Fighting Fires on Semi-Regular Tesselations

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by
Lara-Greta Merling

Annandale-on-Hudson, New York
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Abstract

The Firefighter Problem models how a fire spreads on a graph while firefighters are being placed in an attempt to contain the fire. In this project, we use this model to examine how fires spread on the graphs of each of the eight semi-regular tessellations. Our goal is to determine the minimum number of firefighters needed to contain the fire. We first provide upper bounds on the number of firefighters needed to contain a fire on each semi-regular tessellation. We then establish a lower bound of one firefighter per turn for five of the eight semi-regular tessellations. We formulate a theorem for establishing lower bounds on a graph by using properties of geodesics and apply it successfully to the square grid. We then develop a labeling scheme for the movement of paths on 3.6.3.6 and classify geodesics on that graph.

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Dedication

To my father.

Acknowledgments

I am extremely grateful to Jim Belk for the excellent guidance he has given me throughout this semester. Thank you so much for all the time you have spent helping me finish this project!

I would also like to thank Lauren Rose for helping me get started on this project and giving me valuable advice on how to approach all the work I had to do. I would like to thank Bard College, for the exceptional education it has offered me. Without it, this project would not exist.

Last but not least, I would like to say thank you to Siddhartha, Katya and Diego. Your company and support made my time spent working on this project (and at Bard!) so much more enjoyable.

Introduction

The Firefighter Problem is a simple model for analyzing the way an infection spreads on a graph and how this spread can be stopped. It describes the infection as "fire" and the agents trying to contain it as "firefighters". The model has also been adapted to examine issues such as the spread of a virus, or information through a social network. The Firefighter Problem was introduced for the first time in 1995 at the Manitoba Conference on Combinatorial Mathematics and Computing by Bert Hartnell [2] and its application has since been studied extensively on both finite and infinite graphs. A survey of the main known results of this problem can be found in [2].

The Firefighter Problem has the structure of a game, with the fire and the firefighters taking turns. At turn zero, the fire starts at a vertex on the graph. On each turn, the fire spreads to all unoccupied neighboring vertices. Starting at turn one, a number of firefighters are placed on selected vertices. Typically, on each turn the same number of firefighters will be placed. Once placed at a vertex, a firefighter remains there until the end of the process. Vertices that have a firefighter placed on them are considered occupied and the fire cannot spread to them. This process is repeated on each subsequent turn until the fire is contained and can no longer spread.

The Firefighter Problem has been examined on different classes of graphs, including trees, n-dimensional grids and regular tessellations [2]. In this project, we examine some examples of the Firefighter Problem on infinite graphs that are associated with tessellations, with an ultimate goal of determining what features of a graph make a fire easier, or more difficult to contain.

Tessellations are patterns of shapes that fill up surfaces completely, without the shapes overlapping or leaving any empty spaces. *Regular tessellations* are defined as tessellations that use one type of regular polygon to fill up a plane. There are three types of regular polygons that can tessellate a plane, namely the equilateral triangle, the square and the hexagon. The triangular, square and hexagonal grids are regular tessellations. More information about tessellations can be found in [5].

Previous papers have examined the model of the Firefighter Problem on the graphs of regular tessellations. Ping Wang and Stephanie Moeller [6] showed that at least two firefighters per turn are required to contain a fire on the square grid. Patricia Fogarty [3] proved that at least three firefighters per turn are needed to contain a fire on the regular tessellation formed by equilateral triangles. Fogarty also offered an alternative proof that one firefighter cannot contain a fire on the square grid. Her proof is based on a structural theorem that can also be applied to establish lower bounds for other graphs. For the hexagonal grid, Margaret-Ellen Messinger [4] proved that two firefighters per turn are enough to contain a fire and conjectured that one firefighter per turn is not enough. The problem of whether one firefighter per turn suffices to contain a fire on the hexagonal grid is still open.

After regular tessellations, the simplest tessellations are those that are *semi-regular*, combining two or more types of regular polygons around identical vertex points. There

Name	Graph of
T_1	4.6.12
T_2	4.8.8
T_3	3.12.12
T_4	3.3.3.4.4
T_5	3.3.4.3.4
T_6	3.6.3.6
T_7	3.3.3.3.6
T_8	3.4.6.4

Table 0.0.1: Notations for the Graphs of Semi-regular Tessellations

are eight possible semi-regular tessellations, named according to the types of polygons that surround each vertex. Table 0.0.1 provides the name of each of the eight semi-regular tessellations and the notations that will be used for each of them throughout this project.

In this project, we investigate the Firefighter Problem on the infinite graphs of semi-regular tessellations with the goal of determining the minimum number of firefighters necessary to contain a fire on each of these graphs. Semi-regular tessellations are a class of infinite graphs that have not been examined before in the context of the Firefighter Problem. These graphs are particularly convenient to look at, as all vertices are identical. Thus, where fire starts on the graph does not influence the shape of the resulting fire. By looking at the Firefighter Problem in the context of semi-regular tessellations, we aim to develop a better understanding of how to find the minimum number of firefighters needed to contain a fire on any infinite graph.

In Chapter 1, we introduce the Firefighter Problem, both informally and formally. We illustrate the model with some simple examples on both finite and infinite graphs and provide the formal definitions that we will use throughout the project. In this chapter we also take a close look at the previous results for the Firefighter Problem on regular tessellations.

Chapter 2 discusses the minimum number of firefighters needed to contain a fire on each semi-regular tessellation. For each of the eight semi-regular tessellations we find an upper bound for the number of firefighters per turn that suffice to contain a fire. We find that one firefighter per turn is enough to contain a fire on T_1, T_2, and T_3; two are enough for T_6, T_7, and T_8; and three are necessary for T_4 and T_5. To verify that the suggested number is the minimum number of firefighters needed, we have to prove that less firefighters per turn cannot contain the fire. In this chapter, we also show that one firefighter per turn is not enough to contain a fire on T_4 and T_5. However, we do not have corresponding results for T_6, T_7 and T_8.

In Chapter 3 we discuss an attempt to prove that one firefighter does not suffice to contain a fire on T_6. First, we reformulate and generalize Wang and Moeller's proof for the square grid [6] using geodesic paths. Then we develop a labeling scheme for paths on T_6 and consequently use this scheme to classify geodesics on T_6. Our results in this chapter put strong constraints on how one firefighter could contain a fire on T_6 although we are unable to prove that one firefighter is not enough.

1
Background

1.1 Tessellations

Throughout this project we investigate the Firefighter Problem on graphs of regular and semi-regular tessellations. In this section we will define these tessellations and discuss some of their main properties. All the material in this section can be found in [5].

A *tessellation* is a pattern made up of one or more shapes, completely covering a surface without any gaps or overlaps. The word tessellation is derived from the latin tessella, which was the square stone used in ancient Roman mosaics. There are numerous shapes that can tessellate a plane, however this project is only concerned with tessellations that are formed by regular polygons. Recall that *regular polygons* are those polygons which have congruent sides and angles. A *regular tessellation* is a tessellation that is composed of one single type of regular polygon. There are three regular polygons that tessellate the plane alone, namely the equilateral triangle, the square and the hexagon. The triangular, square, and hexagonal tessellations can be seen in Figure 1.1.1.

In this project we are mostly concerned with semi-regular tessellations.

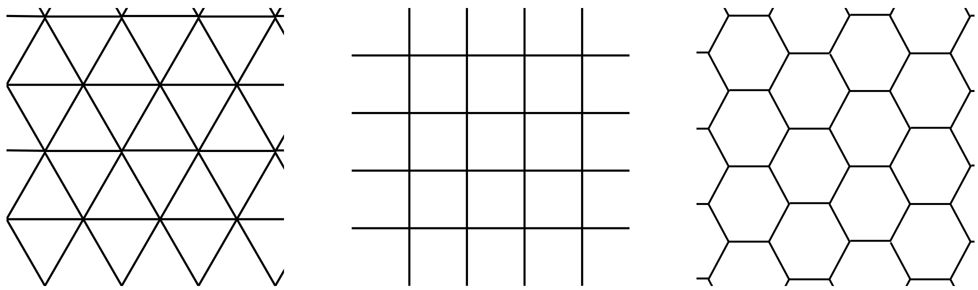


Figure 1.1.1: Regular Tessellations

Definition 1.1.1. A *semi-regular tessellation* (also called Archimedean tessellation) is a tessellation with the following properties:

1. It is composed of two or more kinds of regular polygons,
2. At all the vertex points the regular polygons are arranged identically. △

As with all tessellations, the regular polygons cover the plane without any gaps or overlaps. In order for the regular polygons to tessellate the plane, the sum of the angles they form around one point must add up to $360°$. By looking at the angle measurements of regular polygons, back in 1785, Rev. Mr. Jones was able to conclude that there are only 8 possible semi-regular tessellations.

Theorem 1.1.2. *There are only eight possible semi-regular tessellations.*

We will present a sketch of the proof following Jones [5].

Sketch of Proof. Looking at the measure of interior angles of polygons, we find there are 21 combinations of different polygons that have angles that add up to exactly $360°$ and can fill up the space around one point. From the 21 combinations only 8 form a pattern that can be extended indefinitely. □

These semi-regular tessellations are formed from combinations of triangles, squares, hexagons, and dodecagons. Seymour and Britton [5] assign the semi-regular tessellations names based on the sequence of regular polygons that goes around one point. The eight semi-regular tessellations are (4.6.12), (4.8.8), (3.12.12), (3.3.3.4.4), (3.3.4.3.4), (3.6.3.6), (3.3.3.3.6), and (3.4.6.4). Figure 1.1.3 and Figure 1.1.4 show these tessellations.

The *graph of a tessellation* is formed by the edges and vertices of the tessellation. Throughout this project we will use the notations in Table 1.1.2.

All eight semi-regular tessellations have a high degree of symmetry which we can express using the symmetry group. For the following definition, recall that an isometry of the plane is a transformation that preserves distance (such as rotation and reflection).

Definition 1.1.3. The symmetry group of a tessellation is the group of all isometries of the plane that map the tessellation to itself. △

All three regular tessellations have exactly three centers of rotational symmetry [5]. Each regular tessellation has one rotation at the center of the polygons, one at the vertices and one at the midpoints of the sides.

Semi-regular tessellation also display multiple rotational symmetries. However, there is no fixed pattern for all 8 semi-regular tessellations [5]. We observe that the symmetry groups also act on the vertices.

Name	Graph of
T_1	4.6.12
T_2	4.8.8
T_3	3.12.12
T_4	3.3.3.4.4
T_5	3.3.4.3.4
T_6	3.6.3.6
T_7	3.3.3.3.6
T_8	3.4.6.4

Figure 1.1.2: Notations for the Graphs of Semi-regular Tessellations

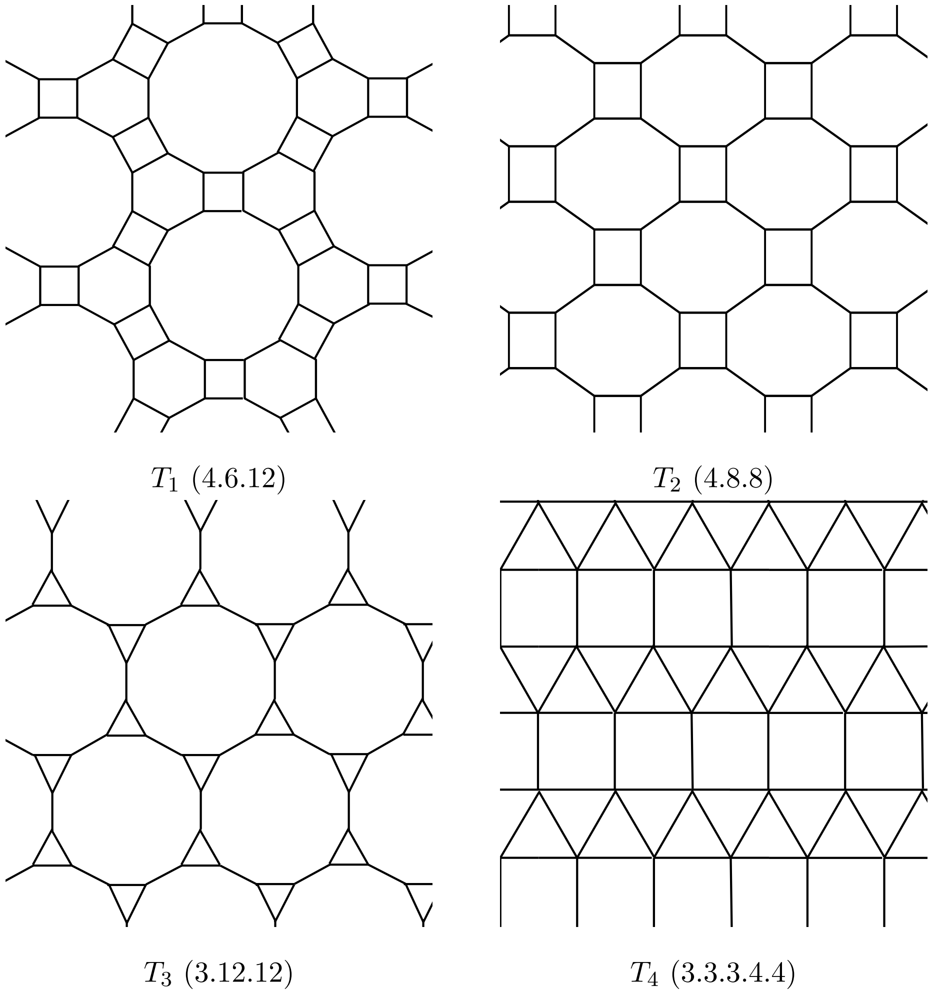
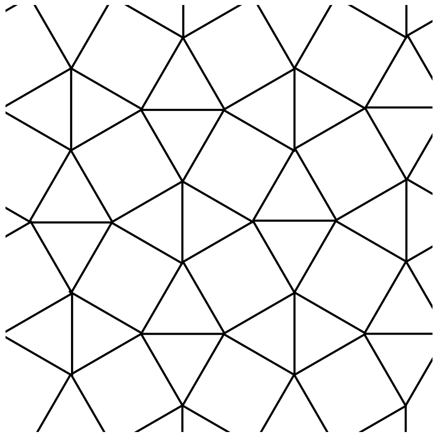


Figure 1.1.3: Semi-Regular Tessellations (4.6.12), (4.8.8), (3.12.12), (3.3.3.4.4)

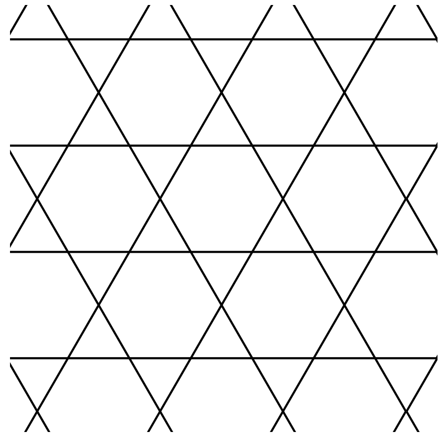
Theorem 1.1.4. *For each of the semi-regular tilings, the symmetry group acts transitively on the set of vertices. That is, for any two vertices v and w, there is an element of the symmetry group f such that $f(v) = w$.*

The proof of the above theorem should be obvious from looking at Figures 1.1.3 and 1.1.4.

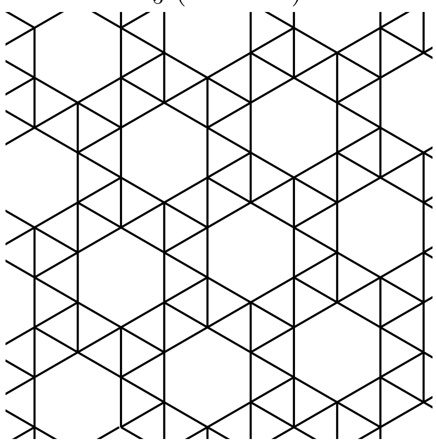
Due to this property, there is only one way to choose a vertex from a semi-regular tessellation. For our purposes this will mean that it does not matter where the fire starts on the graph.



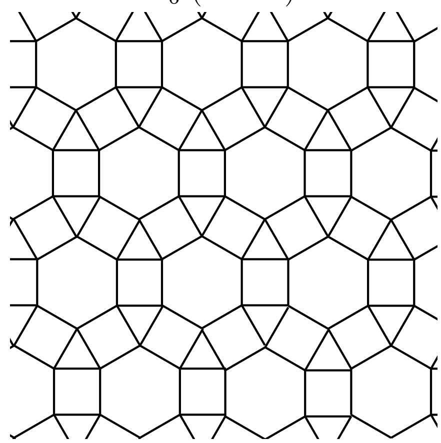
T_5 (3.3.4.3.4)



T_6 (3.6.3.6)



(3.3.3.3.6)



(3.4.6.4)

Figure 1.1.4: Semi-Regular Tessellations (3.3.4.3.4), (3.6.3.6), (3.3.3.3.6), (3.4.6.4)

1.2 The Firefighter Problem

In this section, we introduce the Firefighter Problem both informally in Subsection 1.2.1 and then formally in Subsection 1.2.2.

1.2.1 Basics

We start by presenting the basic rules of the Firefighter Problem and use some simple examples to better illustrate them.

The Firefighter Problem was first presented as follows: a fire breaks out at one vertex of a graph. On each subsequent turn, a number of firefighters are deployed to protect the other vertices from catching fire. Once a firefighter is placed at a vertex, it will remain there until the end of the process. For the purpose of this project we will always assume that the same number of firefighters is placed on each turn. After the firefighters are assigned, the fire spreads to all the neighboring vertices where a firefighter has not been placed. This process continues until the fire is surrounded by firefighters, or all the vertices have been burned. Once the fire can no longer spread, it is *contained*.

There are four types of vertices that we can come across:

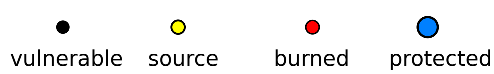
1. The *source* of the fire is the vertex denoted v_0, at which the fire starts.

2. A *protected* vertex has a firefighter assigned to it and can no longer catch fire.

3. A *burned* vertex has caught on fire.

4. A *vulnerable* vertex is any vertex that has not been protected, or burned.

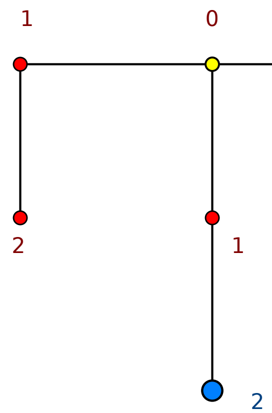
The picture below shows the four different types of vertices we will encounter. We will use these notations to illustrate the Firefighter Problem throughout the project.



To understand how the Firefighter Problem works, below we will show some simple examples. We use different numbers of firefighters per turn to see how that affects the spread of the fire.

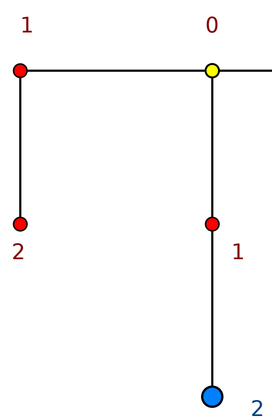
First we will show an example for a finite graph.

Example 1.2.1. Consider the tree T shown below. First we will use one firefighter per turn to contain this fire.



The fire starts at the source that is marked 0. On the first turn, we place a firefighter, marked 1, that blocks the fire from spreading further on the right side of the source. The fire then spreads to the two burning vertices marked 1. We then place our second firefighter at the protected vertex marked 2. The fire spreads again, to the burning vertex marked 2. At this point the fire is contained, it cannot spread to anymore vertices.

With one firefighter per turn we were able to contain the fire in 2 turns. Throughout this process four vertices were burned. Now we use two firefighters per turn to contain the fire on the same graph.



With two firefighters the fire is also contained in 2 steps. However, only two vertices are burned in the process. ◊

Now we will look at some examples for the spread of fire on some infinite graphs.

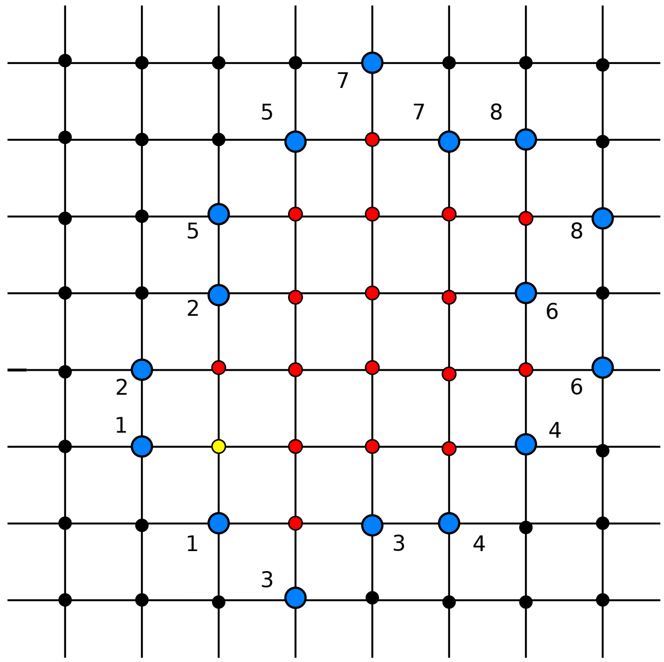
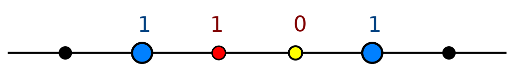


Figure 1.2.1: Two firefighters containing a fire on the square grid

Example 1.2.2. Let L be a line graph and let the fire start at v_0. We will use one firefighter to contain this fire.

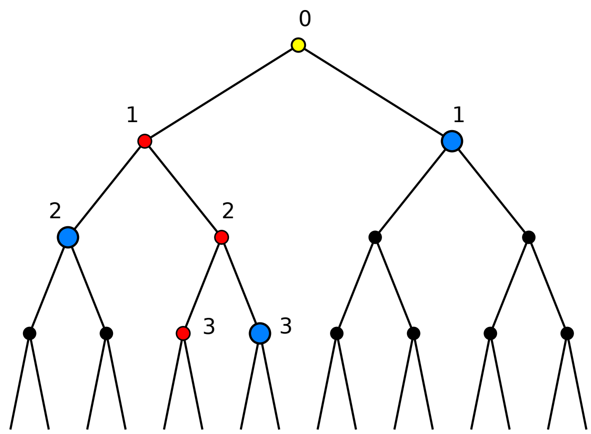


It takes two steps for one firefighter to contain a fire on an infinite line graph. ◊

Example 1.2.3. Let G be a square grid. On each turn we place two firefighters. The fire starts at v_0. It will take eight turns for the firefighters to contain the fire. This example is illustrated in Figure 1.2.1.

◊

Example 1.2.4. Let T_B be an infinite binary tree graph. Suppose we are using one firefighter.



We observe that after three steps the fire is not contained and it looks like one firefighter will not suffice to contain it on this graph.

We can see that 2 firefighters would have been able to contain the fire after the first turn, not allowing it to move further down the tree. ◊

In this project we will explore the way that a fire spreads through the edges and vertices of infinite graphs of regular and semi-regular tessellations. Our goal will be to determine the minimum number of firefighters needed to contain the fire on each of the eight semi-regular tessellations.

1.2.2 Definitions and Notations

In this subsection, we present a formal definition of the Firefighter Problem and the notations used throughout this project.

For the purpose of this project we will use the following notation:

- Let $G = (V, E)$ be a graph. Let $V(G)$ be the set of vertices of G and let $E(G)$ be the set of edges of G.
- Let $t = 0$ be the turn at which the fire starts and $t = \{1, 2, 3, ...n\}\}$ be the turns in which the fire is spreading.
- The fire starts at the source v_0.

- Let f be the number of firefighters used on each turn.
- Let $P_n \subseteq V(G)$ be the set of protected vertices after n turns, where $P_0 = \emptyset$.
- Let $B_n \subseteq V(G)$ be the set of burning vertices after n turns, where $B_0 = \{v_0\}$.

Definition 1.2.5. Let $v, w \in V(G)$. The *distance*, denoted $d(v, w)$ is the length of the shortest path of edges from v to w, where each edge is assumed to have length 1. △

Example 1.2.6. If $v = (x, y)$ and $v' = (x', y')$ are vertices of the infinite square grid, then

$$d(v, v') = |x - x'| + |y - y'|$$

. ♢

A vertex along with all its adjacent vertices form the neighborhood of that vertex. The neighborhood of a set of vertices is the union of the neighborhoods of each individual vertex.

Definition 1.2.7. Let $v \in V(G)$. The *neighborhood* of v is the set $N(v) = \{w \in V(G) \mid d(w, v) \leq 1\}$. More generally, if $S \subseteq V(G)$, the neighborhood of that set, denoted $N(S)$, is

$$N(S) = \bigcup_{v \in S} N(v)$$

. △

We will now define the firefighter strategies that decide the way firefighters are placed throughout the process. At every step, at most f new firefighters are placed.

Definition 1.2.8. Let $f \geq 1$. A *firefighter strategy* for f firefighters is a sequence of vertices $P_1 \subset P_2 \subset P_3 \subset \cdots$, where $|P_n - P_{n-1}| \leq f$ for any n. △

Note that this definition allows for placement of less firefighters per turn than available.

The firefighter strategy is a response to the way the fire spreads, the fire sequence. The fire can only spread to neighboring vertices that have not been protected.

Definition 1.2.9. Given a firefighter strategy P_n, the *fire sequence* $B_0 \subset B_1 \subset ...$ has the following properties:

1. $B_0 = \{v_0\}$ and $P_0 = \emptyset$.

2. $B_n = N(B_{n-1}) \setminus P_n$.

3. $P_n \cap B_{n-1} = \emptyset$. △

In the absence of firefighters, we have the following fire sequence:

$$B_0 = \{v_0\}, \quad B_1 = N(v_0), \quad B_2 = N(N(v_0)), \quad \cdots$$

If at some turn n the fire can no longer spread, then it is contained.

Definition 1.2.10. A firefighter strategy *contains* the fire, if there exists an n such that $B_n = B_{n+1}$. △

It follows from the previous definition that the fire is contained, if and only if $\bigcup_{n=1} B_n$ is finite.

Definition 1.2.11. The *firefighter number* of a graph is the minimum number of firefighters per turn needed to contain a fire on that graph. △

1.3 Previous Results

Several papers have looked at the firefighter problem on infinite graphs of regular tessellations. Here are some of the known results for those graphs.

In [6], Wang and Moeller looked at the Firefighter Problem specifically on the Cartesian square grid. Wang and Moeller showed that you need at least two firefighters per turn and eight turns to isolate a fire on the square grid [6]. First they established an upper bound for the number of firefighters, concluding that two per turn can successfully contain a fire.

We have previously shown how two firefighters per turn can contain a fire on the square grid in Figure 1.2.1.

[6] proposes and proves the following theorem. The proof will be discussed in Chapter 3.

Theorem 1.3.1. *One firefighter per turn does not suffice to contain a fire on the square grid.*

Thus the firefighter number for the square grid is exactly two.

In [3], Fogarty considers the number of firefighters needed to contain a fire on the triangular tessellation. Fogarty proves the following.

Theorem 1.3.2. *The firefighter number for the triangular tessellation is three.*

In [3], Fogarty also proves that no matter how late they arrive, two firefighters can still contain the fire in the infinite square grid and three in an infinite triangular tessellation.

[3] also provides an alternative proof for the firefighter number of the square grid. This proof is based on a structural theorem that can be applied to certain graphs. We will present this theorem by using the notations defined in Section 1.2.2 and the definition below.

The sphere of radius r in G, centered at v_0 includes the vertices that are exactly at distance r from v_0.

Definition 1.3.3. Let $r \geq 0$. The *sphere of radius* r centered at v_0, denoted S_r is the set $\{v \in G \mid d(v, v_0) = r\}$.

Theorem 1.3.4 (Fogarty's Theorem)**.** *Let $f \geq 1$. Suppose that for every $r \geq 0$ and every nonempty set $A \subseteq S_r$, the statement*

$$|S_r, \cap N(A)| \; \geq \; |A| \, + \, f$$

holds true. Then the fire cannot be contained with f firefighters per turn.

The proof for this theorem can be found [3].

Fogarty picks an arbitrary set of vertices A that are in the sphere (Definition 1.3.3) of radius r, of the vertex where the fire starts. If the set of their neighbors $N(A)$ that are in the sphere of radius $r+1$ from the origin is larger than $|A|$ and the numbers of firefighters available at each step, the fire cannot be contained. This simply implies that at each step there are more vertices that will catch fire than vertices that will be protected. Hence the fire cannot be contained.

We will use an infinite binary tree as an example for how Fogarty's theorem works.

Example 1.3.5. Let T_B be the infinite binary tree from Example 1.2.4. Recall that in Example 1.2.4 it seemed like the fire could not be contained with one firefighter per turn. We use Theorem 1.3.4 to prove this.

Let $A \subseteq S_r$ and let $|A| = n$. Then $N(A) \cap S_{r+1} = 2n$. It follows that $2n \geq n+1$, Theorem 1.3.4, it follows that the fire cannot be contained with one firefighter per turn. $\diamondsuit$

Fogarty's theorem can be used to provide a lower bound for the square grid and show one firefighter is not enough. However, it is difficult to use it for many other graphs since one would need to prove that the statement holds for any subset of a sphere.

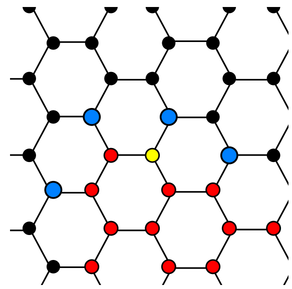
In [4], Messinger looks at the hexagonal grid. Her paper proposes and proves the following theorem:

Theorem 1.3.6. *Two firefighters per turn suffice to contain the fire on the hexagonal grid.*

In [4], Messiger conjectures the following:

Conjecture 1.3.7. *One firefighter is not enough to contain a fire on the hexagonal grid.*

The figure below shows how the fire spreads on the hexagonal grid, when we place one firefighter per turn.



After four turns we observe that the fire keeps getting larger. We can assume the fire will not be contained with one firefighter; however there is no proof for that.

2

Firefighter Number

In this chapter we will examine the spread of fire on the graphs of semi-regular tessellations. The goal is to determine the firefighter number for each of these graphs. Table 2.0.1 lists our main results. We illustrate how the number suggested in the table is enough to contain a fire on each of the eight semi-regular tessellations, thus establishing an upper bound for how many firefighters we need.

Graph	Firefighter Number
T_1	1
T_2	1
T_3	1
T_4	2 or 3
T_5	2 or 3
T_6	1 or 2
T_7	1 or 2
T_8	1 or 2

Table 2.0.1: Firefighter Number for Semi-Regular Tessellations

2.1 Upper Bounds

In this section we establish an upper bound for how many firefighters suffice to contain a fire on each of the eight semi-regular tessellations. We show an illustration for each semi-regular tessellation that proves that the number of firefighters suggested in Table 2.0.1 can contain the fire on each graph. Recall from Section 1.2.2 that f denotes the number of firefighters that are placed on each turn. First we look at the graphs for which one firefighter per turn is enough, then at those for which we need two firefighters per turn and finally at those where we need three firefighters per turn.

Theorem 2.1.1. *If $f = 1$, then we can contain a fire on T_1, T_2 and T_3.*

The proof for this theorem can be found in Figure 2.1.1.

We show that on T_1 we are able to contain the fire with one firefighter per turn after 6 turns. On T_2 we need at least 9 turns to surround the fire when placing on firefighter on each turn. One firefighter per turn can contain a fire on T_3 in only 4 turns.

Theorem 2.1.2. *If $f = 2$, then we can contain a fire on T_6, T_7 and T_8.*

This theorem is proved in Figure 2.1.2.

With two firefighters per turn, on T_6 we are able to contain the fire after 5 turns. On T_7 we contain the fire after 7 turns and on T_8 after 8 turns.

Theorem 2.1.3. *If $f = 3$, then we can contain a fire on T_4 and T_5.*

We prove this theorem in Figure 2.1.3. Three firefighters per turn contain a fire on both T_4 and T_5 after only 3 turns.

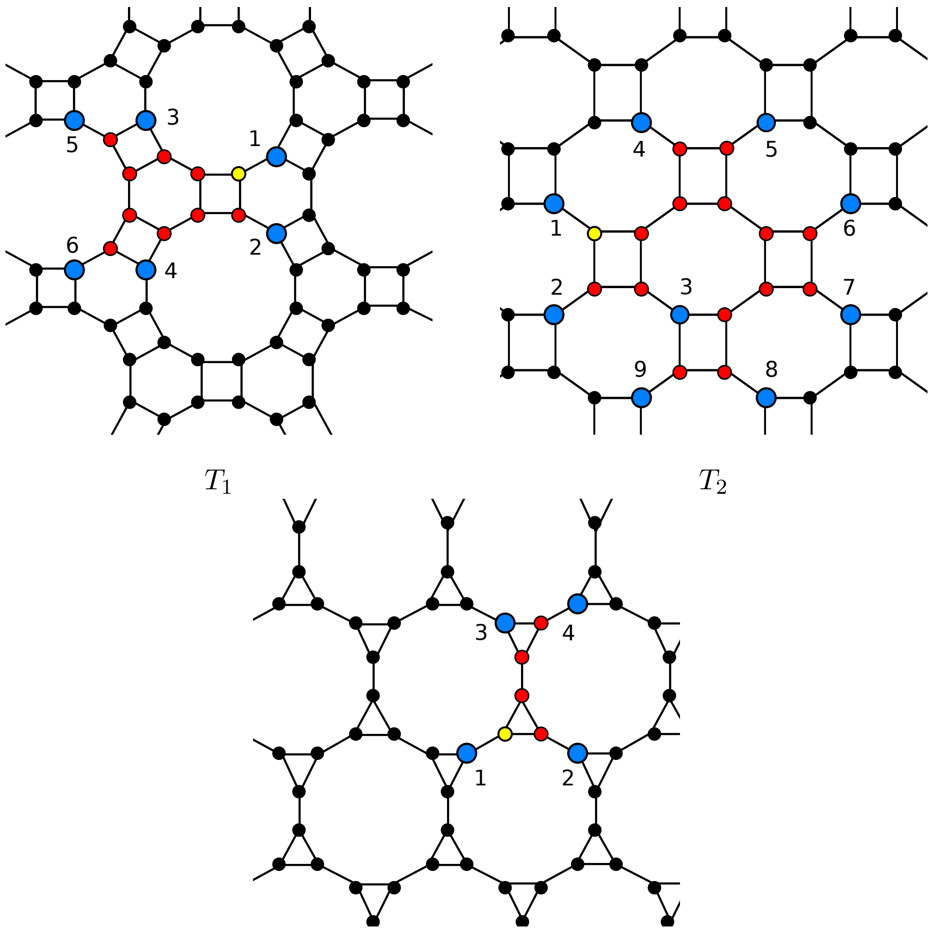
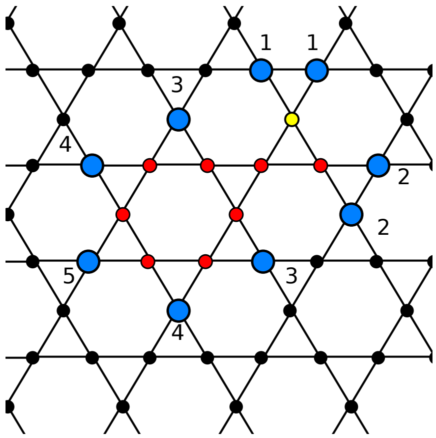
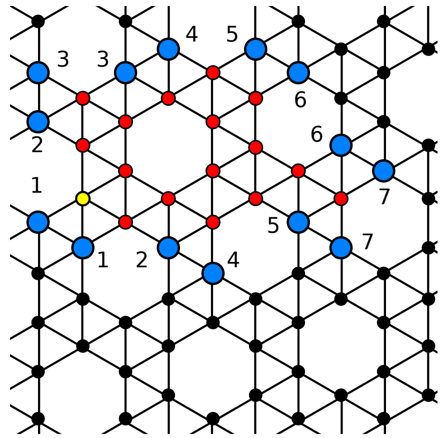


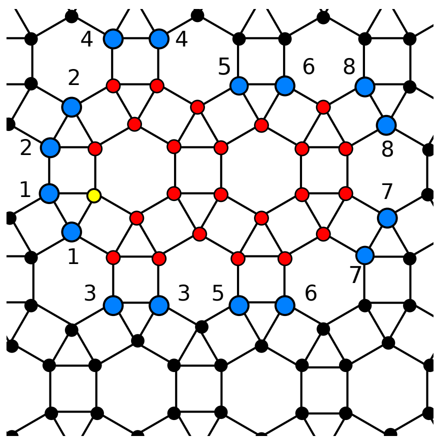
Figure 2.1.1: Containing fires on T_1, T_2 and T_3 with one firefighter



T_6

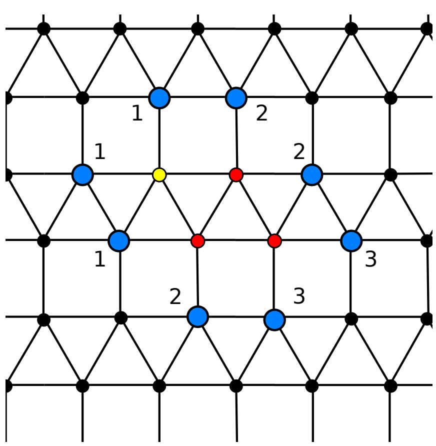


T_7

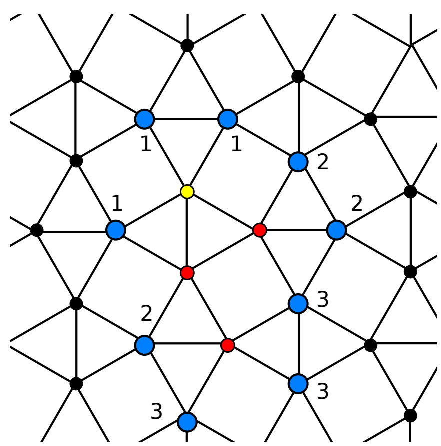


T_8

Figure 2.1.2: Containing a fire with two firefighters



T_4



T_5

Figure 2.1.3: Containing a fire with three firefighters

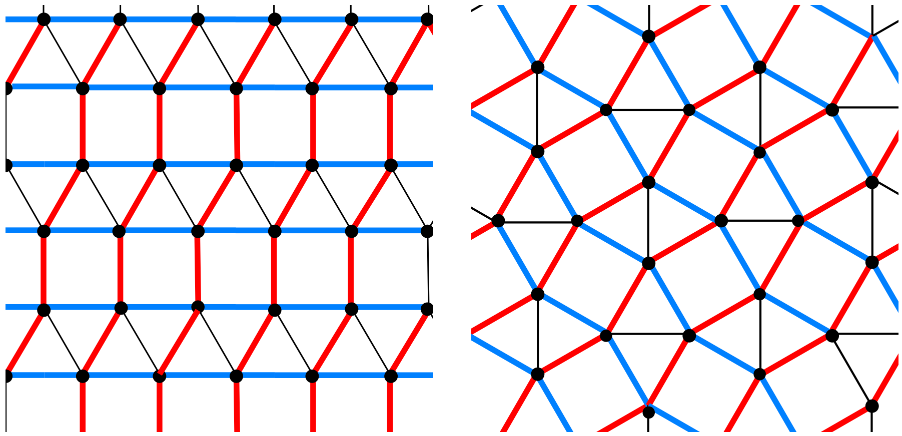


Figure 2.2.1: Square grid as subgraph of T_4 and T_5

2.2 Lower Bounds

Clearly for T_1, T_2, and T_3, where the fire can be contained with only one firefighter per turn, we do not need to establish a lower bound. In this section we will establish that one firefighter per turn cannot contain a fire on T_4 and T_5.

Theorem 2.2.1. *One firefighter per turn does not suffice to contain a fire on T_4 and T_5.*

We observe that both T_4 and T_5 contain the square grid as a subgraph. Figure 2.2.1 illustrates this. To prove Theorem 2.2.1, we first need to prove the following theorem:

Theorem 2.2.2. *Let G' be a subgraph of G and let G' have the same vertex set as G. If f firefighters suffice to contain a fire on G, then they suffice to contain it on G'.*

We will use induction to prove that the set of burning vertices in G' is a subset of the set of burning vertices in G. Throughout this proof we will use the notations provided in Section 1.2.2.

Proof. Let $P_1 \subseteq P_2 \subseteq ... \subseteq P_n$ be a firefighter strategy that contains a fire on G with f firefighters. Let $B_0 \subseteq B_1 \subseteq \cdots$ be the corresponding fire sequence on G. Let $P_1 \subseteq P_2 \subseteq \cdots$

be the firefighter strategy for G' and let $B'_0 \subseteq B'_1 \subseteq B'_2 \subseteq ...$ be the corresponding fire sequence on G'.

We claim that $B'_n \subseteq B_n$. We proceed by induction on n.

Base Case: At turn 0, we have $B'_0 = \{v_0\}$ and $B_0 = \{v_0\}$, so $B'_0 \subseteq B_0$.

Induction Step: Suppose $B'_{n-1} \subseteq B_{n-1}$. Then clearly, $N(B'_{n-1}) \subseteq N(B_{n-1})$. By Definition 1.2.9, $B'_n = N(B'_{n-1}) \setminus P_n$ and $B_n = N(B_{n-1}) \setminus P_n$. It follows that $B'_n \subseteq B_n$.

We have shown that $B'_n \subseteq B_n$ for all n.

Since the given strategy contains a fire on G, we know that $\bigcup_{1 \longrightarrow \infty} B_n$ is finite. Hence $\bigcup_{1 \longrightarrow \infty} B'_n$ is also finite. Therefore, the fire is contained on G'. □

From Theorem 1.3.1, we have the following corollary:

Corollary 2.2.3. *Let G be a subgraph of G and let G' and G have the same vertex set. Let f' be the firefighter number of G' and f be the firefighter number of G. Then $f' \leq f$.*

We can now prove Theorem 2.2.1.

Proof. The square grid is a subgraph with the same vertex set of T_4 and T_5. From [6], we know that one firefighter per turn cannot contain a fire on the square grid. From Corollary 2.2.3, it follows that one firefighter per turn cannot contain a fire on T_4 and T_5.

□

2.3 Fogarty's Theorem on T_6, T_7 and T_8

Recall Theorem 1.3.4 from Section 1.3.

Theorem 2.3.1 (Fogarty's Theorem)**.** *Let $f \geq 1$. Suppose that for every $r \geq 0$ and every nonempty set $A \subseteq S_r$, the statement*

$$|S_{r'} \cap N(A)| \;\geq\; |A| \,+\, f$$

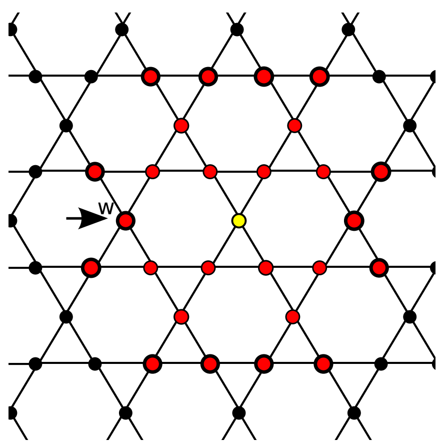
holds true. Then the fire cannot be contained with f firefighters per turn.

This theorem can be applied to graphs in an attempt to establish a lower bound for the number of firefighters per turn needed to contain a fire. We suppose that $f = 1$.

We show that Theorem 1.3.4 cannot establish the lower bound of one firefighter per turn for T_6, T_7 and T_8. To do so we provide a counter example for a vertex set for which the claim in Theorem 1.3.4 is not true.

Theorem 2.3.2. *For T_6, there is a nonempty subset $A \subseteq S_3$ so that $|S_4 \cap N(A)| < |A| + 1$.*

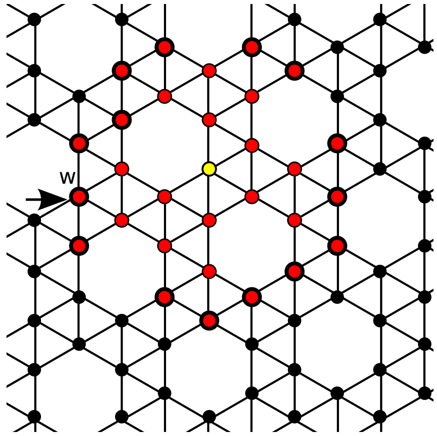
Proof. In the figure below, we can observe the spread of the fire in the first 3 spheres. Note that the highlighted vertices are in S_3, so $w \in S_3$. Let $A = \{w\}$. Then $|S_4 \cap N(A)| = \emptyset$.



□

Theorem 2.3.3. *For T_7, there is a nonempty subset $A \subseteq S_3$ so that $|S_4 \cap |N(A)| < |A| + 1$.*

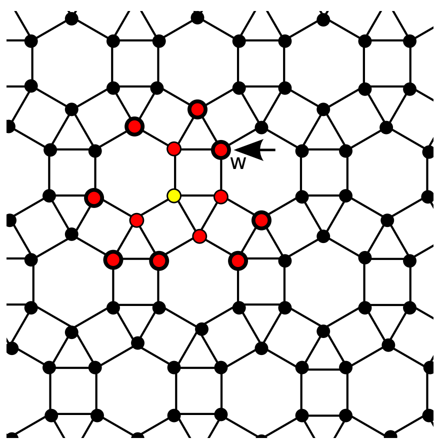
Proof. In the figure below, we can observe the spread of the fire in the first 3 spheres. Note that the highlighted vertices are in S_3, so $w \in S_3$. Let $A = \{w\}$. Then $|S_4 \cap N(A)| = 1$.



□

Theorem 2.3.4. *For T_8, there is a nonempty subset $A \subseteq S_2$ so that $|S_3 \cap N(A)| < |A|+1$.*

Proof. The figure below illustrates the spread of the fire in the first 2 spheres. Note that the highlighted vertices are in S_2, so $w \in S_2$. Let $A = \{w\}$. Then $|S_3 \cap N(A)| = 1$.



□

3
Geodesics

In this chapter we explore the properties of T_6 with the goal of establishing a firefighter number for this graph. We reformulate Wang and Moeller's proof [6] for why one firefighter does not suffice to contain a fire on the square grid in terms of the properties of certain geodesics and propose a general theorem for when one firefighter per turn does not suffice to contain a fire on a graph. Next, we provide a labeling scheme for paths on T_6 and we completely classify the geodesics on that graph. Using this approach we obtain some partial results about firefighter strategies on T_6. However, our results are not strong enough to prove that one firefighter cannot contain a fire.

3.1 Reformulation of Wang and Moeller's Proof for the Square Grid

In this section we will provide the definitions necessary to understand Wang and Moeller's proof for why one firefighter does not suffice to contain a fire on the square grid. We then present and explain a generalized version of their proof that uses certain properties of geodesics.

Throughout this section we assume that G is a graph on which a fire can be contained with one firefighter after n turns. We use the following definitions:

Definition 3.1.1. A *geodesic* between $v, w \in V(G)$ is a path from v to w of length $d(v, w)$. △

Definition 3.1.2. An *infinite geodesic* is an infinite path, on which the subpath between any two vertices is a geodesic. △

Definition 3.1.3. A *fire-path* is a path between two vertices on which all intermediate vertices (not counting the endpoints) are burned. △

Definition 3.1.4. The *fire-distance* $d_f(v)$ is the length of the shortest fire path from v_0 to v. △

Definition 3.1.5. A *major firefighter* is a firefighter placed at a vertex $v \in P_n$, where $d(v_0, v) = d_f(v)$. △

A major firefighter is a protected vertex v for which there exists a geodesic fire-path from v_0. We denote the *set of major firefighters* as M. Note that $M \subseteq P_n$. Wang and Moeller refer to the set M in as the *Interior Boundary Vertices* [6].

Figure 3.1.1 shows an example of M.

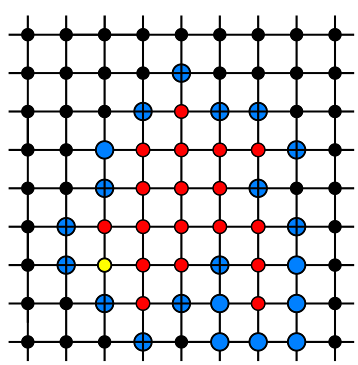


Figure 3.1.1: Contained fire on a grid. Crossed out firefighters are major firefighters, the other firefighters are not.

At the turn we place a major firefighter, the distance from v_0 to that firefighter will be the longest distance from the v_0 to any other vertex that is burned or protected.

Proposition 3.1.6. *Let $v \in P_n$ be a major firefighter. If v was placed on turn k, then $d_f(v) \geq k$.*

Proof. Let v be a major firefighter placed on turn k. Then the fire-path from v_0 to v is a geodesic. Since v was placed on turn k, the fire will have already spread the first k steps. □

We observe that there are at most as many major firefighters as the maximum distance from the fire to all the major firefighters [6].

Proposition 3.1.7. $|M| \leq \max_{v \in M} d_f(v)$.

Proof. Suppose the last major firefighter placed on the vertex v, at turn k. Since we can place at most one firefighter per turn, it follows that $|M| \leq k$. From Proposition 3.1.6, we know that $k \leq d_f(v)$. Hence $|M| \leq \max_{v \in M} d_f(v)$. □

Definition 3.1.8. Let g be a geodesic path from v_0 to a vertex v. Let w be a vertex on g, other than v and let g' be an infinite geodesic path starting at w. We say that g' is a *vulnerable geodesic*, if the following conditions are met:

1. The paths g and g' only intersect at w.

2. The path that starts at v_0 and follows g to w and then follows g' from w is a geodesic. △

Definition 3.1.9. Two vulnerable geodesics are *disjoint*, when they do not have any vertex in common after the initial vertex. △

The following theorem is a generalization of Theorem 1.3.1 found in [6].

Theorem 3.1.10. *Let G be a graph and let the fire start at v_0. Suppose that every geodesic of length k, starting at v_0 has at least k disjoint vulnerable geodesics coming off it. Then the fire cannot be contained with one firefighter per turn.*

Proof. Suppose the fire is contained on G. Note that there is at least one major firefighter, since every geodesic starting at v_0 must contain a major firefighter. Let v be the major firefighter that is furthest away from v_0 and let g be a geodesic fire-path of length k from v_0 to v. By hypothesis, there are k disjoint vulnerable geodesics coming off of g. Since the fire is contained, a major firefighter has to be placed on each of the disjoint vulnerable geodesics. So there are at least k major firefighters besides v. Then $|M| \geq k + 1$. From Proposition 3.1.7, we know that $|M| \leq k$, so we have a contradiction. □

We now use Theorem 3.1.10 to prove that we cannot contain a fire on the square grid with one firefighter.

Theorem 3.1.11. *On the square grid, one firefighter per turn cannot contain a fire.*

Proof. Let G be the graph of a square grid and let the fire start at v_0. Let g be a geodesic of length k from v_0 to a vertex v. We show that there are at least k vulnerable geodesics coming off g.

The path g is a geodesic on the square grid. Hence it can move it at most two directions. Suppose without loss of generality, the two directions are up and right. From any vertex on g we can form a path that is a geodesic. When g moves up, we continue our path by moving right and still have a geodesic. When g moves right, then we continue our path by moving up and still have a geodesic. We let the new geodesics move in only one direction, so they are disjoint. We can observe how the vulnerable geodesics start from an initial geodesic in Figure 3.1.2. Without counting the endpoint vertex v, there are at least k

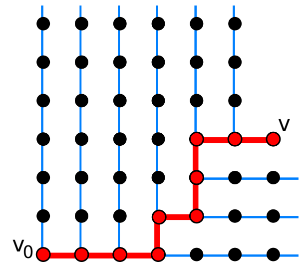


Figure 3.1.2: Geodesic of length 9 on the square grid with 9 disjoint vulnerable geodesics

vulnerable geodesics starting off g. From Theorem 3.1.10 it follows that one firefighter per turn cannot contain a fire on the square grid.

□

3.2 Properties of T_6

In this section, we develop a system of labels for describing paths in T_6. We distinguish between different types of vertices based on the the directions we can move in from them. We then determine what sequences of directions describe a path on T_6 and what sequences of directions describe a geodesic.

Recall the graph T_6 in Figure 3.2.1.

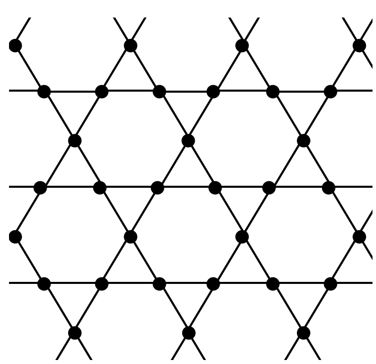


Figure 3.2.1: T_6

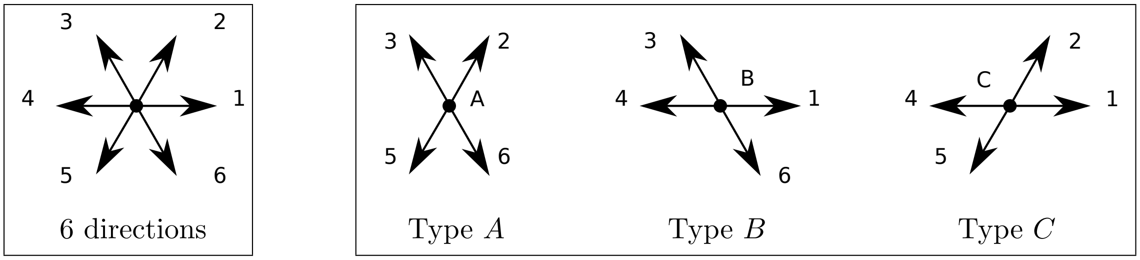


Figure 3.2.2: Directions a path can move in on T_6 and vertex types

3.2.1 Paths on T_6

This subsection will present the directions the paths can move in and explain the way paths move from one vertex to another on T_6.

A path on the graph T_6 can move in 6 different directions, $\{1, 2, 3, 4, 5, 6\}$ as shown in Figure 3.2.2. When looking at T_6, we observe that not all vertices have paths moving in all the 6 possible directions. From each vertex we can move in 4 directions. Considering the directions in which a path can move, three types of vertices can be identified on T_6. We will refer to them as vertices Type A, Type B and Type C. The types of vertices are shown in Figure 3.2.2.

Any path on T_6 will move through the three different types of vertices, following specific rules.

Definition 3.2.1. A vertex is type A if starting from it you can move in directions 2, 3, 5, 6. A vertex is type B if from it you can move in directions 1, 3, 4, 6. A vertex is type C if from it you can move in directions 1, 2, 4, 5. $\triangle$

When we move along a path on T_6, we move in one of the six directions mentioned above. Due to the way the graph is structured, after each move we will arrive at a different type of vertex. For example, if we start at a vertex type A after one move we will be either at a vertex type B, or at a vertex type C, depending on which direction we moved in. To give a better understanding of how a path moves, we will look at the diagram in Figure 3.2.3.

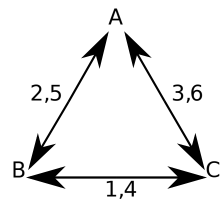


Figure 3.2.3: Movement between types of vertices

We observe the following properties of paths on T_6:

1. Suppose a path starts at a vertex type A. If it moves in directions 2 or 5, then it will arrive at a vertex type C. If it moves in directions 3 or 6, then it will arrive at a vertex type B.

2. Suppose a path starts at a vertex type B. If it moves in directions 1 or 4, then it will arrive at a vertex type C. If it moves in directions 3 or 6, then it will arrive at a vertex type A.

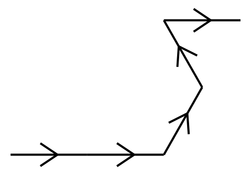
3. Suppose a path starts at a vertex type C. If it moves in directions 2 or 5, then it will arrive at a vertex type A. If it moves in directions 1 or 4, then it will arrive at a vertex type B.

We will describe a path on T_6 through the direction sequence it moves in.

Definition 3.2.2. The *direction sequence* describes a path by listing the directions it moves in from one vertex to the next one. △

We provide some examples to clarify this notation.

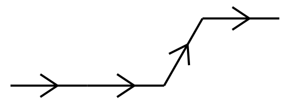
Example 3.2.3. The sequence 11231 describes the path illustrated below.



◊

Note that not all sequences of directions constitute a path on T_6.

Example 3.2.4. The sequence 1121 describes the following path:



By looking at Figure 3.2.1 we can observe this sequence does not describe a path on T_6. ◊

Definition 3.2.5. A *valid path* is described by a direction sequence that constitutes a path on T_6. △

From the above mentioned properties of paths, we can determine which sequences of vertices cannot form a valid path on T_6.

3.2.2 Properties of Paths on T_6

In this subsection, we explain the symmetry and reflection of paths on T_6 by using same basic group theory.

To understand these properties we will provide some definitions following Armstrong [1].

Definition 3.2.6. The *dihedral group* D_6 is the symmetry group of a hexagon. △

This group has two types of elements:

1. Counterclockwise rotations of the hexagon through multiples of $\frac{\pi}{3}$ around its center.

2. Reflections around an axis of symmetry that lies in the plane.

The dihedral group D_6 has twelve elements consisting of:

1. Six rotations $r^0, r^1, r^2, r^3, r^4, r^5$, where each r^k, for $k \in \{0, 1, 2, 3, 4, 5\}$, rotates the hexagon counterclockwise around its center by $\frac{k\pi}{3}$.

2. Six reflections, $s_1, s_2, s_3, s_4, s_5, s_6$. We can see these lines of reflection in Figure 3.2.4

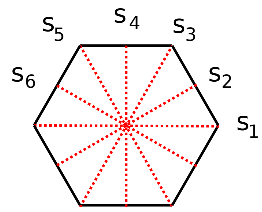


Figure 3.2.4: Symmetry lines of the hexagon

Each of the elements of D_6 permutates the six directions as shown in Figure 3.2.1.

Table 3.2.1: Permutations of directions on T_6 through elements of the dihedral group

Element	Permutation
r^0	identity
r^1	$(1\,2\,3\,4\,5\,6)$
r^2	$(1\,3\,5)\,(2\,4\,6)$
r^3	$(1\,4)\,(2\,5)\,(3\,6)$
r^4	$(1\,5\,3)\,(2\,6\,4)$
r^5	$(1\,6\,5\,4\,3\,2)$

Element	Permutation
s_1	$(2\,6)\,(3\,4)$
s_2	$(1\,2)\,(4\,5)\,(3\,6)$
s_3	$(1\,3)\,(4\,6)$
s_4	$(1\,4)\,(2\,3)\,(5\,6)$
s_5	$(1\,5)\,(2\,4)$
s_6	$(1\,6)\,(2\,3)\,(3\,4)$

We show a simple example of how we apply these elements to sequences of paths.

Example 3.2.7. The sequence 15516 corresponds to a valid path on T_6. We apply s_1 to this sequence. Then $s_1(15516) = 12213$, a valid path. ♢

Note that we can also reverse the direction of a path by switching the start and the endpoint. When reversing the direction, we

- reverse the order of the sequence,
- change each direction to its opposite.

Example 3.2.8. We reverse the direction of the path 15516. We then obtain the path 34224. ♢

Definition 3.2.9. Two direction sequences are *equivalent*, if one can be obtained from the other one through elements of the dihedral group, or through reversing the directions. △

Proposition 3.2.10. *Let $d_1 \cdots d_n$ be a sequence of directions and let $e_1 \cdots e_n$ be equivalent direction sequences.*

1. *If $d_1 \cdots d_n$ corresponds to a valid path, then $e_1 \cdots e_n$ corresponds to a valid path.*

2. *If $d_1 \cdots d_n$ corresponds to a geodesic, then $e_1 \cdots e_n$ corresponds to a geodesic.* △

3.2.3 Forbidden sequences on T_6

Proposition 3.2.11. *No valid path on T_6 contains any of the following sequences or their equivalents:*

1. *121*

2. *1232*

3. *1223*

Proof. We will show that 121 is not a valid sequence for a path on T_6. For a path to be able to move in direction 1, it must start at a vertex type B or C.

Case 1: Suppose we start at a vertex type B and move in direction 1. We arrive at a vertex type C, from which we move further in direction 2. Then we arrive at a vertex type A and cannot further move in direction 1.

Case 2: Suppose we start at vertex type C. We move in direction 1 and arrive at a vertex type B. From there we cannot move further in direction 2.

Similar arguments can be used to show that the sequences 1232 and 1223 do not form valid paths on T_6. □

3.2.4 Geodesics on T_6

In this subsection we define geodesics on T_6. We use the following notations:

- When a path moves in the same direction, for example 1, n times, we use the notation $(1)^n$.

- Any two directions that differ by 1 mod 6 will be referred to as *consecutive directions.*

First we propose what the form of a geodesic on T_6 is.

Theorem 3.2.12. *A path on T_6 is a geodesic, if and only if it is of one of the following types:*

1. *Type I: The path moves in one single direction.*

2. *Type II: The path only moves in two consecutive directions (e.g 2 and 3, or 6 and 1).*

3. *Type III: The path moves in three consecutive directions and takes the form $1(2)^n3$, or the equivalent.*

The proof of this theorem will occupy the remainder of this section. We will introduce rules on how to shorten a path on T_6. We will then show that for any path, when we can no longer apply any shortening rules, we have a geodesic of the form mentioned in Theorem 3.2.12.

Proposition 3.2.13. *Any path on T_6 is either a Type I, Type II or Type III geodesic, or it can be shortened.*

We begin by introducing the rules for shortening paths on T_6.

The following rules and their equivalents (Subsection 3.2.2) can be used to shorten a valid path on T_6 that is not already a geodesic.

Proposition 3.2.14. *Shortening Rules*

1. *In the sequence 14, the two cancel each other.*

2. *A sequence of the form 13 can be shortened to 2.*

3. A sequence of the form $11(2)^n3$ *can be shortened to* $(2)^{n+1}1$.

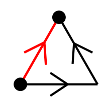
4. A sequence of the form $1(2)^n33$ *can be shortened to* $3(2)^{n+1}$.

5. A sequence of the form 1234 *can be shortened to* 32.

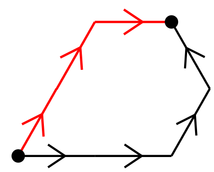
Proof. Shortening Rules

Rule 1: If we move in direction 1 and then in direction 4, we arrive at the same vertex.

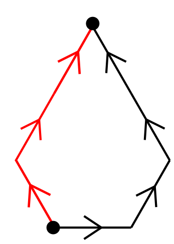
Rule 2: By moving in direction 2 instead of 13, we shorten the path by one step, as seen in the picture below.



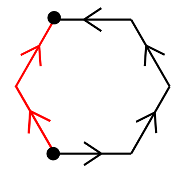
Rule 3: By moving in the directions $(2)^{n+1}1$ instead of $11(2)^n3$ our path will be shortened by one step.



Rule 4: By replacing $1(2)^n33$ with $3(2)^{n+1}$ we have a path that is one step shorter.



Rule 5: By moving in directions 32 instead of 1234 we have a path that is shorter by two steps.



□

As stated in Theorem 3.2.12, any geodesic can move in at most 3 different directions. We will show that if we apply the above rules, any valid path that contains both a step in direction 1 and one in direction 4 can be shortened.

Proposition 3.2.15. *In any geodesic, every pair of directions in a row must be equal or consecutive (i.e.* 33*,* 34 32*).*

Proof. By Shortening Rules 1 and 2 we can shorten any pair of directions that are not equal or consecutive. □

Proposition 3.2.16. *Any path that contains both a step in direction* 1 *and a step in direction* 4 *can be shortened.*

Proof. Suppose we have a valid path on T_6 that contains both a 1 and a 4 in the sequence of directions.

We can also assume without loss of generality that the path starts at 1 and ends at 4, without any other 1 or 4 in between. If there is more than one 1, or more than one 4 we simply pick the sequence that is in between a 1 and a 4. We will look at all the possible forms this path can take.

From Proposition 3.2.15, it follows that the path must contain 2s and 3s, or 5s and 6s in between the 1 and the 4.

Without loss of generality we can suppose the path starts with $1(2)^n3$.

From Proposition 3.2.3 (1), we know that 232 does not constitute a valid path. Hence the next digit in the sequence cannot be 2.

If the next digit in the sequence is 3, then our path starts with $1(2)^n33$, which can be shortened to $3(2)^{n+1}$ by Rule 4 (Proposition 3.2.14).

Thus the next digit in the sequence is 4, so our sequence starts with $1(2)^n33$.

Case 1: If $n = 1$, then we have the sequence 1234. By Shortening Rule 5, we can change that to 34.

Case 2: If $n > 1$, then we have the sequence $1(2)^{n-2}2234$. By Shortening Rule 4, we can change 2234 to 332.

Therefore we can conclude that any path containing both a 1 and a 4 can be shortened to no longer contain both directions. □

Now that we know that we can reduce any path to 3 directions, we will show that any sequence containing three consecutive directions can either be shortened with one of the rules, or is a geodesic. Without loss of generality we will use 1s, 2s and 3s to illustrate this.

Proposition 3.2.17. *We can shorten any path that contains only 1's, 2's and 3's and is not of the form of a geodesic given in Theorem 3.1.10.*

Proof. From Proposition 3.2.14, we know that if the sequence does not contain both 1s, 2s, and 3s we can arrive at a Type I or Type II geodesic. If the sequence takes the form $1(2)^n3$, then we have a Type III geodesic.

From Proposition 3.2.15, it follows that any pair of directions in the sequence must be consecutive. We know that $21(2)^n3$ is not a valid path (Proposition 3.2.3).

Without loss of generality the sequence has to contain $11(2)^n3$. We can shorten that to $3(2)^{n+1}$. □

3.2.5 Disjoint Paths of Geodesics on T_6

In this subsection we will look at the number of disjoint paths coming off geodesics on T_6. We then try to incorporate what we have established about geodesics on T_6 into a proof that one firefighter cannot contain a fire on this graph. We have already established in Chapter 2 that two firefighters per turn suffice to contain a fire on T_6.

We establish a minimum number of disjoint paths that come of any geodesic of length k, for any $k \in \mathbb{N}$.

Theorem 3.2.18. *There are at least k disjoint paths coming out of any geodesic of length k on T_6.*

Proof. Let g be a geodesic of length k on T_6.

Assume g is a Type I or Type II geodesic. We can suppose without loss of generality that the path only uses directions 2 and 3.

Case 1: Suppose the path starts and ends at a vertex type A. Hence there are $\frac{k}{2}$ vertices of type A and $\frac{k-2}{2}$ vertices of type B or C. From any vertex type B or C we can start 2 disjoint paths, moving in directions 1 and 4. This means there are at least $2(\frac{k-2}{2}) = k - 4$ disjoint paths starting from these vertices. There are two disjoint paths in directions 2 and 3 from one endpoint and two disjoint paths in directions 5 and 6 from the other endpoint. It follows there are at least k disjoint paths.

Case 2: Suppose the path starts at a vertex type A and ends a vertex type B or C. Then there are $\frac{k-1}{2}$ vertices type B or C. Hence there are at least $k - 2$ disjoint paths starting from those vertices. There are 2 disjoint paths starting from the endpoint that is type A. Therefore, we can conclude that there are at least k disjoint paths.

Case 3: Suppose the path starts and ends at a vertex type B or C. Then there are $\frac{k}{2}$ vertices type B or C. Hence there are at least k disjoint paths.

Now suppose g is a Type III geodesic. Without loss of generality, we can assume that the path only uses directions 4, 3, and 2. From Theorem 3.1.10, we know the path will take the form $4(3)^{k-3}2$. Then the path starts at a vertex type B and moves in direction 4 to a vertex type C. From the starting vertex we can start 2 geodesics, one moving in direction 1, and one in direction 5. From the next vertex in the sequence, we can start a geodesic in direction 4. We are left with a path of length $k-2$ that only moves in directions 3 and 2. On that path there are $\frac{k-4}{2}$ vertices type B or C. From each we can start 2 disjoint geodesics. From the endpoint we can also start a geodesic moving in direction 2. It follows there are at least k disjoint geodesics coming off g. □

Remark 3.2.19. Note that not all disjoint paths coming off a geodesic on T_6 are vulnerable geodesics. ♢

Unfortunately, this implies that not all the firefighters that need to be placed on each of those paths to contain the fire are major firefighters. This means that for T_6 an argument similar to that made in Proof 3.1.10 does not hold.

We have however learned a few things about T_6 that can be used in the future to prove that one firefighter cannot contain a fire on this graph.

Assuming we cannot show that the fire is contained, then the last firefighter placed could not have been a major firefighter. If that was the case, we would be able to show that there are not enough firefighters to place them along the disjoint path coming off the geodesic that leads to that major firefighter.

Corollary 3.2.20. *If one firefighter per turn can contain a fire on T_6, then not all the firefighters are major firefighters.*

However, assuming the fire is contained with only one firefighter, then there are firefighters that surround the fire. In that case there should be a way to optimize the use of firefighters and remove all the firefighters that are not major firefighters.

Conjecture 3.2.21. *If we can contain the fire on T_6 with 1 firefighter, we should be able to do so by only using major firefighters.*

Proof. Suppose the fire can be contained with only major firefighters. Let v be the major firefighter that is furthest away from v_0 and let g be a geodesic fire-path of length k from v_0 to v. From Theorem 3.2.18, we know there are k disjoint vulnerable geodesics coming off of g. Since the fire is contained, a major firefighter has to be placed on each of the disjoint vulnerable geodesics. So there are at least k major firefighters besides v. Then $|M| \geq k+1$. From Proposition 3.1.7, we know that $|M| \leq k$, so we have a contradiction. □

Together Corollary 3.2.20 and Conjecture 3.2.21 imply that the fire cannot be contained on T_6 with one firefighter.

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