is a graph where there are n vertices and every vertex is connected to every other vertex in the graph. $\triangle$

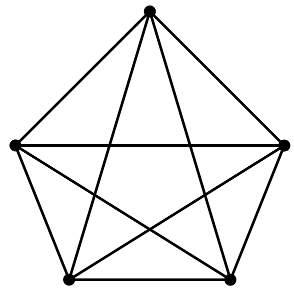


Figure 3.1.1. Complete Graph K_5

NOTE: For all the games played on complete graphs in this section, we will assume throughout that $u \neq v$. Therefore from Proposition 2.3.6, this indicates that every complete graph is a non-negative game. Since we assume $u \neq v$ then for all complete graphs, due to automorphism as the graph has symmetry, the position of u and v does not affect the game.

Definition 3.1.2. Let $\phi\colon \mathbb{N} \longrightarrow \mathbb{N}$ be the function defined by:

$$\phi(n) = \min\{m \mid K_m \text{ is positive with } n \text{ cops and one robber}\}$$

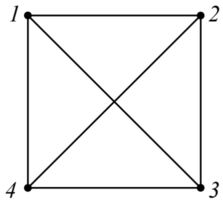
$\triangle$

We will denote each edge of a graph as a two element set $\{a, b\}$ such that the edge is between vertex a and vertex b.

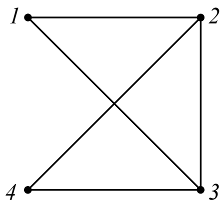
Theorem 3.1.3. $\phi(1) = 4$.

Proof. In order for us to illustrate this Theorem, we need to show that K_4 is a positive game and K_3 is a non-positive game.

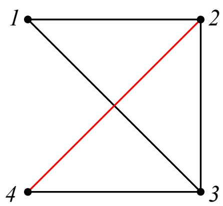
The following sequence of graphs illustrate that K_4 is a positive game. The vertices are labelled $1, 2, 3, 4$. The robber is trying to get from 1 to 4. As we are trying to show that this is a positive game, we must show that the robber wins when the cop starts.



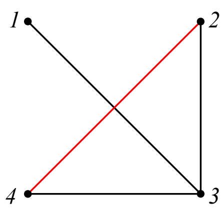
Complete Graph K_4



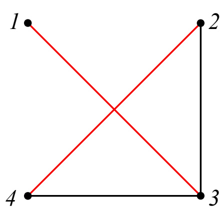
Cop's first turn: The cop is forced to delete $\{1, 4\}$ to prevent the robber from winning.



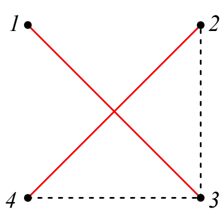
Robber's first turn: The robber should mark $\{2, 4\}$.



Cop's second turn: The cop is forced to delete $\{1, 2\}$.

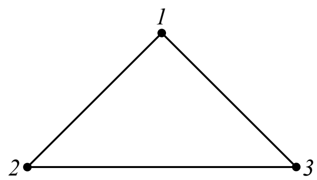


Robber's second turn: The robber should mark $\{1, 3\}$.

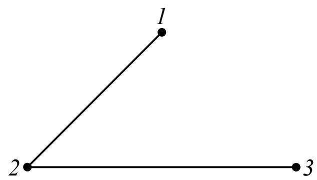


Cop's third turn: The cop is forced to delete two edges: $\{2, 3\}$ and $\{3, 4\}$ but only has one move. Therefore the robber wins on his next turn.

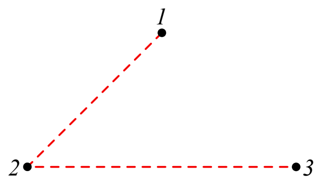
We now need to show that K_3 is a non-positive game. This means that the cop can win if the cop starts. The following sequence of graphs illustrate this game. The goal of the robber is to get from vertex 1 to vertex 3.



Complete Graph K_3



Cop's first move: The cop should delete $\{1, 3\}$ to prevent the robber from winning.



Robber's first move: The robber needs to mark two edges: $\{1, 2\}$ and $\{2, 3\}$ but only has one move. Therefore the cop will disconnect the graph on his next turn and win the game.

Thus we have shown that $\phi(1) = 4$. □

Theorem 3.1.4. *K_{n+3} is a non-positive game for n cops and one robber for $n \geq 2$.*

Proof. Let the vertices of K_{n+3} be labelled $1, \ldots, n+3$. Suppose the robber is trying to get from 1 to $n+3$. We must show that the cops win when they start.

Cops' First Turn: The cops should delete the edge $\{1, n+3\}$. With the remaining $n-1$ moves, the cops should delete $\{2, n+3\}$,...,$\{n, n+3\}$.

Robber's First Turn: The robber is forced to mark one of the two remaining edges connected to vertex $n+3$, say $\{n+1, n+3\}$. Due to symmetry it does not make a difference if the robber chose the other edge.

Cops' Second Turn: The cops are forced to delete the edge between vertex 1 and the vertex the robber previously used so $\{1, \{n+1\}$ With the $n-1$ moves remaining, the cops should delete $n-1$ edges connected to vertex 1 so $\{1, 2\}$,...,$\{1, n\}$.

Robber's Second Turn: The robber is forced to mark the last remaining edge connected to vertex 1, say $\{1, n+2\}$.

Cops' Third Turn: The cops are forced to delete the edge between the two marked edges done by the robber so $\{n+1, n+2\}$. The cops are also forced to delete the edge between $\{n+3\}$ and the vertex the robber just used so $\{n+2, n+3\}$. With the remaining $n-2$ moves, the cops should delete $n-2$ edges connected to the vertex the robber used to connect to 1. So therefore $\{n+2, 2\}$,...,$\{n+2, n-1\}$.

Robber's Third Turn: The robber is forced to mark the last remaining edge connected to the vertex he connected to 1 so $\{n+2, n\}$.

Cops' Fourth Turn: The cops should now delete the $n-1$ edges remaining that are connected to the vertex the robber previously used. Therefore the cops are deleting $n-1$ edges connected to vertex n. The cops have now disconnected the graph and therefore win.

Thus K_{n+3} is non-positive. □

Corollary 3.1.5. $\phi(n) > n+3$.

Theorem 3.1.6. *K_m is a positive game for $m = 2n^2 + n + 1$ with n cops and one robber.*

Proof. By Theorem 3.1.3 we know that $\phi(1) = 4$ so we can assume here that $n \geq 2$. Let the vertices of K_m be labelled $1, \ldots, m$. Without loss of generality, suppose the robber is trying to get from vertex 1 to vertex 2. We must show that the robber wins when the cops start. We claim the robber wins on his $n+2$ turn.

Cops' First Turn: On every turn, the cops will be deleting a total of n edges which uses a total of $2n$ vertices. On his first move he is forced to delete edge $\{1, 2\}$ to prevent the robber from winning. The cops' next moves will involve deleting edges using most $2n-2$ unused vertices.

Robber's First Turn: The robber should now mark an edge between vertex 2 and an unused vertex. Without loos of generality, we can assume $\{2, 3\}$.

Cops' Second Turn: The cops are now forced to delete $\{1, 3\}$. With the cops's remaining $n-1$ moves, they will use at most $2n-2$ unused vertices.

The plays continue in this fashion until the end of the cops' n_{th} turn. The sequence of plays continue for the robber where he is to mark an edge between his previously used vertex to another unused vertex. Therefore without loss of generality, the robber is marking $\{k+1, k+2\}$ on his k_{th} turn. This continues until he has marked a path of $n-1$ edges say $2, \ldots, n$ and has made $n-1$ turns. The cops also continue their sequence of plays and on the k_{th} turn, are forced to delete $\{1, k+1\}$. The cops then continue to delete edges using at most $2n-2$ unused vertices. After the cops have made their n_{th} turn let us count the number of vertices that have been used up this far.

- vertex 1 and 2 in the cops' first turn
- $2n-2$ vertices in the cops' first turn
- after the first turn, the cops have $n-1$ turns and deleted at most $2n-2$ unused vertices on each turn so used a total of $(n-1) \times (2n-2)$ vertices
- $n-1$ vertices from the robber

This gives a total of $2+(2n-2)+(n-1) \times (2n-2)+(n-1) = 2n^2 - n + 1$ used vertices thus far. Therefore there remain $(2n^2 + n + 1) - (2n^2 - n + 1) = 2n$ unused vertices.

Robber's nth Turn: The robber should now mark an edge between his previously used vertex from his $n-1$ turn to one of the $2n$ unused vertices, leaving $2n-1$ unused vertices. Therefore the robber now has a path of $(n-1)+1 = n$ edges.

Cops' n+1 Turn: The cops are now forced to delete the edge between vertex 1 and the previously used vertex by the robber to prevent the robber from winning. With the cops's remaining $n-1$ moves, they will use at most $2n-2$ unused vertices.

Robber's n+1 Turn: There now remain $(2n1)-(2n-2) = 1$ unused vertex. The robber should mark the edge between vertex 1 and this unused vertex.

Cops' n+2 Turn: The cops are now forced to delete $n+1$ edges but only have n moves. Therefore the robber wins on his $n+2$ turn.

Thus K_m is a positive game for $m = 2n^2 + n + 1$. $\square$

Corollary 3.1.7. *$\phi(n) \leq 2n^2 + n + 1$.*

From Corollary 3.1.5 and Corollary 3.1.7, we can observe the following inequalities:

$$n + 4 \leq \phi(n) \leq 2n^2 + n + 1.$$

These inequalities provide lower and upper bounds for $\phi(n)$. The following table shows numerical values for $n + 4$ and $2n^2 + n + 1$.

n	2	3	4	5
n+4	6	7	8	9
$\boldsymbol{2n^2+n+1}$	11	22	37	56

The values indicate that $\phi(n)$ is somewhere between a wide range of values as n gets larger. Let us look at the same table but for larger values of n.

n	100	200	300	400
n+4	104	204	304	404
$\boldsymbol{2n^2+n+1}$	20,101	80,201	180,301	320,401

As we can see the larger the value of n, the wider the range for $\phi(n)$ becomes. We can also observe that $\phi(n)$ lies in the range between a linear equation and a quadratic equation. I believe $\phi(n)$ is a quadratic equation for larger values of n. Due to the fact that $n+3$ is the value for which the game is non-positive and $n+4$ is very close to this value it gives further reason to believe that $\phi(n)$ is closer to being quadratic especially for larger values of n. It also seems as though for larger integer values of n, as the sequence goes from $n \to \infty$, the value of n does not depend on any number of finite numbers in the sequence.

Conjecture 3.1.8. *$\phi(2) \leq 9$ and $\phi(3) \leq 14$.*

The following are some questions that can be later on explored:

1. Is $\phi(n)$ linear or quadratic? For what values of n does $\phi(n)$ become quadratic?

2. What is the relationship, if any, between $\phi(n)$ and the Disjoint Spanning Tree Theorem 1.3.2? Does $\phi(n)$ contain three or more disjoint spanning trees and how does this affect the robber's strategy?

3.2 Complete Bipartite Graphs

After playing on complete graphs and trying to find values for $\phi(n)$, it seemed time to consider a different family of graphs to investigate. Complete bipartite graphs are similar to complete graphs so therefore I chose to investigate the n-cop game on *complete bipartite graphs*.

Definition 3.2.1. Let $n \in \mathbb{N}$. A *complete bipartite graph* $K_{m,n}$ is a bipartite graph with edges in two disjoint sets say A and B. All the vertices in A are connected to all the vertices in B however the vertices within the same set are not connected. Here we only consider complete bipartite graphs when $m = n$. $\triangle$

Given that the robber's goal is to get from vertex u to vertex v and $u \neq v$, we may consider two cases:

1. u and v are in the same set so $u, v \in A$ or $u, v \in B$ (same side)

2. u and v are in different sets so $u \in A$ and $v \in B$ (opposite side)

Definition 3.2.2. Let $\alpha, \beta, \gamma \colon \mathbb{N} \longrightarrow \mathbb{N}$ be the functions defined by:

$\alpha(n) = \min\{m \mid K_{m,m}$ is positive with n cops and u, v on opposite side$\}$

$\beta(n) = \min\{m \mid K_{m,m}$ is positive with n cops and u, v on same side$\}$

$\gamma(n) = \min\{m \mid K_{m,m}$ is non-negative with n cops and u, v on same side$\}$ $\triangle$

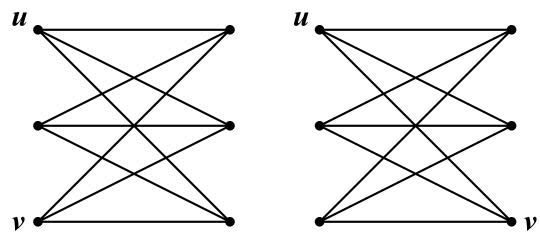
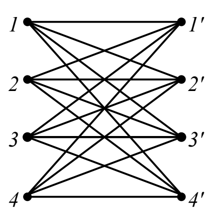


Figure 3.2.1. $K_{3,3}$ with u and v on the same side and opposite side

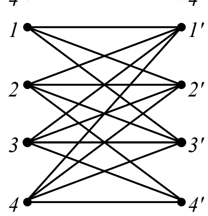
NOTE: All graphs $K_{m,m}$ with u, v on opposite sides are non-negative by Proposition 2.3.6 as there will be a direct edge between the two vertices regardless of where u, v are positioned. Therefore when investigating these games, the cops should always start first as otherwise the robber wins.

Theorem 3.2.3. $\alpha(1) = 4$.

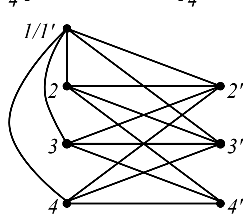
Proof. For us to prove that $\alpha(1) = 4$ we must show that $K_{4,4}$ is positive and $K_{3,3}$ is non-positive for u, v on opposite sides. The following sequence of graphs illustrates that $K_{4,4}$ is a positive game using the deletion-contraction method. We must show that the robber wins when the cop starts. The robber is trying to get from vertex 1 to vertex $4'$. So in this example vertices $1 = u$ and $4' = v$.



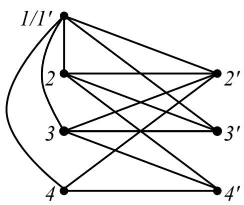
Complete Bipartite Graph $K_{4,4}$



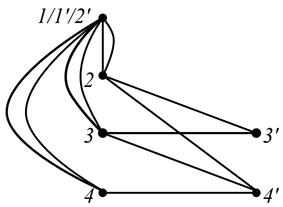
Cop's first turn: The cop is forced to delete the edge {1,4'} as otherwise the robber can contract this edge and win the game.



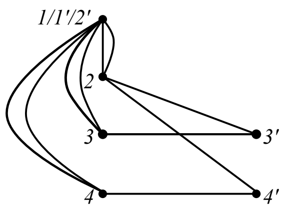
Robber's first turn: The robber should choose to contract an edge between vertices 1 and an unused edge in B say $\{1, 1'\}$. Now vertex 1 becomes $1/1'$.



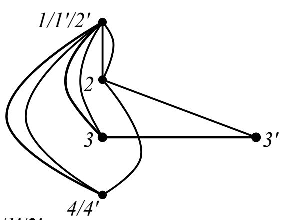
Cop's second turn: Cop chooses to delete edge $\{4, 3'\}$



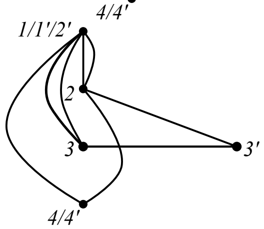
Robber's second turn: The robber should choose to contract an edge between vertices 1 and an unused edge in B say $\{1, 2'\}$. Now vertex $1/1'$ becomes $1/1'/2'$



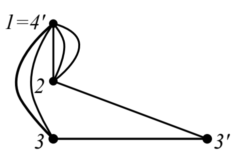
Cop's third turn: The cop chooses to delete edge $\{3, 4'\}$



Robber's third turn: The robber should now contract an edge between vertices $4'$ and any unused vertex in A say $\{4, 4'\}$. Now vertex 4 becomes $4/4'$.

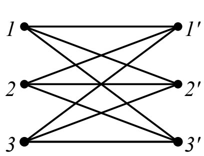


Cop's fourth turn: At this point it is clear the robber will win on his next turn because the cop needs to delete two edges but only has one move so chooses to delete edge $\{1, 4'\}$.

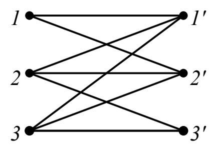


Robber's fourth turn: The robber now wins by contracting the edge between vertices 1 and $4'$ thus $1 = 4'$.

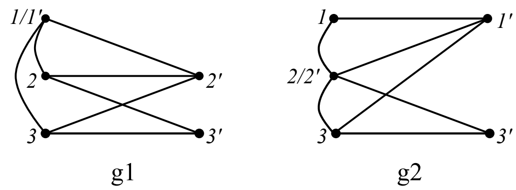
We now need to show that $K_{3,3}$ is a non-positive game. We must therefore show that the cop wins then the cop starts. Assume the robber is trying to get from 1 to $3'$.



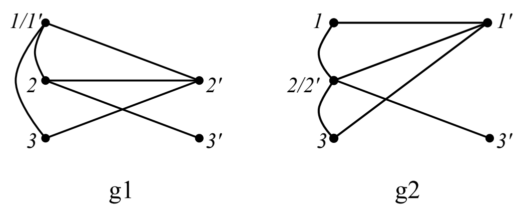
Complete Bipartite Graph $K_{3,3}$



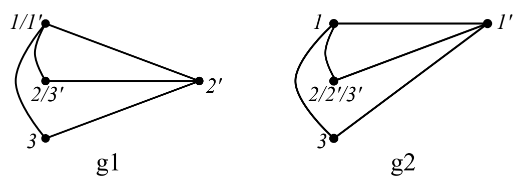
Cop's first turn: The cop is forced to delete $\{1, 3'\}$.



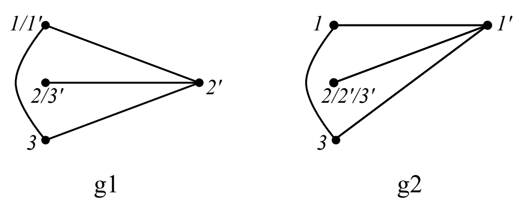
Robber's first turn: Without loss of generality we can assume the robber contracts either $\{1, 1'\}$ or $\{2, 2'\}$ (due to symmetry any other move is equivalent to one of these two moves). Graph *g1* shows the graph for the first case and *g2* shows the graph for the second case.



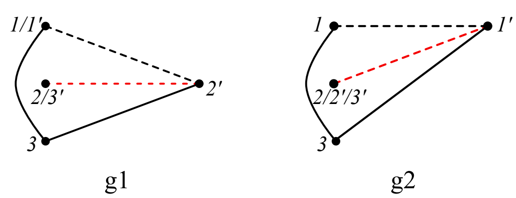
Cop's second turn: In either case the cop should now delete an edge connected to $3'$ so $\{3, 3'\}$ on both *g1* and *g2*.



Robber's second turn: On both *g1* and *g2*, the robber is forced to contract $\{2, 3'\}$ as it is the only remaining edge connected to $3'$.



Cop's third turn: In either case, the cop is now forced to delete $\{2, 1'\}$



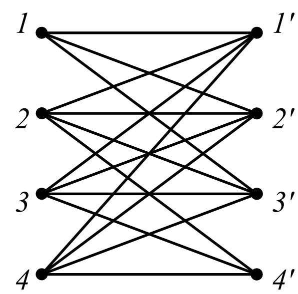
Robber's third turn: The robber only has one edge available to contract $\{2, 2'\}$ on *g1* and $\{2, 1'\}$ on *g2*. However on the cop's next turn he will disconnect the graph by deleting $\{1, 2'\}$ on *g1* and $\{1, 1'\}$ on *g2* and therefore the cop wins the game.

□

Theorem 3.2.4. $\beta(1) = 4$.

Proof. For us to prove that $\beta(1) = 4$ we must show that $K_{4,4}$ is positive and $K_{3,3}$ is non-positive for u, v on the same side. The following graphs show the graph $K_{4,4}$. By

Theorem 1.3.2, to show that $K_{4,4}$ is positive, me must show two disjoint spanning trees . Assume the robber is trying to get from 1 to 4.



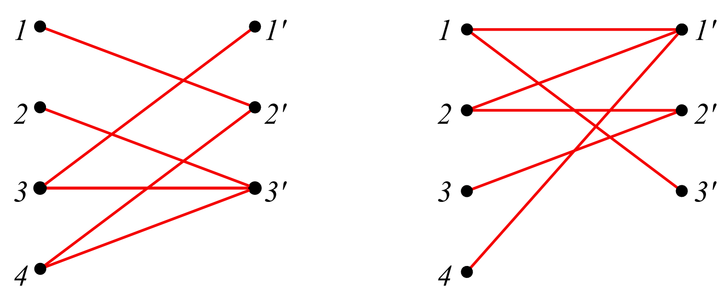


Figure 3.2.2. $K_{4,4}$ is a positive game

Now we must show that $K_{3,3}$ is a non-positive game with u, v on the same side. We must show that the cop wins when the cop starts. Let the set $A = \{1, 2, 3\}$ and the set $B = \{1', 2', 3'\}$. The robber is trying to get from 1 to 3.

Turn	**Cop**	**Robber**
First	$\{1, 1'\}$	$\{2, 2'\}$
Second	$\{1, 2'\}$	$\{1, 3'\}$
Third	$\{3, 3'\}$	$\{2, 3'\}$
Fourth	$\{3, 2'\}$	$\{3, 1'\}$

Turn	**Cop**	**Robber**
First	$\{1, 1'\}$	$\{3, 3'\}$
Second	$\{3', 1\}$	$\{1, 2'\}$
Third	$\{2', 3\}$	$\{2, 2'\}$
Fourth	$\{2, 3'\}$	robber loses

The tables above show the sequence of moves for two different games played on $K_{3,3}$. Due to symmetry of the graph, without loss of generality the robber on his first turn may choose to mark $\{3, 3'\}$ or $\{2, 2'\}$. We will only be describing the moves for the first table. However, the reader is invited to play the game following the sequence in table 2 on their own. On the first turn the cop should delete $\{1, 1'\}$. The robber then marks $\{2, 2'\}$. The cop on his second turn should delete $\{1, 2'\}$ forcing the robber to mark $\{1, 3'\}$ as this is the only remaining edge connected to 1. The cop on his third turn should delete $\{3, 3'\}$ forcing the robber to mark $\{3', 2\}$ as otherwise the cop on his next turn can delete it disconnecting the graph. On the fourth turn, the cop should delete $\{3, 2'\}$ forcing the robber to mark $\{3, 1'\}$ as this is the only remaining edge connected to 3. On the cop's final turn, we should delete $\{2, 1'\}$ therefore winning the game. □

Theorem 3.2.5. $\gamma(1) = 3$.

Proof. Assume we are playing on $K_{3,3}$. We must show that the robber wins when the robber starts. Let the set $A = \{1, 2, 3\}$ and the set $B = \{1', 2', 3'\}$. The robber is trying to get from 1 to 3.

Turn	**Robber**	**Cop**
First	$\{1, 1'\}$	$\{1', 3\}$
Second	$\{3, 3'\}$	$\{3', 1\}$
Third	$\{3', 2\}$	$\{2, 1'\}$
Fourth	$\{1, 2'\}$	$\{2, 2'\}$ $\{3, 2'\}$

On the first turn the robber should mark $\{1, 1'\}$ forcing the cop to delete $\{1', 3\}$. The robber on his second turn should then mark $\{3, 3'\}$ forcing the cop to delete $\{3', 1\}$. The robber on his third turn should mark $\{3', 2\}$ forcing the cop to delete $\{2, 1'\}$. On the fourth turn, the robber should mark $\{1, 2'\}$. The cop needs to delete $\{2, 2'\}$ and $\{3, 2'\}$ but only has one move so therefore the robber wins on his next turn.

Now we must show that $K_{2,2}$ is a negative game therefore the cop wins when the robber starts. However, $K_{2,2}$ is obviously a negative game as there exists two edges that once deleted, disconnect the graph such as in Example 1.3.3. □

Theorem 3.2.6. *$K_{m,m}$ is a positive game when $m = (n + 1)^2$ for n cops with u, v on opposite sides.*

Proof. By Theorem 3.2.3 we know that $\alpha(1) = 4$ so we can assume here that $n \geq 2$. We must show that the robber wins when the cops start. Let us assume the set $A = \{1, \ldots, m\}$ and the set $B = \{1, \ldots, m'\}$. The robber is trying to get from vertex 1 to vertex m'. The game begins with $n^2 + 2n + 1$ unused vertices in each vertex set A and B respectively. We claim that the robber wins on his $n + 3$ turn.

Cops' First Turn: The cops are forced to delete $\{1, m'\}$ to prevent the robber from winning. The cops can choose to delete any $n - 1$ edges and will use at most $n - 1$ unused vertices from each set .

Robber's First Turn: The robber should choose to mark an edge between vertex 1 and any unused vertex in B. Without loss of generality we can assume $\{1, 1'\}$.

Cops' Second Turn: The cops can choose to delete any n edges and will use at most n unused vertices from each set .

Robber's Second Turn: The robber should now choose to mark an edge between vertex 1 and any unused vertex in B. Without loss of generality we can assume $\{1, 2'\}$.

Cops' Third Turn: The cops can choose to delete any n edges and will use at most n unused vertices from each set .

The plays continue in this fashion until the end of the cops' $n+1$ turn. This sequence of plays continue for the robber where he is to mark an edge between vertex 1 and an unused vertex in B. This continues until the robber has marked n edges from 1 to B. Therefore without loss of generality we can assume his last marked edge in this sequence of plays is $\{1, n'\}$. The cops also continue their sequence of plays until their $n + 1$ turn and will delete at most n unused vertices.

After the cops have made their $n+1$ turn let us count the number of vertices that have been used up thus far in A and B respectively:

Set A is as follows:

- vertex 1 from the cop's first move
- $n-1$ vertices from the cop's first move
- after the first turn, the cops at this point have n turns and deleted at most n unused vertices on each turn so used a total of n^2 vertices

This gives a total of $1+(n-1)+n^2 = n^2+n$ used vertices thus far in A.

Set B is as follows:

- vertex m from the cop's first move
- $n-1$ vertices from the cop's first move
- n vertices from the robber
- after the first turn, the cops at this point have n turns and deleted at most n unused vertices on each turn so used a total of n^2 vertices

This gives a total of $1+(n-1)+n+n^2 = n^2+2n$ used vertices thus far in B.

The game started with a total of n^2+2n+1 vertices. Therefore after the cops' $n+1$ turn there remain $(n^2+2n+1)-(n^2+n) = n+1$ unused vertices in A and $(n^2+2n+1)-(n^2+2n) = 1$ unused vertex in B.

Robber's n+1 Turn: The robber should now mark the edge between 1 and the one remaining vertex in B. Let us refer to this vertex as *unused*. Therefore the robber marks $\{1, \mathit{unused}\}$

Cops' n+2 Turn: There now remain no unused vertices in B thus with the cops n moves they will delete at most n unused vertices in A.

Robber's n+2 Turn: There is now at least $(n+1)-n = 1$ unused vertex remaining in A. The robber should mark the edge between m' and this one remaining unused vertex in

A. Without loss of generality we can assume this unused vertex is m. Therefore the robber marks $\{m, m'\}$.

Now on the cops' $n+3$ turn, they are forced to delete $n+1$ edges connected from m to B but they only have n moves. Therefore the robber wins on his $n+3$ turn. Figure 3.2.3 is an illustration of how this proof works. The robber's $n+2$ turn is shown as the dotted red lines from m' to m. As you can see, the cop needs to delete $\{m, [\{1', \ldots, n'\} + unused]\}$ but only has n moves. Therefore the robber wins.

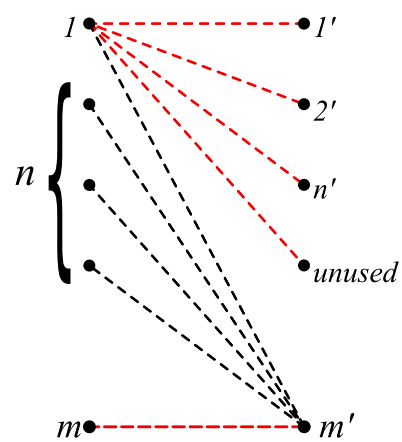


Figure 3.2.3. Illustration of Theorem 3.2.6

□

Corollary 3.2.7. $\alpha(n) \leq (n+1)^2$.

Theorem 3.2.8. *$K_{m,m}$ is a positive game when $m = (n+1)^2$ for n cops with u, v on the same side.*

Proof. By Theorem 3.2.4 we know that $\beta(1) = 4$ so we can assume here that $n \geq 2$. We must show that the robber wins when the cops start. Let us assume $A = \{1, \ldots, m\}$ and $B = \{1', \ldots, m'\}$. The robber is trying to get from vertex 1 to vertex m. The game begins with $n^2 + 2n + 1$ unused vertices in each vertex set A and B respectively. The robber wins on his $n+4$ turn.

Cops' First Turn: The cops can choose to delete any n edges and will use at most n unused vertices from each set .

Robber's First Turn: The robber should choose to mark an edge between vertex 1 and any unused vertex in B. Without loss of generality we can assume $\{1, 1'\}$.

Cops' Second Turn: The cops are now forced to delete the edge $\{1', m\}$ to prevent the robber from winning. The cops can choose to delete any $n-1$ edges and will use at most $n-1$ unused vertices from each set .

Robber's Second Turn: The robber should now choose to mark an edge between vertex 1 and any unused vertex in B. Without loss of generality we can assume $\{1, 2'\}$.

Cops' Third Turn: The cops are now forced to delete $\{2', m\}$ to prevent the robber from winning. The cops can choose to delete any $n-1$ edges and will use at most $n-1$ unused vertices from each set .

The plays continue in this fashion until the end of the cops' $n+2$ turn.This sequence of plays continue for the robber where he is to mark an edge between vertex 1 and an unused vertex in B. This continues until the robber has marked $n+1$ edges from 1 to B. Therefore without loss of generality we can assume his last marked edge in this sequence of plays is $\{1, (n+1)'\}$. The cops also continue their sequence of plays until their $n+2$ turn where they are forced to delete $\{1, (n+1)'\}$ and at most $n-1$ unused vertices.

After the cops have made their $n+2$ turn let us count the number of vertices that have been used up thus far in A and B respectively:

Set A is as follows:

- vertex 1 and m

- n vertices from the cops' first turn

- after the first turn, the cops have $n+1$ turns and deleted at most $(n-1)$ unused vertices on each turn so used a total of $(n+1) \times (n-1)$ vertices

This gives a total of $2 + n + (n+1) \times (n-1) = n^2 + n + 1$ used vertices thus far in A.

Set B is as follows:

- n vertices from the cops' first turn
- $n+1$ vertices from the robber's turns
- after the first turn, the cops have $n+1$ turns and deleted at most $(n-1)$ unused vertices on each turn so used a total of $(n+1) \times (n-1)$ vertices

This gives a total of $n + (n+1) + (n+1) \times (n-1) = n^2 + 2n$ used vertices thus far in B.

The game started with a total of $n^2 + 2n + 1$ vertices. Therefore after the cops' $n+2$ turn there remain $(n^2 + 2n + 1) - (n^2 + n + 1) = n$ unused vertices in A and $(n^2 + 2n + 1) - (n^2 + 2n) = 1$ unused vertex in B.

Robber's n+2 Turn: The robber should now mark the edge between m and the one remaining vertex in B. Let us refer to this vertex as *unused*. Therefore the robber marks $\{unused, m\}$

Cops' n+3 Turn: The cops are forced to delete the $\{1, unused\}$. There now remain no unused vertices in B thus with the cops $n-1$ moves they will delete at most $n-1$ unused vertices in A.

Robber's n+3 Turn: There is now at least $n-(n-1) = 1$ unused vertex remaining in A. The robber should mark the edge between *unused* and this one remaining unused vertex in A. Without loss of generality we can assume this unused vertex is $(m-1)$. Therefore the robber marks $\{unused, m-1\}$.

Now on the cops' $n+4$ turn, they are forced to delete $n+1$ edges connected from $m-1$ to B but they only have n moves. Therefore the robber wins on his $n+4$ turn.

Figure 3.2.4 is an illustration of how this proof works. The robber's $n+3$ turn is shown as the dotted red lines from *unused* to $m-1$. As you can see, the cop needs to now delete $\{m-1, \{1', 2', \ldots, (n+1)'\}\}$ but only has n moves. Therefore the robber wins.

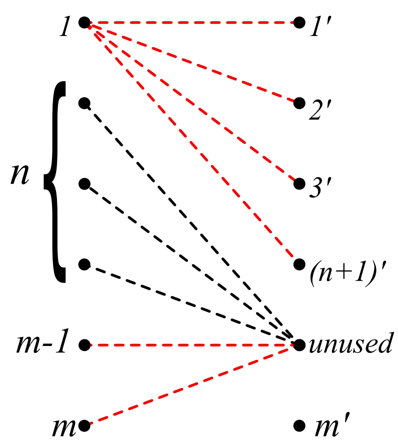


Figure 3.2.4. Illustration of Theorem 3.2.8

□

Corollary 3.2.9. $\beta(n) \leq (n+1)^2$.

Theorem 3.2.10. *$K_{m,m}$ is a non-negative game when $m = n^2 + n + 1$ for n cops with u, v on the same side.*

Proof. By Theorem 3.2.5 we know that $\gamma(1) = 3$ so we can assume here that $n \geq 2$. We must show that the robber wins when the robber starts. Let us assume $A = \{1, \ldots, m\}$ and $B = \{1', \ldots, m'\}$. The robber is trying to get from vertex 1 to vertex m. The game begins with n^2+n+1 unused vertices in each vertex set A and B respectively. The robber wins on his $n+4$ turn.

Robber's First Turn: The robber should choose to mark an edge between vertex 1 and any unused vertex in B. Without loss of generality we can assume $\{1, 1'\}$.

Cops' First Turn: The cops are now forced to delete the edge $\{1', m\}$ to prevent the robber from winning. The cops can choose to delete any $n-1$ edges and will use at most $n-1$ unused vertices from each set .

Robber's Second Turn: The robber should now choose to mark an edge between vertex 1 and any unused vertex in B. Without loss of generality we can assume $\{1, 2'\}$.

Cops' Third Turn: The cops are now forced to delete $\{2', m\}$ to prevent the robber from winning. The cops can choose to delete any $n-1$ edges and will use at most $n-1$ unused vertices from each set .

The plays continue in this fashion until the end of the cops' $n+1$ turn. This sequence of plays continue for the robber where he is to mark an edge between vertex 1 and an unused vertex in B. This continues until the robber has marked $n+1$ edges from 1 to B. Therefore without loss of generality we can assume his last marked edge in this sequence of plays is $\{1, (n+1)'\}$. The cops also continue their sequence of plays until their $n+1$ turn where they are forced to delete $\{1, (n+1)'\}$ and at most $n-1$ unused vertices.

After the cops have made their $n+1$ turn let us count the number of vertices that have been used up thus far in A and B respectively:

Set A is as follows:

- vertex 1 and m
- the cops have $n+1$ turns and deleted at most $(n-1)$ unused vertices on each turn so used a total of $(n+1) \times (n-1)$ vertices

This gives a total of $2 + (n+1) \times (n-1) = n^2 + 1$ used vertices thus far in A.

Set B is as follows:

- $n+1$ vertices from the robber's turns

- the cops have $n+1$ turns and deleted at most $(n-1)$ unused vertices on each turn so used a total of $(n+1) \times (n-1)$ vertices

This gives a total of $(n+1) + (n+1) \times (n-1) = n^2 + n$ used vertices thus far in B.

The game started with a total of n^2+n+1 vertices. Therefore after the cops' $n+1$ turn, there remain $(n^2+n+1)-(n^2+1) = n$ unused vertices in A and $(n^2+n+1)-(n^2+n) = 1$ unused vertex in B.

Robber's n+2 Turn: The robber should now mark the edge between m and the one remaining vertex in B. Let us refer to this vertex as *unused*. Therefore the robber marks $\{unused, m\}$

Cops' n+3 Turn: The cops are forced to delete the $\{1, unused\}$. There now remain no unused vertices in B thus with the cops $n-1$ moves they will delete at most $n-1$ unused vertices in A.

Robber's n+3 Turn: There is now at least $n-(n-1) = 1$ unused vertex remaining in A. The robber should mark the edge between *unused* and this one remaining unused vertex in A. Without loss of generality we can assume this unused vertex is $(m-1)$. Therefore the robber marks $\{unused, (m-1)\}$.

Now on the cops' $n+3$ turn, they are now forced to delete $n+1$ edges from B connected to vertex $m-1$ but only has n moves. Therefore the robber wins on his $n+4$ turn. □

Corollary 3.2.11. $\gamma(n) \leq n^2 + n + 1$.

Proposition 3.2.12. *If the graph $K_{m,m}$ is a positive game with n cops and u, v on opposite side, then $K_{m+1,m+1}$ is a non-negative game with n cops and u, v on the same side.*

Proof. Assume $K_{m,m}$ is a positive game for n cops and u, v on the opposite side. Consider the graph $K_{m+1,m+1}$ with u, v on the same side. Let the disjoint sets be defined as $A = \{1, \ldots, m+1\}$ and $B = \{1', \ldots, (m+1)'\}$. The robber is trying to get from 1 to $m+1$.

Therefore $u = 1$ and $v = m+1$. Since we are trying to prove that $K_{m+1,m+1}$ is non-negative, we must show that the robber wins when the robber starts. Let the robber's first move be to contract the edge e between vertices $m+1$ and m'. Therefore the resulting graph is $K_{m+1,m+1}/e$ such that $m+1 = m'$. The new disjoint sets are now $A_{new} = \{1, \dots, m\}$ and $B_{new} = \{1, \dots, (m+1) = m'\}$. Thus in $K_{m+1,m+1}/e$, every vertex in A_{new} is connected to every vertex in B_{new}. Therefore this resulting graph contains a subgraph that is isomorphic to $K_{m,m}$. Since $m+1 = m'$ then $m+1$ is now in B_{new} and therefore on the opposite side. Since we have assumed $K_{m,m}$ is a positive game for u, v on opposite sides which means the robber wins when the cop starts, then this implies that $K_{m+1,m+1}$ is non-negative with u, v on the same side. □

I have proven that $K_{m,m}$ is a non-positive game when $m = n+2$ for u, v on the same side and on opposite sides. These proofs are not included in the write up however it gives us some lower bounds for $\alpha(n)$ and $\beta(n)$ such that $n+3 \leq \alpha(n)$ and $n+3 \leq \beta(n)$.

From the theorems we therefore know the following bounds for α, β and γ:

1. $n+3 \leq \alpha(n) \leq n^2 + 2n + 1$
2. $n+3 \leq \beta(n) \leq n^2 + 2n + 1$
3. $\gamma(n) \leq n^2 + n + 1$

The table below allows us to observe inputs for the inequalities of the functions:

$\mathbf{n}$	2	3	4	5
$\mathbf{n+3}$	5	6	7	8
$\boldsymbol{n^2+2n+1}$	9	16	25	36
$\mathbf{n^2+n+1}$	7	13	21	31

The table therefore tells us that $5 \leq \alpha(2) \leq 9$, $6 \leq \alpha(3) \leq 16$, $7 \leq \alpha(3) \leq 25$,…,$103 \leq \alpha(100) \leq 10,201$. The same applies for β. The sequence of bounds therefore has an increasing difference of $n^2, n^2+2, n^2+3, \dots, n^2+\infty$. Therefore, with the same reasoning as in the case of complete graphs, I hold the assumption that $\alpha(n)$ and $\beta(n)$ are quadratic equations.

From Proposition 3.2.12 we know that if the graph $K_{m,m}$ is a positive game with n cops and u, v on opposite side, then $K_{m+1,m+1}$ is a non-negative game with n cops and u, v on the same side. Therefore if $m \geq \alpha(n)$ then $m+1 \geq \gamma(n)$. In particular, since $\alpha(n) \geq \alpha(n)$, then this implies that $\alpha(n) + 1 \geq \gamma(n)$. We also know that $n^2 + 2n + 1 \geq \alpha(n)$ therefore it is true that $n^2 + 2n + 1 + 1 \geq \alpha(n) + 1$. Therefore $n^2 + 2n + 2 \geq \alpha(n) + 1 \geq \gamma(n)$.

Let us now look at the relationship between the results found for complete graphs and the results found for complete bipartite graphs.

Something we have not proven but which is true is that $K_{m,m} \subseteq K_{2m}$. We know that $\phi(n) \leq 2n^2 + n + 1$, $\alpha(n) \leq n^2 + 2n + 1$ and $\beta(n) \leq n^2 + 2n + 1$. Therefore from Proposition 2.3.3 part 2, we know that if $K_{m,m}$ is positive, then K_{2m} is also positive. Therefore this means that if $m \geq \alpha(n)$, then $2m \geq \phi(n)$. This implies that $2\alpha(n) \geq \phi(n)$. Since β and α are both complete bipartite graphs $K_{m,m}$, the same applies to $\beta(n)$. We also know that $n + 3 \leq \phi(n)$ thus $n + 3 \leq \phi(n) \leq 2\alpha(n)$.

The following are some questions that can be explored:

1. For what values of n are $\alpha(n)$ and $\gamma(n)$ quadratic?

2. What is a lower bound for $\gamma(n)$?

3. How does the game differ on bipartite graphs?

4. What is the relationship, if any, between $\alpha(n)$, $\beta(n)$ and the Disjoint Spanning Tree Theorem 1.3.2 for n-cop game?

4
N-Cop, N-Robber Game

After exploring the n-cop game, it seemed logical for us to investigate a game with multiple robbers that we will call n robbers. This chapter explores the n-cop, n-robber game played on complete bipartite graphs. Introducing n robbers is equivalent to one robber having n moves. Therefore, like the n cops, the robbers are moving collectively together and mark (or contract) a total of n edges on each turn. The n cops and n robbers alternate turns until either the n cops have disconnected all paths in the graph or the n robbers have marked a path from u to v. The definitions for the types of games remain the same, however now the robber gets as many moves as the cop.

Theorem 4.1. Let $\epsilon, \delta \colon \mathbb{N} \longrightarrow \mathbb{N}$ be the functions defined by:

$$\epsilon(n) = \min\{m \mid K_{m,m} \text{ is positive with } n \text{ cops and } n \text{ robbers and } u, v \text{ on opposite sides}\}$$

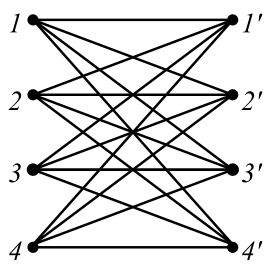
$$\delta(n) = \min\{m \mid K_{m,m} \text{ is positive with } n \text{ cops and } n \text{ robbers and } u, v \text{ on same side}\}$$

We already know from Section 3.2 that $\epsilon(1) = 4$ and $\delta(1) = 4$.

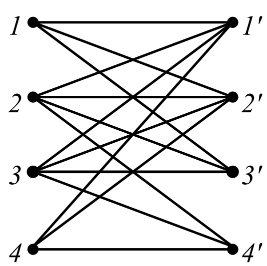
Theorem 4.2. $\epsilon(2) = 4$.

In order for us to illustrate this Theorem, we need to show that $K_{4,4}$ is positive and $K_{3,3}$ is a non-negative game with u, v on opposite sides with 2 cops and 2 robbers.

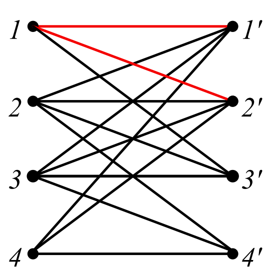
The following is an illustration of a game played on $K_{4,4}$ with two cops and two robbers. We must show that this is a positive game so therefore the robbers win when the cops start. The two robbers are trying to get from vertex 1 to vertex $4'$.



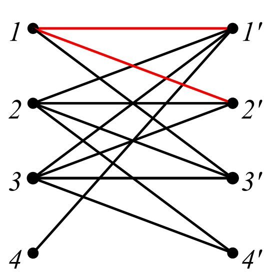
Complete Bipartite Graph $K_{4,4}$



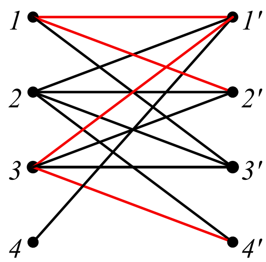
Cops' first turn: The cops delete the edge $\{1, 4'\}$. They can now delete one more edge (say $\{4, 3'\}$) that will delete at most two unused vertices.



Robbers' first turn: There are now at least two unused vertices in B. The robbers should mark these two unused vertices to vertex 1. Thus marking edges $\{1, 1'\}$ and $\{1, 2'\}$.

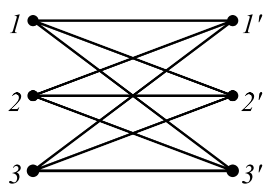


Cops' second turn: At this point it is obvious that on the robbers' next turn, they can mark many edges that can create a path between vertices 1 and $4'$. The cops choose to delete the edges $\{4, 2'\}$ and $\{4, 4'\}$.

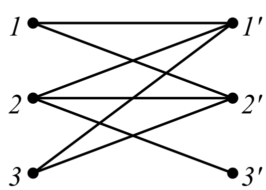


Robbers' second turn: Now the robbers will have at least one vertex in A that can be marked to $4'$ say $\{4, 4'\}$. Then on second move connect this vertex in A to one of the two vertices they marked in B on their first turn say $\{4, 2'\}$. The robbers have won thus the game is positive.

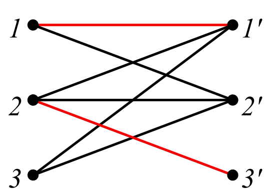
We now need to show that $K_{3,3}$ is a non-positive game. This means that the cops can win if the cops starts. The following illustrates this game and therefore shows a strategy for the cops to win on $K_{3,3}$ against the two robbers. The goal of the robbers is to get from vertex 1 to vertex $3'$.



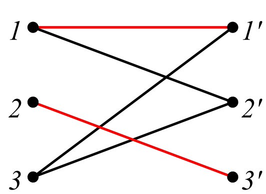
Complete Bipartite Graph $K_{3,3}$



Cops' first turn: Cops' delete $\{1, 3'\}$ and should then delete an edge between vertex $3'$ and a vertex in set A say $\{3, 3'\}$



Robbers' first turn: The robbers are forced to mark $\{2, 3'\}$ as it is the only remaining edge connected to $3'$. The robbers are also forced the mark either $\{1, 1'\}$ or $\{1, 2'\}$ as they are the only remaining vertices connected to 1. Due to symmetry, without loss of generality, the robbers mark $\{1, 1'\}$.



Cop's second turn: Finally the cops should delete the two edges connected to vertex 2. At this point the cops win as they have disconnected the graph.

Theorem 4.3. $\epsilon(n) = n + 1$ for $n \geq 3$.

Proof. For us to prove that $\epsilon(n) = n + 1$, we need to show that $K_{m,m}$ is positive for $m = n + 1$ with u, v on opposite sides for n cops and n robbers and we also need to show that $K_{m,m}$ is non-positive when $m = n$. To show this is positive, we must show that the robbers win when the cops start. In this game, we label the vertices as $A = \{1, 2, \ldots, m\}$ and $B = \{1', 2', \ldots, m'\}$.

Cops' First Turn: The cops are forced to delete $\{1, m'\}$ as otherwise the robber will win. With the remaining $n - 1$ moves, the cops can delete any edge and will use at most $n - 1$ unused vertices from each set.

Robbers' First Turn: The robbers will now have at least one edge that they can mark between vertices 1 and an unused vertex in set B. Then on the next move, the robbers will have at least one unused vertex in A and and should mark the edge between this vertex and the vertex previously used from B. So now the robbers have marked a path from vertex 1 to another vertex in set A. From this point the robbers obviously win as the robbers should now connect this vertex in A to vertex m' therefore marking a direct path from 1 to m'.

Now we must show that $K_{m,m}$ is a non-negative game with n cops and n robbers when $m = n$ for $n \geq 3$ with u, v on opposite sides. We must therefore show that the cops wins when they start.

The robbers are trying to get from vertex 1 to vertex m'. Therefore on the cops' first turn, they should choose to delete the n edges connected from vertex m' to the n vertices in set A. Since $m = n$, they have deleted all possible edges connected to m' therefore disconnecting the graph. Thus the cops win on first turn. $\square$

Theorem 4.4. $\delta(n) = n + 1$ for $n \geq 2$.

Proof. For us to prove that $\delta(n) = n + 1$ for $n \geq 2$, we must prove that $K_{m,m}$ is positive when $m = n+1$ with u, v on the same side. We also must prove that $K_{m,m}$ is non-positive when $m = n$. To show the game is positive, we must show that the robbers win when the cops start. In this game, we label the vertices as $A = \{1, 2, \ldots, m\}$ and $B = \{1', 2', \ldots, m'\}$. The robbers are trying to get from vertex 1 to m.

Cops' First Turn: The cops can delete any two edges. They will use at most n unused vertices from each set A and B.

Robbers' First Turn: The robbers will now have at least one edge that they can mark between vertices 1 and an unused vertex in set B and should mark this edge. Then on their next move, they should mark the edge between the unused vertex previously used from B to vertex m. Thus the robbers win the game.

Now we must show that $K_{m,m}$ is a non-positive game with n cops and n robbers when $m = n$ for $n \geq 2$ with u, v on the same side. This means we must show the cops can win when they start.

The robber is trying to get from vertex 1 to vertex m. On the cops' first turn, they should choose to delete the n edges connected from vertex m to the n vertices in B. Since $m = n$, they have deleted all possible edges connected to m therefore disconnecting the graph. Thus the cops win on first turn and game is non-positive. $\square$

Now let us pose an interesting question:

Consider playing a game on $K_{14,14}$ with u, v on the same side. Suppose the robber is trying to get from vertex 1 to 12. If you were the robber and had the choice of the following, which one would you choose to have this be a positive game?

1. 2 robbers versus 13 cops or

2. 1 robber versus 6 cops

Let us now look at the results for ϵ and δ for different values of n.

$\mathbf{n}$	2	3	4	5	100
$\epsilon(\mathbf{n})$	4	4	5	6	101
$\boldsymbol{\delta(n)}$	3	4	5	6	101

As we can can see $\epsilon(2) \neq \delta(2)$. Also notice that for $\delta(n)$, when $n \geq 2$, the robbers win on their first turn and only need 2 moves. Therefore for $\delta(n) \geq 3$, two robbers are sufficient to win the game. Also notice that for $\epsilon(n)$, when $n \geq 3$, the robbers win on their first turn also and only need 3 moves. Therefore for $\delta(n) \geq 4$, three robbers are sufficient to win the game. Therefore even when the robbers are up against 100 cops in the case of $\delta(100)$, the robbers only need to use 2 of their men to escape successfully. This shows the advantage of having more than one robber and how much easier it is to win the game. Let us revisit the question posed in the beginning? Would you rather 1 robber versus 6 cops or 2 robbers versus 13 cops. Since u, v are on opposite sides for $K_{14,14}$ we must consider $\delta(13)$ and $\beta(6)$. We know that $\delta(13) = 14$ therefore this means that 13 robbers can win against 13 cops however we also know that only 2 of these robbers are needed to beat 13 cops. However, $\beta(6)$ which means there are 6 cops against 1 robber, the range of values is $9 \leq \beta(6) \leq (49)$. This is a wide range of values to consider for the minimum for the game to be positive, therefore with this reasoning it would be a safer bet to choose 2 robbers versus 13 cops.

Further questions that can be explored are:

1. How does the n-cop, n-robber game relate to complete graphs?

2. What is the relationship, if any, between $\epsilon(n)$, $\delta(n)$ and the Disjoint Spanning Tree Theorem 1.3.2 for n-cop, n-robber game?

There is much more research that can be done investigating the Shannon switching game. The Spanning Tree Theorem is a very interesting theorem solved. How does the number of spanning trees in a given graph affect the n-cop game? What are other variations of this game that can be explored? What are some results of the n-cop game when played on other families of graphs? Mathematicians often study the Shannon switching game using matroid theory. What if the n-cop game was studied using matroids? How does the study on matroids relate to the game and other possible variations? The results in this project only mark a start for the many variations that can further be explored.

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