# A Three-Element Generating Set for Thompson's Group $F$

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# Abstract

In this paper we investigate Cayley graphs, which provide a geometric structure to algebraic groups. Specifically, we apply this structure to Thompson's group $F$, which is a certain group of piecewise linear homeomorphisms typically seen as generated by two elements. We include a third generator and examine the effect this generator has on the Cayley graph of $F$. With respect to the new generating set, we then find a length formula for elements of $F$, and show that the Cayley graph of $F$ has no dead ends. Finally, we show that there exists a non-constant bounded harmonic function on this graph, and provide some lower bounds for it.

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# Dedication

To Sneakers, my oldest friend.

# Acknowledgments

I would like to thank my advisor Jim Belk for the great ideas and long hours he donated to make this project happen. I would like to thank Lauren Rose for encouraging me to attempt a math senior project in the first place — this literally would not have been possible without her! I would also like to thank Rebecca Thomas for agreeing to be a member of my board. I thank all the faculty and students of the Bard Math department for their helpfulness, intelligence, and enthusiasm. I thank my family for their unwavering support and encouragement. Finally, I thank my friends for making a valiant attempt to understand and be interested in my project.

# 1
# Background

## 1.1 Cayley Graphs

Given a finitely generated group $G$, the Cayley graph of $G$ is a graph which describes the structure of the group. Informally, a Cayley graph has one vertex for element of a group, and edges representing multiplication by generators of the group.

**Definition 1.1.1.** Let $G$ be a group generated by a finite set $X$. The *Cayley graph* $\Gamma = \Gamma(G, X)$ is a directed graph constructed as follows.

1. The vertices of $\Gamma$ are the elements of $G$.
2. If $g, g \in G$, then $g$ and $h$ are connected by an edge in $\Gamma$ if and only if $h = gx$ for some $x \in X$. $\triangle$

If $g$ is a vertex of a Cayley graph, it can be seen that for each generator $x$ of $G$, $g$ will have an edge going from $g$ to $gx$. Likewise, for each generator there will be an edge from $gx^{-1}$ to $g$. As such, if $X$ is our generating set, each vertex will have $2|X|$ edges extending from it.

Cayley graphs are interesting primarily because they give a geometric structure to a group, which in turn allows us to use geometric techniques to analyze algebraic objects. Specifically, a Cayley graph on a group provides a metric, or a notion of distance, on the group.

**Definition 1.1.2.** Let $v, w$ be vertices of a Cayley graph $\Gamma$. The *distance* between $v$ and $w$ is the number of edges on the shortest path between them. $\triangle$

This concept of distance then allows us to ask some interesting questions about a given Cayley graph, such as: what is the formula which computes the number of elements within a given distance of the identity?

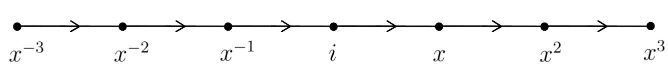
**Definition 1.1.3.** Let $\Gamma$ be a Cayley graph. If $n \in \mathbb{N}$, the *$n$-ball* of $\Gamma$ is the set $B(n)$ of all vertices within $n$ edges of the identity. The *growth function* for $\Gamma$ is the function $g\colon \mathbb{N} \to \mathbb{N}$ defined by

$$g(n) = |B(n)|$$

$\triangle$

We will now give some examples of Cayley graphs and growth functions on them.

**Example 1.1.4.** Consider the group $C_\infty = \langle x \rangle$. As this group has one generator, the Cayley graph of this group is a singe line extending infinitely in both directions. A portion of this Cayley graph is shown below. The growth function for $\Gamma(C_\infty)$ is straightforward to compute. When $n = 1$, we have 3 elements in our 1-ball. When $n = 2$, we have 5. In general, $|B(n)| = 2n + 1$.




$C_\infty = \langle x \rangle$

$\lozenge$

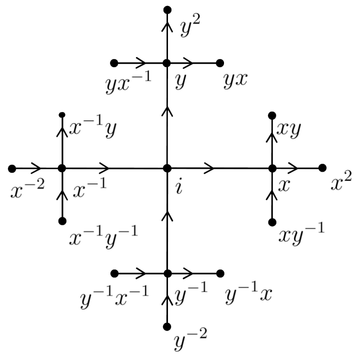



Figure 1.1.1. $\langle x, y \rangle$

**Example 1.1.5.** Consider the free group on two generators, $F_2 = \langle x, y \rangle$. This group has no relations, that is, there is no non-trivial reduced word which equals the identity. Thus, we see that the Cayley graph of this group will have trees branching infinitely in all directions. A picture of a portion of this graph is shown in figure 1.1.1.

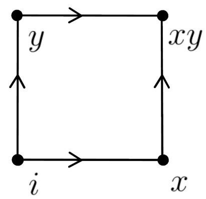
We will now consider the growth function $g$ on this graph. It is evident that $g(1) = 5$, since we have $i, x, y, x^{-1}$ and $y^{-1}$ in the set $B(1)$. We can then compute $g(2) = 17$ as follows; in addition to the 5 vertices of $B(1)$, we see that each of $x, y, x^{-1}, y^{-1}$ have 3 edges extending from them, and thus the total number of elements within 2 edges will be $5 + 4(3) = 17$. It turns out that the growth function for this group is actually

$$g(n) \; = \; 1 + \sum_{i=1}^{n} 3^{i-1} \cdot 4 \; = \; 2 \cdot 3^n - 1$$

This is an example of an exponential growth function.

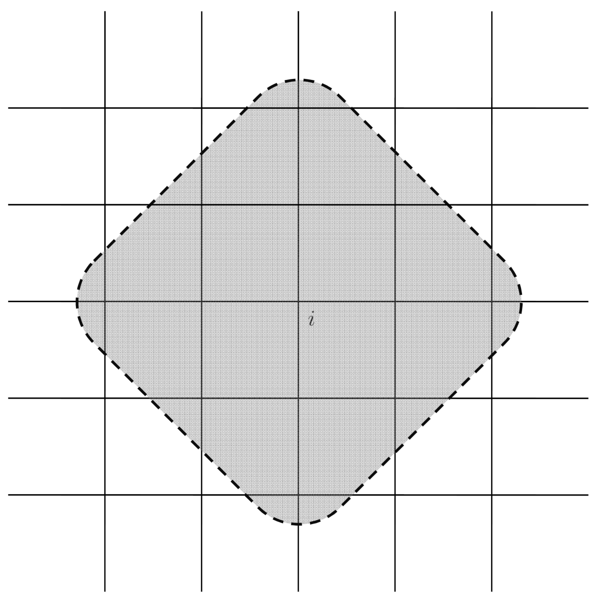
$\Diamond$

**Example 1.1.6.** Consider the group $\mathbb{Z} \times \mathbb{Z}$, with presentation $\langle x, y \mid xy = yx \rangle$. Like the free group in the previous example, this group is generated by two elements. However, this group is distinguished from the free group by the relation $xy = yx$. This addition of commutativity to our group changes the structure of the Cayley graph to the one seen below. Specifically, we now see that traveling first along an $x$ edge, and then along a $y$ edge, will take us to the same place as if we first travel along a $y$ edge, and then an $x$ edge.




$$\langle x, y \mid xy = yx \rangle$$

For the growth function of this graph, we see that $g(1) = 5$ and $g(2) = 13$. The image below shows a portion of $\Gamma(\mathbb{Z} \times \mathbb{Z})$, with the 2-ball of the graph within the diamond.

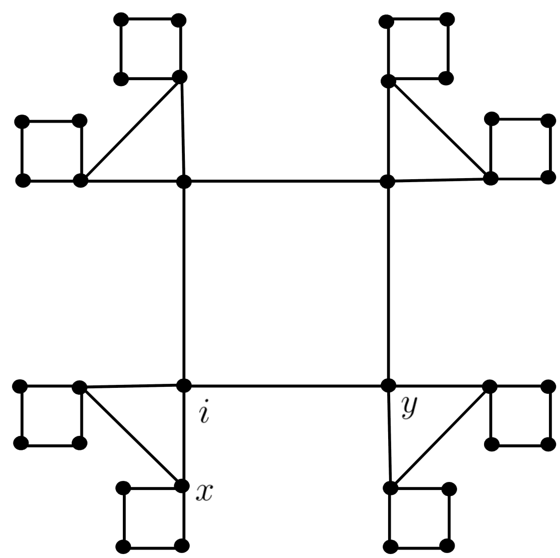



Comparing the values of the growth function for $\Gamma(\mathbb{Z} \times \mathbb{Z})$ with those for $\Gamma(F_2)$, we see that for a given $n \in \mathbb{N}$ $(n \geq 2)$, the size of an $n$-ball in $\Gamma(\mathbb{Z} \times \mathbb{Z})$ is smaller than that of an $n$-ball in $\Gamma(F_2)$. This is due to the relation $xy = yx$ which is present in $\mathbb{Z} \times \mathbb{Z}$. Whereas in $F_2$ the elements $xy$ and $yx$ are distinct, in $\mathbb{Z} \times \mathbb{Z}$ these elements are the same, and thus only add one to the cardinality of an $n$-ball. The growth function for $\Gamma(\mathbb{Z} \times \mathbb{Z})$ is in fact

$$g(n) = 1 + \sum_{i=1}^{n} 4i \;=\; 2n^2 + 2n + 1$$

$\Diamond$

**Example 1.1.7.** Consider the group $\langle x, y \mid x^3 = y^4 = i\rangle$, where $i$ is the identity (the free product of $\mathbb{Z}_3$ and $\mathbb{Z}_4$). This group has two relations, namely $x^3 = i$ and $y^4 = i$. Since $x^3 = i$, we know that multiplying any element by $x$ three times will bring us back to that element. This can be represented by drawing a triangle from each element to describe multiplication by $x$ three times. Likewise, since $y^4 = i$, we know that multiplying any element by $y$ four times will bring us back to that element. This can be represented by drawing a square from each element to describe multiplication by $y$. A portion of the Cayley graph of this group can be seen below.




$\langle x, y \mid x^3 = y^4 = i\rangle$

Note that each square is connected to four triangles, and each triangle to three squares. These triangles and squares branch off infinitely in all directions, and thus constitute a graph which is in some sense similar to $\Gamma(F_2)$ shown above. Like $\Gamma(F_2)$, this graph has exponential growth. . ◊

Because the free group has an exponential growth function, the growth function of any finitely generated group is at most exponential. Roughly speaking, the finitely generated groups fall into two classes: those with *exponential* growth and those with *polynomial* growth (there are also groups with *intermediate* growth, although we are not concerned with these). This is the most basic geometric classification of groups. For a more in depth discussion of growth functions, see [Me].

A more subtle geometric property of groups is the notion of amenability.

**Definition 1.1.8.** Let $G$ be a finitely generated group with Cayley graph $\Gamma(G)$. If $S$ is a finite subset of $G$, the *boundary* of $S$, denoted $\delta S$, is the number of edges leading from a vertex in $S$ to a vertex outside of $S$. $|\delta S|$ denotes the number of elements in the boundary of $S$.

A group $G$ is *amenable* if

$$\text{glb } \frac{|\delta S|}{|S|} = 0$$

where $S$ ranges over all finite sets of vertices in the Cayley graph of $G$. △

Intuitively, a graph $\Gamma(G)$ is amenable if, as one considers larger and larger subsets $S$ of $\Gamma(G)$, the number of edges leading out of $S$ get arbitrarily small relative to the number of elements in $S$.

**Example 1.1.9.** We return to $\Gamma(C_\infty)$. Recall that we found $g(n) = 2n+1$ for this Cayley graph. It is also easy to see that any $n$-ball will have two elements in its boundary. If we let $S$ be an $n$-ball, we then see

$$\frac{|\partial S|}{|S|} = \frac{2}{2n+1}$$

Note that $\lim_{n\to\infty} \dfrac{2}{2n+1} = 0$, and thus $\mathbb{Z}$ is amenable. ◊

**Example 1.1.10.** We now reconsider $\Gamma(\mathbb{Z} \times \mathbb{Z})$. Recall that the growth function for $\Gamma(\mathbb{Z} \times \mathbb{Z})$ is

$$g(n) = 1 + \sum_{i=1}^{n} 4i$$

Thus, for any $n \in \mathbb{N}$, we can calculate $g(n) - g(n-1) = 4n$. Thus we see that $4n$ vertices of our $n$-ball will have edges leading out of the ball. Furthermore, a look at $\Gamma(\mathbb{Z}\times\mathbb{Z})$ shows us that exactly four of these vertices will have 3 edges leading out of the $n$-ball, while the rest will have 2. We can then calculate $\partial B(n) = 4n \cdot 2 + 4 = 8n + 4$. We then see

$$\frac{\partial B(n)}{|B(n)|} = \frac{8n+4}{2n^2+2n+1}$$

Clearly $\lim_{n\to\infty} \frac{8n+4}{2n^2+2n+1} = 0$, and thus $\Gamma(\mathbb{Z} \times \mathbb{Z})$ is amenable. $\Diamond$

As exemplified by the previous example, any group with polynomial growth is amenable, as the relative size of $\frac{\partial B(n)}{|B(n)}$ will go to 0 as $n$ increases. For groups with exponential growth, the relative size of $\frac{\partial B(n)}{|B(n)|}$ will not go to zero. However, there may sets other than $n$-balls which can be used to show amenability for groups with exponential growth. It known that any group which is not amenable has exponential growth, although there are known groups which have exponential growth and are also amenable. See [Wag] for more information on amenability.

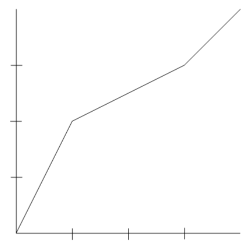
Growth functions and amenability are relatively simple to compute for most groups. There are, however, some groups which are easy to describe, but quite difficult to determine the amenability of. One such group is Thompson's group $F$, a group with exponential growth. We will now give an overview of $F$ and introduce some of the as-yet unsolved problems stemming from it.

## 1.2 Thompson's Group $F$

**Definition 1.2.1.** Thompson's Group $F$ is the group of all piecewise-linear homeomorphisms $f\colon [0,1] \to [0,1]$ such that:

1. Each linear segment of $f$ has a slope which is a power of 2.

2. The breakpoints of $f$ have dyadic rational coordinates. $\triangle$



The image above shows a typical element of Thompson's group. This element is defined as

$$f(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{4} \\ \frac{1}{2}x + \frac{3}{8} & \frac{1}{4} \leq x \leq \frac{3}{4} \\ x & \frac{3}{4} \leq x \leq 1 \end{cases}$$

Thompson's Group $F$ was first described by Richard Thompson in the 1960's, and later rediscovered by topologists Freyd and Heller in 1969 (see [Fr] and [FrHe]). It has since become an important object of study in topology and geometric group theory (see [CFP] or [Be] for a comprehensive introduction to $F$). Though we have described $F$ as acting on the unit interval, there are other ways of thinking about elements of $F$. One such way is through forest diagrams, which were introduced by Belk and Brown [BeBr]. Forest diagrams interact particularly well with the generators of $F$. To understand how forest diagrams are constructed, we will first describe how $F$ can be though of as acting on the real line.

**Theorem 1.2.2.** *Thompson's group $f$ is the group of all piecewise-linear homeomorphisms $f\colon \mathbb{R} \to \mathbb{R}$, which satisfy the following conditions:*

1. *Each linear segment of $f$ has a slope which is a power of 2.*
2. *The function $f$ has finitely many breakpoints, and each breakpoint has dyadic rational coordinates.*
3. *The leftmost segment of $f$ is of the form $f(t) = t - m$, and the rightmost segment is of the form $f(t) = t - n$ for $m, n \in \mathbb{Z}$.*

Elements of $F$ can be described by their action on certain subdivisions of the real line.

**Definition 1.2.3.** A *dyadic subdivision* of the real number line is a division obtained as follows:

1. Divide the real line into intervals of the form $[n, n+1]$ for $n \in \mathbb{N}$.

2. Divide some finite number of those intervals in half.

3. Divide some finite number of the intervals just obtained in half again.

4. Continue dividing such intervals in half some finite number of times.

The result will be a dyadic subdivision of $\mathbb{R}$. $\triangle$

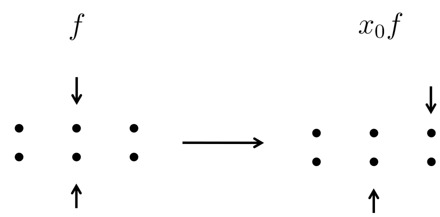
Every interval of the subdivision will be of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ for $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $n \geq 0$. These intervals are easily represented by binary trees. Each tree represents one of the intervals $[n, n+1]$ for $n \in \mathbb{N}$, and each leaf of a tree represents an interval of the dyadic subdivision of $[n, n+1]$.

The pointers in the domain and codomain of our function represent the interval $[0, 1]$. The first and second conditions ensure that each linear segment of $f$ will map one such interval linearly onto another one, stated in the following theorem.

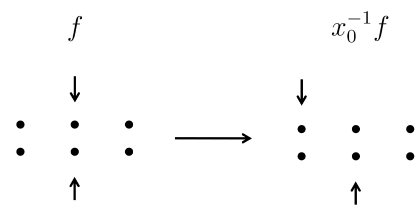
**Theorem 1.2.4.** *Every element of $F$ maps linearly between the intervals of two dyadic subdivisions.*

This theorem allows us to represent elements of $F$ through pairs of opposing dyadic subdivisions. By convention, the domain appears on the bottom and the codomain on the top. The pointers in the domain and codomain of our function represent the interval $[0, 1]$. It is clear that through such diagrams, any piecewise-linear map from one dyadic subdivision to another can be obtained, and thus all elements of $F$ are representable as such.

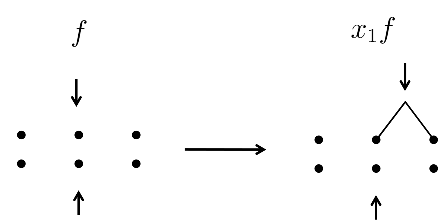
The set of all such forests as described above is $F$. While not obvious, it turns out that all such forests are generated by two elements, labeled $x_0$ and $x_1$. For any forest diagram $f$, the action of $x_0$ on $f$ is represented by the forest diagram obtained by moving the pointer in the codomain of $f$, or the upper pointer, one interval to the right. This is shown below.



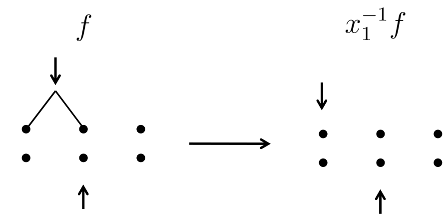

Likewise, $x_0^{-1}$ moves the upper pointer one interval to the left, as seen below.



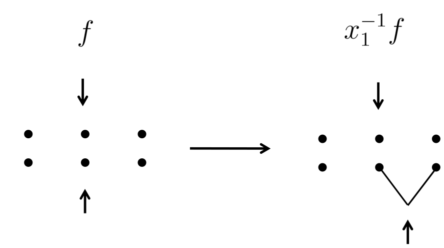

The generator $x_1$ attaches a caret which connects the tree that the top pointer is on and the tree to the right of the top pointer. The top pointer then points to the new tree made by the caret. This is shown in below.



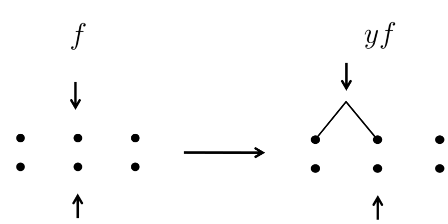

The generator $x_1^{-1}$ actually attaches a "negative caret" to the position the top pointer of $f$. If the top pointer of $f$ is currently on a non-trivial tree, then $x_1^{-1} f$ removes the top caret of that tree, as seen below.



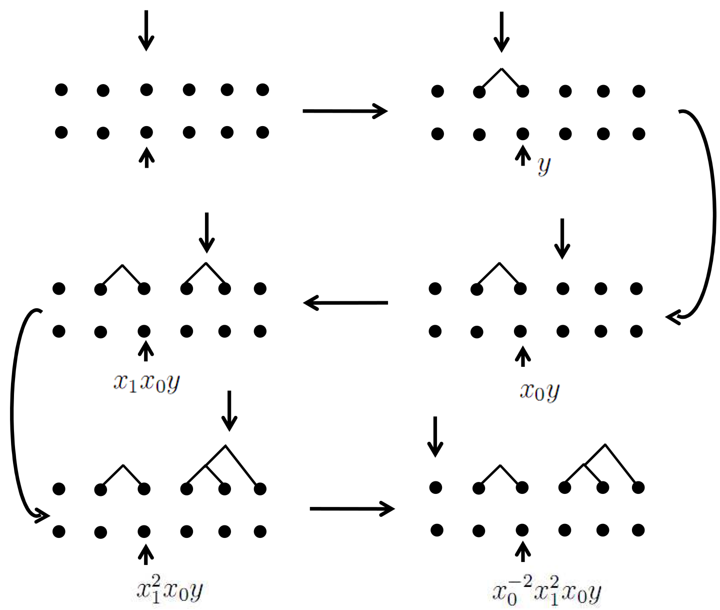

If the top pointer of $f$ is on a trivial tree, then $x_1^{-1}f$ drops a caret to the domain forest of $f$, shown below.




The group $F$ has traditionally been viewed with regards to these two generators, $x_0$ and $x_1$. However, it is possible to describe $F$ in terms of three generators as well: $x_0, x_1$ and $y$, where $y = x_1 x_0^{-1}$. The introduction of $y$ as a generator serves to make $F$ more "symmetrical", in that trees to the left of the top pointer can now be made as easily as trees to the right of the top pointer. The figure below displays the action of $y$ on an element of $f$. Since $y = x_1 x_0^{-1}$, we must have $y^{-1} = x_0 x_1^{-1}$. As such, $y^{-1}$ has the effect of first acting on an element $f$ by $x_1^{-1}$, and then moving one space to the right. In this sense again we see that the $y$ generator balances out the left and right movements of the top pointer.



**Example 1.2.5.** To get a better sense of how forest diagrams are constructed, consider $f = x_0^{-2}x_1^2x_0y$. Since we are using left multiplication, we would start to construct $f$ by first diagraming the action of $y$ on the identity. We would then move the top pointer one interval to the right, representing the action of $x_0$ on $y$. After this, we construct two carets, one on top of the other, as dictated by $x_1^2$. Finally, we move to the left two times. The construction of $f$ is diagramed below.




$\diamondsuit$

We stated previously that $F$ is finitely generated. With respect to the three element generating set, $F$ has presentation

$$F = \langle x_0, x_1, y \mid y = x_1x_0^{-1}, x_0^{-1}x_1x_0y = yx_0^{-1}x_1x_0, x_0^{-2}x_1x_0^2y = yx_0^{-2}x_1x_0^2\rangle$$

It should be noted that the growth function of $F$ is currently unknown, although it has been shown that $F$ has exponential growth. Likewise, it is unknown whether $F$ is amenable or not. The Cayley graph of $F$ is in fact very poorly understood on the whole, and it is to that end that we next develop a length formula for $F$ in terms of the 3-element generating set.

# 2

# Lengths of Elements of $F$

## 2.1 Length Formula For The Three Element Generating Set

Given a group $G$, a length formula for an element gives the length of the shortest word representing that element. The following definition is equivalent. In terms of a Cayley graph, a length formula for a vertex $v \in V(\Gamma)$ will tell you how many distinct vertices you must travel through on the shortest path from $i$ to $v$.

**Example 2.1.1.** For example, in the Cayley graph of the free group on 2 generators, a length formula $l\colon G \to \mathbb{N}$ would tell us $l(y^2x^3) = 5$. This can be confirmed by looking at the Figure 2.1.1, in which the shortest (and only non-reducible) path from the identity to $y^2x^3$ is represented by a dotted line. $\Diamond$

In general, parts of words which are not reduced correspond to cycles on a Cayley graph, and thus can artificially inflate the length formula.

Belk and Brown [BeBr] have already found a length formula in terms of forest diagrams for the $\{x_0, x_1\}$ generating set for $F$. To find this formula, rather than investigating the little-understood Cayley graph of $F$, they looked at forest diagrams and the effect that generators have on these diagrams. We will follow in their footsteps and prove a length

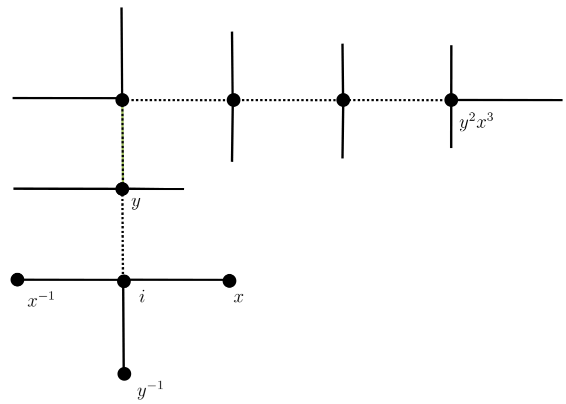



Figure 2.1.1.

formula for the $\{x_0, x_1, y\}$ generating set by examining forest diagrams for this three element generating set.

Before stating the length formula, we will provide some definitions.

**Definition 2.1.2.** A *space* in a forest diagram is the area between one interval and the next. An *interior space* is a space which is underneath a caret. An *exterior space* is a space which is not interior. The *support* of a forest diagram of an element $f \in F$ is the portion of the diagram corresponding to the linear segments of $f$ which are not the leftmost nor the rightmost segments. △

Informally, the support of a diagram is the section of the diagram where trees exist. All spaces to the right of the rightmost tree, and to the left of the leftmost tree, are not in the support of the diagram.

**Definition 2.1.3.** The *weight* of a space in a forest diagram $f$ is the number of times the top pointer will have to cross over that space in constructing $f$ in the shortest manner possible. △

To calculate the length formula, we must weight each space in the support of a forest diagram. Let $f \in F$, and let $\mathfrak{f}$ be its reduced forest diagram. Label the spaces of each forest of $\mathfrak{f}$ as follows:

1. Label a space $\mathbf{L}_R$ if it is exterior, to the left of the pointer, and immediately to the right of some caret.

2. Label a space $\mathbf{L}_0$ if it is exterior, to the left of the pointer, and not already labeled $\mathbf{L}_R$.

3. Label a space $\mathbf{R}_L$ if it is exterior, to the right of the pointer, and immediately to the left of some caret.

4. Label a space $\mathbf{R}_0$ if it is exterior, to the right of the pointer, and not already labeled $\mathbf{R}_L$.

5. Label a space $\mathbf{I}_{LR}$ if it is interior, immediately to the left of some caret, and immediately to the right of some caret.

6. Label a space $\mathbf{I}_L$ if it is interior, immediately to the left of some caret, and not already labeled $\mathbf{I}_{LR}$.

7. Label a space $\mathbf{I}_R$ if it is interior, immediately to the right of some caret, and not already labeled $\mathbf{I}_{LR}$.

8. Label a space $\mathbf{I}_0$ if it is interior and not already labeled $\mathbf{I}_{LR}$, $\mathbf{I}_L$, or $\mathbf{I}_R$.

We can then weight the spaces in the support of $\mathfrak{f}$ according to their labels, using the following chart:

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{L}_0$ | 2 | 2 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\mathbf{L}_R$ | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\mathbf{R}_0$ | 1 | 1 | 2 | 2 | 0 | 1 | 0 | 1 |
| $\mathbf{R}_L$ | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 |
| $\mathbf{I}_0$ | 0 | 1 | 0 | 1 | – | 0 | 0 | 1 |
| $\mathbf{I}_L$ | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\mathbf{I}_R$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| $\mathbf{I}_{LR}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Note that a space labeled $\mathbf{I}_0$ can never be opposite another space labeled $\mathbf{I}_0$ in a reduced forest diagram, as that would correspond to two opposing carets, which by definition do not exist in a reduced forest digram.

Our chart has the interesting and intuitively reasonable feature of being symmetric along the diagonal, so that the weight of label $A$ over label $B$ is the same as the weight of label $B$ over label $A$ for any labels $A, B$. This is a consequence of the symmetry of the top and bottom forests in a forest diagram, and in a more general sense, represents the fact that an element $f \in F$ and its inverse $f^{-1}$ are both the same distance from the identity.

Another interesting symmetry exists between the $L$ and $R$ labels. That is, if all $L$'s in the chart are replaced by $R$'s, and vice versa, the chart will remain the same. This is a consequence of the newfound symmetry between left and right movements on forest diagrams obtained by adding the $y$ generator. Specifically, creating a caret to the left of the top pointer now requires only one generator, whereas before it required two. This removes the rightward "bias" in terms of lengths of elements.

Using the above chart, the length formula is as follows:

**Theorem 2.1.4.** *Let $f \in F$, and let $\mathfrak{f}$ be its reduced forest diagram. Then the $\{x_0, x_1, y\}$-length of $f$ is*

$$l(f) = l_0(f) + l_1(f)$$

*where:*

1. *$l_0(f)$ is the sum of the weights of all spaces in the support of $\mathfrak{f}$.*
2. *$l_1(f)$ is the total number of carets in $\mathfrak{f}$.*

To prove that this is a length formula for $F$, we use the following proposition, given by Fordham [Ford].

**Proposition 2.1.5.** *A function $l\colon F \to \mathbb{N}$ is a length formula with respect to the $\{x_0, x_1, y\}$ generating set if and only if:*

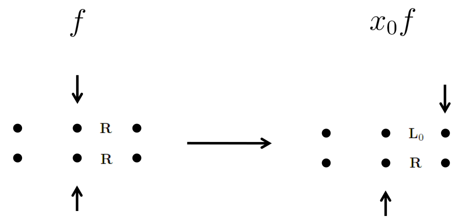
1. *For any $f \in F$ and $x \in \{x_0, x_1, y\}$, we have $|l(xf) - l(f)| \leq 1$.*
2. *For any non-identity $f \in F$, there exists some $x \in \{x_0, x_1, y\}$ such that $l(xf) < l(f)$.*

The proofs of these statements will be deferred until we have provided a multitude of theorems and lemmas about the effect of various generators on elements of $F$.

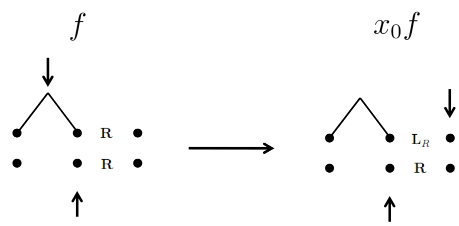
**Definition 2.1.6.** Let $\mathfrak{f}$ be a forest diagram of an element $f \in F$. The *current tree* of $\mathfrak{f}$ is the tree which the top pointer points to. The *left*(*right*) space of $\mathfrak{f}$ is the exterior space directly to the left (right) of the top pointer. Likewise, the *double left* (*double right*) space of $\mathfrak{f}$ is the exterior space to the left (right) of the left (right) space of $\mathfrak{f}$. $\triangle$

We will use the notation $\mathbf{R}$ to represent a space which has either $\mathbf{R}_0$ or $\mathbf{R}_L$ for a label. Likewise, we will use the notation $\mathbf{L}$ to represent a space which has either $\mathbf{L}_0$ or $\mathbf{L}_R$ for a label, and $\mathbf{I}$ to represent a space with $\mathbf{I}_R, \mathbf{I}_L, \mathbf{I}_{LR}$, or $\mathbf{I}_0$.

Also, throughout the following proofs we will frequently distinguish between the current tree of $f$ being trivial or non-trivial. Note that if the current tree of $f$ is trivial, then the right space of $f$ must have top label $\mathbf{L}_0$ in $x_0 f$. This is shown below.



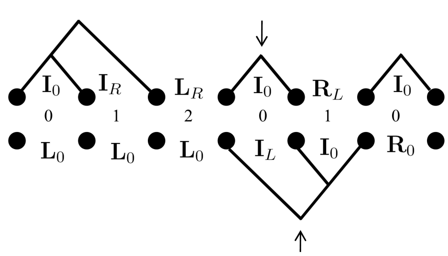

Likewise, if the current tree of $f$ is non-trivial, then the right space of $f$ will have top label $\mathbf{L}_R$ in $x_0 f$, as can be seen below .




In general, knowing whether the current tree of $f$ is trivial or not gives some useful information about what the labelings of the left and right spaces of $f$ will be in $xf$ for some $x$ in the three element generating set of $F$.

Before beginning our proof, we will provide an example of using the length formula to compute the length of an element of $F$.

**Example 2.1.7.** Consider the element $f \in F$ with support shown below. After labeling each space and calculating the weights for each opposing pair of spaces, we see that $l_0(f) = 4$, and that $l_1(f) = 6$. Thus, we have $l(f) = l_0(f) + l_1(f) = 10$. This is confirmed by noting that $x_0 x_1 y x_0^{-2} y x_0^{-1} x_1 y^{-2}$ is a reduced word for $f$.



$\Diamond$

We now prove some propositions which in turn allow us to prove Theorem 3.5.

**Proposition 2.1.8.** *Let $f \in F$. Then $|l(x_0 f) - l(f)| \leq 1$. Specifically, $l(x_0 f) = l(f) + 1$ if and only if one of the following conditions hold:*

*1. $x_0 f$ has larger support than $f$.*

*2. The right space of $f$ has bottom label $\mathbf{L}$, and left-multiplication by $x_0$ does not remove this space from the support.*

*3. The right space of $f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{I}_R \end{bmatrix}$.*

*4. The current tree of $f$ is non-trivial and the right space of $f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{I}_0 \end{bmatrix}$.*

*Also, $l(x_0 f) = l(f)$ if and only if one of the following conditions hold:*

*1. The right space of $f$ has label $\begin{bmatrix} \mathbf{R}_L \\ \mathbf{I}_R \end{bmatrix}$.*

*2. The right space of $f$ has bottom label $\mathbf{I}_{LR}$.*

*3. The right space of $f$ has top bottom label $\mathbf{I}_L$ and the current tree of $f$ is non-trivial.*

*4. The right space of $f$ has label $\begin{bmatrix} \mathbf{R}_L \\ \mathbf{I}_0 \end{bmatrix}$, and the current tree of $f$ is non-trivial.*

*5. The right space of $f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{I}_0 \end{bmatrix}$, and the current tree of $f$ is trivial.*

*Finally, $l(x_0 f) = l(f) - 1$ if and only if one of the following conditions hold:*

*1. $x_0 f$ has smaller support than $f$.*

*2. The right space of $f$ has bottom label $\mathbf{R}$.*

*3. The right space of $f$ has bottom label $\mathbf{I}_L$ and the current tree is trivial.*

*4. The right space of $f$ has label $\begin{bmatrix} \mathbf{R}_L \\ \mathbf{I}_0 \end{bmatrix}$ and the current tree is trivial.*

It is obvious that $l_1(x_0 f) = l_1(f)$, since $x_0$ neither adds nor removes a caret from $f$. Furthermore, we see that the only space whose label changes in $x_0 f$ is the right space of $f$, and that only the top label of that space will change.

**Case 1**: Suppose that $x_0 f$ has larger support than $f$. Then the right space of $f$ is unlabeled, but has label $\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$ in $x_0 f$. The space will have this label because it will be to the left of the top pointer, since $x_0$ moves the top pointer one space to the right, and because it will necessarily be to the right of the bottom pointer, or else it would have previously been in the support of $f$. Note that $\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$ always has weight 1, and thus $l_0(x_0 f) = l_0(f) + 1$.

**Case 2**: Suppose that $x_0 f$ has smaller support than $f$. Then the right space of $f$ has label $\begin{bmatrix} \mathbf{R} \\ \mathbf{L} \end{bmatrix}$ in $f$, since it is clearly to the right of the top pointer and to the left of the bottom pointer. Again, note that $\begin{bmatrix} \mathbf{R} \\ \mathbf{L} \end{bmatrix}$ always has weight 1. This label will be destroyed in $x_0 f$, and thus $l_0(x_0 f) = l_0(f) - 1$.

**Case 3:** Suppose that $x_0 f$ has the same support as $f$. Then the right space of $f$ has top label $\mathbf{R}$, but top label $\mathbf{L}$ in $x_0 f$. The relevant rows of the weight table are:

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{L}_0$ | 2 | 2 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\mathbf{L}_R$ | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\mathbf{R}_0$ | 1 | 1 | 2 | 2 | 0 | 1 | 0 | 1 |
| $\mathbf{R}_L$ | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 |

It is first apparent from the table that for any space with top label $\mathbf{R}$, switching that label to $\mathbf{L}$ will change the weight by at most 1. Thus $|l(x_0 f) - l(f)| \leq 1$. Specifically, we see that if the bottom label is an $\mathbf{R}$, then $x_0 f$ will always reduce the length by 1, so $l(x_0 f) = l(f) - 1$. If the bottom label is an $\mathbf{L}$, then $x_0 f$ will always increase the length by 1, so $l(x_0 f) = l(f) + 1$.

When the bottom label is an $\mathbf{I}$ of some sort, determining the effect of $x_0$ on $f$ becomes a bit more complicated. However, it is evident that the only ways that $x_0$ can increase the length is if the right space of $f$ has top label $\mathbf{R}_0$, bottom label $\mathbf{I}_0$, and the current tree is non-trivial, or if the right space of $f$ has top label $\mathbf{R}_0$ and bottom label $\mathbf{I}_R$.

Likewise, the only ways that $x_0$ can decrease the length is if the right space of $f$ has top label $\mathbf{R}$, bottom label $\mathbf{I}_L$, and the current tree of $f$ is trivial, or if the right space of $f$ has top label $\mathbf{R}_L$, bottom label $\mathbf{I}_0$, and the current tree is trivial.

Finally, we see that $x_0$ keeps the length constant when: the right space of $f$ has bottom label $\mathbf{I}_{LR}$, or when the right space of $f$ has label $\begin{bmatrix} \mathbf{R}_L \\ \mathbf{I}_R \end{bmatrix}$, or when the right space of $f$ has bottom label $\mathbf{I}_L$ and the current tree is non-trivial, or when the right space of $f$ has bottom label $\mathbf{I}_0$ and top label $\mathbf{R}_L$ if the current tree is non-trivial, or label $\mathbf{R}_0$ if the current tree is trivial.

We will now provide a corollary concerning $x_0^{-1}$, which follows directly from the first four conditions of Proposition 2.1.8.

**Proposition 2.1.9.** *$l(x_0^{-1}f) = l(f) - 1$ if:*

1. *$x_0^{-1}f$ has smaller support than $f$.*
2. *The left space of $f$ has label $\begin{bmatrix} \mathbf{L} \\ \mathbf{L} \end{bmatrix}$.*
3. *The left space of $f$ has label $\begin{bmatrix} \mathbf{L} \\ \mathbf{I}_R \end{bmatrix}$ and the current tree is trivial.*
4. *The left space of $f$ has label $\begin{bmatrix} \mathbf{L}_R \\ \mathbf{I}_0 \end{bmatrix}$ and the current tree is trivial.*

For our next theorem, we introduce some new terminology. Let the double right space of $f$ be the space to the right of the right space of $f$. Likewise, let the double left space of $f$ be the space to the left of the left space of $f$.

**Proposition 2.1.10.** *Let $f \in F$. If left multiplying $f$ by $x_1$ cancels a caret in the bottom forest, then $l(x_1 f) = l(f) - 1$.*

First, note that $l_1(x_1 f) = l_1(f) - 1$, since we are assuming that $x_1$ removes a caret from $f$. We will now show that $l_0(x_1 f) = l_0(f)$. Since left multiplication by $x_1$ cancels a caret in the bottom forest, we know that the right space of $f$ is destroyed. That space must have been labeled $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{I}_0 \end{bmatrix}$, since if it did not have top label $\mathbf{R}_0$, then $x_1$ would have built onto the caret to the right of the top pointer instead of canceling a bottom caret. Note that this space has weight 0, and thus its destruction does not affect the $l_0$ weight of $x_1 f$. Note that both the left space of $f$ and the double right space of $f$ could have their labels affected by $x_1$ multiplication. We first consider the effect of $x_1$ on the left space of $f$. If this space is not in the support of $f$, then it remains outside of the support of $x_1 f$, and thus does not affect $l_0$. Otherwise, we see that its top label must be $\mathbf{L}_0$ or $\mathbf{L}_R$ in both $f$ and $x_1 f$. Consider the relevant row of the weight table :

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{L}_0$ | 2 | 2 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\mathbf{L}_R$ | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |

If the bottom label of the left space of $f$ is an $\mathbf{L}$, then that label will not be changed by cancelling a caret in the bottom forest, and thus the weight will remain the same. If the bottom label is an $R$, it is possible that the label could be changed from an $\mathbf{R}_L$ to an $\mathbf{R}_0$ when the caret is canceled. However, both of these labels have the same weight when there is an $\mathbf{L}$ label on top, and thus the weight would still remain the same.

We also know that if the bottom label of the left space of $f$ was an $I$ of some sort, it must have been either an $\mathbf{I}_L$ or an $\mathbf{I}_{LR}$, since it must have been to the left of the caret which was destroyed. If the bottom label was an $\mathbf{I}_L$, then it will be changed to an $\mathbf{I}_0$, since it will no longer be to the left of a caret. Note that both $\mathbf{I}_L$ and $\mathbf{I}_0$ have the same weight in the above rows, so changing from $\mathbf{I}_L$ to $\mathbf{I}_0$ will not affect $l_0$. If the bottom label was an $\mathbf{I}_{LR}$, then it will become an $\mathbf{I}_R$. Note that $\mathbf{I}_{LR}$ and $\mathbf{I}_R$ both have the same weight in the above rows as well, so changing from $\mathbf{I}_{LR}$ to $\mathbf{I}_R$ will not affect $l_0$.

We now consider the affect of $x_1$ on the double right space of $f$. Firstly, if the double right space of $f$ is not in the support of $f$, then it remains outside of the support of $x_1 f$, and thus does not affect $l_0$. Otherwise, we know that the top label of the double right space of $f$ must be $\mathbf{R}$, for if it were not then multiplying by $x_1$ would have added onto the caret which existed there. It is also important to note that multiplication by $x_1$ will not change the top label of the double right space of $f$ for the following reason: if the label is an $\mathbf{R}_0$, then multiplication by $x_1$ will not add a caret to the left of the double right space, nor will it create a caret over it, so the space will retain its label. Likewise, if the label is an $\mathbf{R}_L$, multiplication will not remove the caret to the left of the double right space, nor will it create a caret over it. Thus the top label of the double right space of $f$ will remain unchanged.

First suppose that the bottom label of the double right space of $f$ is an $\mathbf{L}$. It is easily deduced that if the bottom label of the double right space of $f$ is an $\mathbf{L}$, it must be $\mathbf{L}_R$, since it is to the right of the caret which is destroyed by $x_1$. We then see that multiplication by $x_1$ could leave that space with label $\mathbf{L}_R$ if the bottom current tree of $f$ consists of more than one caret, or it could be changed to $\mathbf{L}_0$ if the bottom current tree is only one caret. In either situation, we can see that the weight of any $\mathbf{R}$ opposite any $\mathbf{L}$ will always be 1, and thus the weight of the double right space of $f$ will remain unchanged.

Now suppose that the bottom label of the double right space of $f$ is an $\mathbf{R}$. Then removing a caret to the right of that space will not change it to an $\mathbf{L}$ or an $\mathbf{I}$. Note that an $\mathbf{R}$ opposite an $\mathbf{R}$ will always have weight 1, and thus the weight of the double right space of $f$ will remain unchanged.

Finally, suppose that the bottom label of the double right space of $f$ is an $\mathbf{I}$. Note that the label must have been either an $\mathbf{I}_R$ or an $\mathbf{I}_{LR}$, as the space was directly to the right of the caret which was destroyed. If the bottom label of the space was $\mathbf{I}_R$, then it will be labeled either $\mathbf{I}_R$ or $\mathbf{I}_0$ in $x_1 f$, since multiplication by $x_1$ will certainly not add a caret to the right of the space. Note that both $\begin{bmatrix} \mathbf{R} \\ \mathbf{I}_0 \end{bmatrix}$ and $\begin{bmatrix} \mathbf{R} \\ \mathbf{I}_R \end{bmatrix}$ have the same weight for a fixed value of $\mathbf{R}$. Thus changing the bottom label of the double right space of $f$ from an $\mathbf{I}_R$ to an $\mathbf{I}_0$ will not change the weight of the space.

If the bottom label of the double right space of $f$ was an $\mathbf{I}_{LR}$, then it will be labeled either $\mathbf{I}_{LR}$ or $\mathbf{I}_L$ in $x_1 f$, as $x_1$ will not change whether the space is to the left of a caret or not. We then see that both $\begin{bmatrix} \mathbf{R} \\ \mathbf{I}_{LR} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{R} \\ \mathbf{I}_{L} \end{bmatrix}$ are of weight 1. Thus changing the bottom label of the right space from $\mathbf{I}_{LR}$ to $\mathbf{I}_L$ will not change the weight of the space.

We have heretofore shown that the weights of both the left space of $f$ and of the double right space of $f$ will not change when $f$ is multiplied by $x_1$. We can therefore conclude that $l_0(x_1 f) = l_0(f)$. We also know that $l_1(x_1 f) = l_1(f) - 1$. Thus, we see $l(x_1 f) = l(f) - 1$.

**Proposition 2.1.11.** *If left multiplication by $y$ removes a caret from the bottom forest, then $l(yf) = l(f) - 1$.*

Clearly, $l_1(yf) = l_1(f) - 1$. The rest of the proof is practically identical to the above proof, except that we consider the effects of $y$ on the right space and double left space of $f$, as opposed to the left space and double right space.

**Proposition 2.1.12.** *Let $f \in F$, and suppose that left multiplying by $x_1$ creates a caret in the top forest. Then $|l(x_1 f) - l(f)| \leq 1$. Specifically, $l(x_1 f) = l(f) - 1$ if and only if the right space of $f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{R}_0 \end{bmatrix}$. Also, $l(x_1 f) = l(f)$ if:*

*1. The right space of $f$ has bottom label $\mathbf{L}_0$ or $\mathbf{I}_L$, and the current tree is trivial.*

*2. The right space of $f$ has top label $\mathbf{R}_L$ and bottom label $\mathbf{R}_0$.*

*3. The right space of $f$ has bottom label $\mathbf{R}_L$.*

*4. The right space of $f$ has top label $\mathbf{R}_L$, bottom label $\mathbf{I}_0$, and the current tree is trivial.*

*Finally, $l(x_1 f) = l(f) + 1$ if:*

*1. The current tree of $f$ is trivial, the right space of $f$ has top label $\mathbf{R}_0$, and bottom label $\mathbf{L}_R$, $\mathbf{I}_R$, or $\mathbf{I}_{LR}$.*

*2. The current tree of $f$ is trivial, the right space of $f$ has top label $\mathbf{R}_L$, and bottom label $\mathbf{L}_R$, $\mathbf{I}_L$, or $\mathbf{I}_{LR}$.*

*3. The current tree of $f$ is non-trivial, the right space of $f$ has top label $\mathbf{R}_0$ or $\mathbf{R}_L$, and bottom label $\mathbf{L}$ or $\mathbf{I}$.*

*4. The right space of $f$ is not in the support of $f$.*

It is evident that $l_1(x_1 f) = l_1(f) + 1$. As far as $l_0$, we will show that left multiplication by $x_1$ can only change the label of the right space of $f$. First, we consider the effect of $x_1$ on the left space of $f$. If the left space of $f$ has top label $\mathbf{L}_0$, then that space will remain $\mathbf{L}_0$ in $x_1 f$, since left multiplication by $x_1$ will not add a caret to the double left space of $f$. If the left space of $f$ has top label $\mathbf{L}_R$, then multiplication by $x_1$ surely will not remove a caret to the left of the left space of $f$, so the label will remain $\mathbf{L}_R$. We now consider the effect of $x_1$ on the double right space of $f$. If the top label of the double right space of $f$ is $\mathbf{R}_L$, then multiplication by $x_1$ will not change whether there is a caret to the right of the double right space of $f$, so the label will remain $\mathbf{R}_L$. Finally, if the top label of the double right space of $f$ is $\mathbf{R}_0$, then multiplication by $x_1$ cannot add a caret to the right of the double right space of $f$, so the label will remain $\mathbf{R}_0$.

We will now go through 5 cases which comprise the entirety of ways in which $x_1$ could affect the length of $f$.

**Case 1:** Suppose that $x_1 f$ has a larger support than $f$. Then the right space of $f$ is unlabeled, but has label $\begin{bmatrix} \mathbf{I}_0 \\ \mathbf{R}_0 \end{bmatrix}$ in $x_1 f$. Note that such a label has weight 0, and thus does not affect $l_0$.

Now suppose that $x_1 f$ has the same support as $f$. Note that the right space of $f$ must have top label $\mathbf{R}_0$ or $\mathbf{R}_L$. If the right space has label $\mathbf{R}_0$, then it must be labeled either $\mathbf{I}_0$ or $\mathbf{I}_R$ in $x_1 f$. If the right space has label $\mathbf{R}_L$, then it must be labeled $\mathbf{I}_L$ or $\mathbf{I}_{LR}$ in $x_1 f$. For each case we will include the relevant rows of the weight table.

**Case 2:**

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{R}_0$ | 1 | 1 | 2 | 2 | 0 | 1 | 0 | 1 |
| $\mathbf{I}_0$ | 0 | 1 | 0 | 1 | – | 0 | 0 | 1 |

Suppose that the right space of $f$ has top label $\mathbf{R}_0$, and that the current tree of $f$ is trivial. Then that space will be labeled $\mathbf{I}_0$ in $x_1 f$. We see that if the right space of $f$ has bottom label $\mathbf{R}_0$, then the right space of $f$ will have weight 2 in $f$, but weight 0 in $x_1f$, and thus $l_0(x_1f) = l_0(f) - 2$, so $l(x_1f) = l(f) - 1$. If the right space of $f$ has bottom label $\mathbf{L}_0$, $\mathbf{R}_L$, or $\mathbf{I}_L$, then it is likewise clear that the weight of the space will be decreased by 1 when $\mathbf{R}_0$ changes to $\mathbf{I}_0$, so $l(x_1f) = l(f)$. Finally, if the right space of $f$ has bottom label $\mathbf{L}_R$, $\mathbf{I}_R$, or $\mathbf{I}_{LR}$, then the weight of the that space will stay the same when the top label changes to $\mathbf{I}_0$, so $l(x_1f) = l(f) + 1$.

**Case 3:**

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{R}_0$ | 1 | 1 | 2 | 2 | 0 | 1 | 0 | 1 |
| $\mathbf{I}_R$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

Suppose that the right space of $f$ has top label $\mathbf{R}_0$, and that the current tree of $f$ is non-trivial. Then the right space of $f$ will have top label $\mathbf{I}_R$ in $x_1f$. It is then evident that if the right space of $f$ has bottom label $\mathbf{R}_0$, then that space will have weight 2 in $f$, but weight 0 in $x_1f$, and thus $l(x_1f) = l(f) - 1$. If the right space of $f$ has bottom label $\mathbf{R}_L$, then that space will have weight 2 in $f$, but weight 1 in $x_1f$, and thus we see $l(x_0f) = l(f)$. In all other cases, namely when the right space of $f$ has bottom label $\mathbf{L}$ or $\mathbf{I}$, we see that changing $\mathbf{R}_0$ to $\mathbf{I}_R$ will not change the weight of the space, and thus we have $l(x_0f) = l(f) + 1$.

**Case 4:**

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{R}_L$ | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 |
| $\mathbf{I}_R$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

Suppose that the right space of $f$ has top label $\mathbf{R}_L$, and that the current tree of $f$ is trivial. Then that space will be labeled $\mathbf{I}_L$ in $x_1f$. Looking at the table, we see that if the right space of $f$ has bottom label $\mathbf{L}_0$, $\mathbf{R}$, $\mathbf{I}_0$, or $\mathbf{I}_L$, then changing the top label from $\mathbf{R}_L$ to $\mathbf{I}_L$ will reduce the weight of the space by 1, and thus for those cases we have $l(x_1f) = l(f)$. On the other hand, when the right space of $f$ has bottom label $\mathbf{L}_R$, $\mathbf{I}_L$, or

$\mathbf{I}_{LR}$, then changing the top label from $\mathbf{R}_L$ to $\mathbf{I}_L$ will not change the weight of the space, so $l(x_1 f) = l(f) + 1$.

**Case 5:**

| | $\mathbf{L}_0$ | $\mathbf{L}_R$ | $\mathbf{R}_0$ | $\mathbf{R}_L$ | $\mathbf{I}_0$ | $\mathbf{I}_L$ | $\mathbf{I}_R$ | $\mathbf{I}_{LR}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{R}_L$ | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 |
| $\mathbf{I}_{LR}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Suppose that the right space of $f$ has top label $\mathbf{R}_L$ and that the current tree of $f$ is non-trivial. Then that space will be labeled $\mathbf{I}_{LR}$ in $x_1 f$. Looking at the table, we see that if the right space of $f$ has bottom label $\mathbf{R}$ then it will have weight 2 in $f$, but weight 1 in $x_1 f$, and thus $l(x_1 f) = l(f)$. Alternatively, if the right space of $f$ has bottom label $\mathbf{L}$ or $\mathbf{I}$, then changing the top label to $\mathbf{I}_{LR}$ will have no effect on the weight of the space, and thus $l(x_1 f) = l(f) + 1$.

We now state a corollary which follows directly from the above theorem.

**Proposition 2.1.13.** *Let $f \in F$, and suppose that left multiplying $f$ by $y$ creates a caret in the top forest. Then $|l(yf) - l(f)| \leq 1$. Specifically, $l(yf) = l(f) - 1$ if and only if the left space of $f$ has label $\begin{bmatrix} \mathbf{L}_0 \\ \mathbf{L}_0 \end{bmatrix}$. Also, $l(yf) = l(f)$ if:*

1. *The left space of $f$ has bottom label $\mathbf{R}_0$ or $\mathbf{I}_R$, and the current tree is trivial.*
2. *The left space of $f$ has top label $\mathbf{L}_R$ and bottom label $\mathbf{L}_0$, and the current tree is non-trivial.*
3. *The left space of $f$ has bottom label $\mathbf{L}_R$.*
4. *The left space of $f$ has top label $\mathbf{L}_R$, bottom label $\mathbf{I}_0$, and the current tree is trivial.*

Given the Propositions which we have proven so far, we are now able to prove the following theorem.

**Theorem 2.1.14.** *Let $f \in F$, and let $x$ be any generator from the set $\{x_0, x_1, y\}$. Then $|l(xf) - l(f)| \leq 1$.*

**Proof.** Let $f$ be an element of $F$, and let $x \in \{x_0, x_1, y\}$. If $x = x_0$, then Proposition 2.1.8 tells us that $|l(x_0 f) - l(f)| \leq 1$. This immediately implies that if $x = x_0^{-1}$, then $|l(x_0^{-1} f) - l(f)| \leq 1$ as well. Now suppose that $x = x_1$. It is evident that $x_1$ will either remove a caret in the bottom forest of $f$ or create a caret in the top forest of $f$. If $x_1$ removes a caret in the bottom forest, then Proposition 2.1.10 gives us $l(x_1 f) - l(f) = -1$. On the other hand, if $x_1$ creates a caret in the top forest, then Proposition 2.1.12 tells us that $|l(x_1 f) - l(f)| \leq 1$. These immediately imply that $|l(x_1^{-1} f) - l(f)| \leq 1$ as well. Likewise, Propositions 2.1.11 and 2.1.13 show that $|l(yf) - l(f)| \leq 1$ and that $|l(y^{-1} f) - l(f)| \leq 1$. This proves our theorem. $\square$

**Theorem 2.1.15.** *Let $f \in F$ be a nonidentity element.*

1. *If the current tree of $f$ is nontrivial, then either $l(x_1^{-1} f) < l(f)$ or $l(x_0 f) < l(f)$.*
2. *If left-multiplication by $x_1$ would remove a caret from the bottom tree, then $l(x_1 f) < l(f)$.*
3. *Otherwise, either $l(x_0 f) < l(f)$ or $l(x_0^{-1} f) < l(f)$ or $l(yf) < l(f)$.*

**Proof. Case 1:** First, suppose that the current tree of $f$ is nontrivial. Thus $x_1^{-1}$ will remove a caret from the upper forest of $f$. Suppose that $l(x_1^{-1} f) > l(f)$. From Proposition 2.1.12, we see that $l(x_1^{-1} f) > l(f)$ only when the right space of $x_1^{-1} f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{R}_0 \end{bmatrix}$. Thus we know that the right space of $f$ has label $\begin{bmatrix} \mathbf{R} \\ \mathbf{R} \end{bmatrix}$. Proposition 2.1.8 then tells us that $l(x_0 f) < l(f)$. On the other hand, if $l(x_1^{-1} f) = l(f)$, then we see from Proposition 2.1.12 that either the right space of $f$ has top label $\mathbf{R}_L$ and bottom label $\mathbf{R}_0$, or that the right space of $f$ has bottom label $\mathbf{R}_L$. Proposition 2.1.8 then gives us $l(x_0 f) < l(f)$.

**Case 2:** This follows directly from Proposition 2.1.10.

**Case 3:** Suppose that $l(x_0 f) > l(f)$. By Proposition 2.1.8, we see that there are 4 conditions which could cause this. However the fourth condition assumes that the current tree is non-trivial, which contradicts our first assumption. We will now present subcases for each of the three possible conditions:

*Subcase 1:* Suppose that $x_0 f$ has larger support than $f$, so that the right space of $f$ is not in the support of $f$. Note that the current tree of $f$ must be trivial by assumption. If the left space of $f$ has bottom label $\mathbf{I}_R$, then the left space of $f$ will have label $\begin{bmatrix} \mathbf{L} \\ \mathbf{I}_R \end{bmatrix}$, and thus $l(x_0^{-1} f) < l(f)$ by Proposition 2.1.9. If the left space of $f$ has bottom label $\mathbf{I}_0$, we also know that the top label must be either $\mathbf{L}_R$ or $\mathbf{L}_0$. If the top label is $\mathbf{L}_R$, then by Proposition 2.1.9 we see that $l(x_0^{-1} f) < l(f)$. If the top label is $\mathbf{L}_0$, then it is evident that multiplication by $y$ will remove the bottom caret, and thus by Proposition 2.1.11 we see that $l(yf) < l(f)$. If the left space of $f$ has bottom label $\mathbf{L}$, then the left space of $f$ will be labeled $\begin{bmatrix} \mathbf{L} \\ \mathbf{L} \end{bmatrix}$, and thus $l(x_0^{-1} f) < l(f)$ by Proposition 2.1.9. If the left space of $f$ has bottom label $\mathbf{R}$, then it is evident that multiplication by $x_0^{-1}$ will remove the left space of $f$ from the support of $f$, and thus by Proposition 2.1.9 we see that $l(x_0^{-1} f) < l(f)$.

*Subcase 2:* Suppose that the right space of $f$ has bottom label $\mathbf{L}$, and that left multiplication by $x_0$ does not remove this space from the support. Recall that the current tree of $f$ is trivial by assumption. We then see that the left space of $f$ must have bottom label $\mathbf{L}$, $\mathbf{I}_0$, or $\mathbf{I}_R$, and top label $\mathbf{L}$. If the bottom label of the left space is $\mathbf{L}$ or $\mathbf{I}_R$, then we see by Lemma 3.4 that $l(x_0^{-1} f) < l(f)$. If the bottom label of the left space is $\mathbf{I}_0$, then it is evident that multiplication by $y$ will remove the caret from the bottom forest of the left space of $f$, and thus $l(yf) < l(f)$ by Proposition 2.1.12.

*Subcase 3:* Suppose that the right space of $f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{I}_R \end{bmatrix}$. Then the left space of $f$ will have top label $\mathbf{L}_R$ or $\mathbf{L}_0$ and bottom label $\mathbf{I}_R$ or $\mathbf{I}_0$. If the top label is $\mathbf{L}_R$, then Proposition 2.1.9 tells us that $l(x_0^{-1} f) < l(f)$. If the top label is $\mathbf{L}_0$ and the bottom label is $\mathbf{I}_R$, then we see again by Proposition 2.2.9 that $l(x_0^{-1} f) < l(f)$. If the top label is $\mathbf{L}_0$ and the bottom label is $\mathbf{I}_0$, then it is evident that multiplication by $y$ will remove the caret from the bottom forest of the left space of $f$, and thus $l(yf) < l(f)$ by Proposition 2.1.12. $\square$

## 2.2 Dead Ends

**Definition 2.2.1.** An element $f \in F$ is a *dead end* if multiplication on $f$ by any of the three generators of $F$ decreases the length of $f$, or equivalently, if $l(xf) < l(f)$ for all $x \in \{x_0, x_1, y, x_0^{-1}, x_1^{-1}, y^{-1}\}$. $\triangle$

The terminology "'dead end"' makes sense when viewed in context of the Cayley graph of $F$, where moving in any direction from such an element will bring you closer to the identity. Cleary and Taback [ClTa2] proved that there exist dead ends in the $\{x_0, x_1\}$ generating set for $F$. However, the addition of the $y$ generator ensures that, for any element $f \in F$, there will always be some $xf$ which is no farther from the identity than $f$ is. The proof is as follows.

**Theorem 2.2.2.** *There exist no dead ends in $F$ given the $\{x_0, x_1, y\}$ generating set for $F$.*

**Proof.** We will prove the result by contradiction: assume that there exists some $f \in F$ such that $f$ is a dead end. It is evident that multiplication by $x_1$ will either remove a caret from the bottom forest of $f$ or create a caret in the top forest of $f$. If multiplication by $x_1$ removes a caret from the bottom forest of $f$, then the right space of $f$ must have bottom label $\mathbf{I}_0$,top label $\mathbf{R}_0$, and the current tree must be trivial. However, we then see by Proposition 2.1.8 that if the right space of $f$ has label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{I}_0 \end{bmatrix}$, and if the current tree is trivial, then $l(x_0 f) = l(f)$, which contradicts the fact that $f$ is a dead end.

Now suppose that multiplication by $x_1$ creates a caret in the top forest of $f$. We then see by Proposition 2.1.12 that the only way in which multiplication by $x_1$ could decrease the length of $f$ would be if the right space of $f$ had label $\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{R}_0 \end{bmatrix}$. Given this, there are two ways in which $y$ could decrease the length of $f$: if multiplication by $y$ removes a caret from the bottom forest, or if the left space of $f$ has label $\begin{bmatrix} \mathbf{L}_0 \\ \mathbf{L}_0 \end{bmatrix}$. First, suppose that multiplication by $y$ removes a caret from the bottom forest. Then the left space of $f$ must have bottom label $\mathbf{I}_0$, and the current tree must be trivial. We then see by Proposition 2.1.9 that in order for $x_0^{-1}$ to decrease the length of $f$, we must have the top label of the left space of $f$ be $\mathbf{L}_R$. However, if the left space of $f$ has top label $\mathbf{L}_R$, then multiplication by $y$ will not remove a caret in the bottom forest of $f$, which provides a contradiction.

Now suppose that the left space of $f$ has label $\begin{bmatrix} \mathbf{L}_0 \\ \mathbf{L}_0 \end{bmatrix}$. We will now show that the current tree must be non-trivial. Suppose that the current tree is trivial. Then it is evident that multiplication by $x_1^{-1}$ will create a caret in the bottom forest of $f$. However, as an obvious corollary to Proposition 2.1.10, we see that if multiplication by $x_1^{-1}$ creates a bottom caret, then $l(x_1^{-1}f) = l(f) + 1$. Thus the current tree must be non-trivial. Since the current tree is non-trivial, it is clear that multiplication by $x_1^{-1}$ or $y^{-1}$ will remove a caret from the current tree. We now have two cases.

First, suppose that the current tree of $f$ is comprised of one caret. Considering the bottom label of the current space of $f$, we see that this label cannot be $\mathbf{I}_0$, since we are assuming that $f$ is reduced. Also, since the space immediately to the right of the current bottom space has label $\mathbf{R}_0$, the current bottom space cannot have label $\mathbf{I}_L$. Likewise, since the space immediately to the left of the current bottom space has label $\mathbf{L}_0$, we know the current bottom space cannot have label $\mathbf{I}_R$, and thus it cannot have $\mathbf{I}_{LR}$ either. The same line of reasoning tells us that the current bottom space cannot be a $\mathbf{R}_L$, since the space to the right of it is labeled $\mathbf{R}_0$. Nor can it be a $\mathbf{L}_R$, since the space to the left of it is labeled $\mathbf{L}_0$. Thus the label of the current bottom space must be either $\mathbf{L}_0$ or $\mathbf{R}_0$. If the current bottom label is $\mathbf{L}_0$, then multiplication by $y^{-1}$ will remove the top caret and change the label of the upper space from $\mathbf{I}_0$ to $\mathbf{L}_0$. Clearly, removing a caret will mean $l_1(y^{-1}f) = l_1(f) - 1$. We can then see an $\mathbf{I}_0$ opposite a $\mathbf{L}_0$ has weight 0, but an $\mathbf{L}_0$ opposite a $\mathbf{L}_0$ has weight 2. Thus $l_0(y^{-1}f) = l_0(f) - 2$. We can then compute $l(y^{-1}f) = l(f) + 1$, and therefore $f$ is not a dead end. If the current bottom label is $\mathbf{R}_0$, then multiplication by $x_1^{-1}$ will have the same effect as $y^{-1}$ in the previous lines, and will increase the length by 1.

Now suppose the current tree is comprised of multiple carets. If we multiply $f$ by either $x_1^{-1}$ or $y^{-1}$, then the top caret of the current tree will be removed. Removing this caret will create an additional space, either to the right or the left of the current tree depending on which generator was last used to make the caret. Suppose that removing the top caret creates an additional space to the right of the current tree. We then see that the space "opened up" by the removal of that caret must have had upper label $\mathbf{I}_R$ in $f$. We will now go through some subcases which address the different possibilities for the lower label of the space which was just "opened up".

*Subcase 1* First suppose that the space "opened up" by removing a top caret has bottom label $\mathbf{L}$. We then see that $y^{-1}$ will change the top label of that space from $\mathbf{I}_R$ to $\mathbf{L}_R$. Note that $\mathbf{I}_R$ opposite $\mathbf{L}$ will always have weight 1, but $\mathbf{L}_R$ opposite $\mathbf{L}$ will always have weight 2. Thus $l_0(y^{-1}f) = l_0(f) + 1$. Since we are removing a caret with $y^{-1}$, we know that $l_1(y^{-1}f) = l_1(f) - 1$. Thus we see that $l(y^{-1}f) = l(f)$, which contradicts the fact that $f$ is a dead end.

*Subcase 2* Now suppose that the bottom label of the space "opened up" by removing a top caret has bottom label $\mathbf{R}$. Recall that the space immediately to the right of this space has bottom label $\mathbf{R}_0$, and thus this space cannot have bottom label $\mathbf{R}_L$. Thus the bottom label of this opened space must be $\mathbf{R}_0$. We then see that multiplication by $y^{-1}$ will change the top label of the opened space from $\mathbf{I}_R$ to $\mathbf{L}_R$. Note that an $\mathbf{I}_R$ opposite an $\mathbf{R}_0$ has weight 0, but an $\mathbf{L}_R$ opposite an $\mathbf{R}_0$ has weight 1. Thus we see that $l_0(y^{-1}f) = l_0(f) + 1$. Since multiplication by $y^{-1}$ removes a caret from $f$, we also know $l_1(y^{-1}f) = l_1(f) - 1$. Thus $l(y^{-1}f) = l(f)$, which contradicts the fact that $f$ is a dead end.

*Subcase 3* Now suppose that the bottom label of the space opened up is an $\mathbf{I}$ of some sort. Note that the space to the right of this opened space has label $\mathbf{R}_0$, and thus we know that the opened space cannot have label $\mathbf{I}_L$. Thus it has label $\mathbf{I}_0$ or $\mathbf{I}_R$. We then see that multiplication by $y^{-1}$ will change the top label of the opened space from $\mathbf{I}_R$ to $\mathbf{L}_R$. Note that an $\mathbf{I}_R$ opposite an $\mathbf{I}_0$ or an $\mathbf{I}_R$ will have weight 0, but an $\mathbf{L}_R$ opposite an $\mathbf{I}_0$ or an $\mathbf{I}_R$ will have weight 1. Thus $l_0(y^{-1}f) = l_0(f) + 1$. Since multiplication by $y^{-1}$ removes a caret from $f$, we also know $l_1(y^{-1}f) = l_1(f) - 1$. Thus $l(y^{-1}f) = l(f)$, which contradicts the fact that $f$ is a dead end.

Thus we see that there are no dead ends in $F$.

$\square$

# 3

# Harmonic Functions

## 3.1 Background on Harmonic Functions

Another currently unanswered question concerning $F$ pertains to the existence of a non-constant, bounded harmonic function on the Cayley graph of $F$.

**Definition 3.1.1.** Let $\Gamma$ be a graph with vertices $V(\Gamma)$. A *Harmonic function* is a function $f : V(\Gamma) \to \mathbb{R}$ such that

$$f(v) = \frac{f(v_1) + f(v_2) + \cdots + f(v_n)}{n}$$

where $v_1, \ldots, v_n$ are all the vertices adjacent to $v$.

$\triangle$

In other words, the value assigned to each vertex by a harmonic function is the average of the weights of all vertices which share an edge with it. Clearly, such a function can be obtained merely by assigning the same value to each vertex in $G$. However, we are interested in non-constant harmonic functions, as these are closely related to the geometry of the Cayley graph of $F$, and by extension the amenability of $F$.

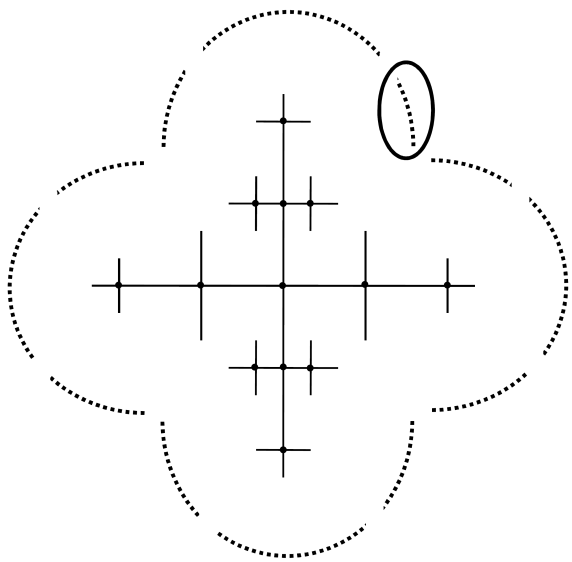
To describe the relation between harmonic functions and amenability, we must first speak about random walks and the Poisson boundary of a Cayley graph. A random walk on a graph $\Gamma$ is what one would likely expect it to be — a path on the graph that is determined randomly. To formalize this concept we have the following definition.

**Definition 3.1.2.** Let $\Gamma$ be a graph with $V$ the set of vertices of $\Gamma$. A *random walk* on $\Gamma$ is a sequence $v_1, v_2, v_3, \ldots$ of elements of $V$, where $v_1$ is called the starting vertex of the random walk, and where $v_i$ is chosen randomly from the neighbors of $v_{i-1}$ for all $i > 1$. $\triangle$

As a random walk has no final vertex, we cannot talk about the end of a random walk. However, we can talk about the section of a Cayley graph which a random walk will almost surely (with probability 1) go to "at infinity". Central to this is the concept of a Poisson boundary.

The Poisson boundary of a Cayley graph is a certain measure space, which can be thought of as a "boundary at the infinity" of a Cayley graph. A Poisson boundary represents the outcome, or finishing point, of a transient random walk on $\Gamma$. Transient, as opposed to recurrent random walks, are random walks which do not repeat themselves after a finite number of vertices. Although we will not define the concept of a Poisson boundary rigorously, we will provide an example of a Poisson boundary to elucidate this concept.

**Example 3.1.3.** We return to $F_2$. Below we see a small portion of the Cayley graph of $F_2$. The dotted lines around the edge of the graph correspond to the Poisson boundary of $F_2$. Note that the Poisson boundary is first separated into four sections: an upper, lower, left, and right section. These sections correspond, respectively, to the set of all random walks which end in the upper, lower, left, and right sections of the Cayley graph. Likewise, each of these four sections are split into three additional sections. For a given quadrant of the Cayley graph, these three additional sections of the Poisson boundary correspond to the subsections of that quadrant. For example, if a random walk starting at the center vertex first travels up, then to the right, and never travels back, then it will be represented by the upper right section of the Poisson boundary (which we have circled for convenience). It turns out that the Poisson boundary of $F_2$ is in fact isomorphic to the Cantor set.



$\lozenge$

This idea of random walks being represented by sections of the Poisson boundary is supported by the next theorem, which provides us with a direct relation between harmonic functions on a Cayley graph and the Poisson boundary of that Cayley graph.

**Theorem 3.1.4.** *Let $\Gamma$ be a Cayley graph with Poisson boundary $P$. There there is a one-to-one correspondence between bounded harmonic functions on $\Gamma$ and bounded measurable (i.e. $L^\infty$) functions on $P$.*

Thus information about the Poisson boundary can inform us about harmonic functions on $\Gamma$, and vice versa. Importantly, if we know that there exists a bounded measurable function on the Poisson boundary, then there must also exist a bounded harmonic function on $\Gamma$. This theorem, in conjunction Theorem 3.1.6, explains our search for a harmonic function on the Cayley graph of $F$.

The next theorem relates Poisson boundaries with the concept of amenability.

**Theorem 3.1.5.** *Let $P$ be the Poisson boundary of the Cayley graph $\Gamma(G)$ for some group $G$. If $P$ is trivial (i.e. consists of a single point), then $G$ is amenable.*

This theorem should seem intuitively reasonable, as a trivial Poisson boundary would imply that all random walks "end up" in the same place. Thus if we consider a large subgraph of $\Gamma$, all edges leading out of this subgraph will be headed towards this ending place, and thus would vanish proportional to the size of our subgraph. Of course, one of the defining features of $F$ is its resistance to common methods of proving amenability, and in this respect it does not disappoint us here.

**Theorem 3.1.6.** *The Poisson boundary of $F$ is non-trivial.*

It is important to note that the converse of Theorem 3.1.6 is untrue - there are known groups with non-trivial Poisson boundaries which are amenable. While the Poisson boundary of $F$ cannot be used with regards to Theorem 3.1.5 to prove amenability, the non-triviality of this boundary leads us on a search for a harmonic function on $F$.

Creating a harmonic function on a Cayley graph of a group so complicated as $F$ from thin air would be difficult. Thankfully, the following theorem provides us with just such a function, although it tells us little about the values that the function takes. Before stating the theorem, however, we will discuss what it means for two random walks to have the same tail.

**Definition 3.1.7.** If $R$ and $S$ are two random walks where $R = v_1, v_2, \ldots$ and $S = w_1, w_2, \ldots$, then $R$ and $S$ have the same tail if there exists $m, n \in \mathbb{N}$ such that $v_{m+i} = w_{n+i}$ for all $i \in \mathbb{Z}$ with $i \geq 0$. $\triangle$

Therefore, two random walks have the same tail if, after some point, they follow the exact same path from vertex to vertex. We can now state a theorem which will assist us greatly in finding a harmonic function on $F$.

**Theorem 3.1.8.** *Let $\Gamma(G)$ be a Cayley graph. Let $E$ be an event (in the sense of probability theory) that depends only on the tail of a random walk. Define $f : V(\Gamma) \to \mathbb{R}$ by*

$$f(v) = P(E \textit{ occurs for a random walk starting at } v)$$

*where $v$ is any vertex of $\Gamma$. Then $f$ is a bounded harmonic function on $\Gamma$.*

Thus, to find a harmonic function on $F$, we need only to find an event which does not depend on any finite number of moves in a random walk. In the next section we will provide an example of one such event.

## 3.2 Existence of and Lower Bounds for a Harmonic Function on $F$

**Example 3.2.1.** We again look at $F_2$. Consider the following event $E$, consisting of all random walks eventually lying in the right quadrant of the Cayley graph of $F_2$ at infinity. Clearly, this event depends only on the tail of a walk. Thus, the probability that a random walk starting at any vertex $v$ ends in the right quadrant provides a harmonic function on $\Gamma(F_2)$.

For a reason that such a function need be harmonic, suppose that the probability of $E$ happening starting at some vertex $v$ is $p$. Let $v_1, v_2, v_3, v_4$ be the vertices adjacent to $v$. In a random walk, it is clear that we have a $\frac{1}{4}$ probability of first moving to $v_1$. If we let $p_1$ denote the probability of $E$ starting a $v_1$, then one-fourth of the time, $p$ will equal $p_1$. Repeating the same argument for $v_2, v_3$, and $v_4$, and noting that $v$ will necessarily move to one of those four vertices in a random walk starting at $v$, shows us that $p = \frac{1}{4}p_1 + \frac{1}{4}p_2 + \frac{1}{4}p_3 + \frac{1}{4}p_4$. ◊

Such events which depend only on the tails of random walks are not difficult to come by. For $F$, one such event is as follows:

**Definition 3.2.2.** Let $E$ be the event consisting of all random walks on $\Gamma(F)$ such that the bottom pointer eventually always points to a trivial caret. Let $V$ be the set of all vertices of $\Gamma(F)$. We define $\sigma\colon V \to \mathbb{R}$ as

$$\sigma(v) = P(E \text{ occurs for a random walk starting at } v)$$

△

Any element $f \in F$ for which the bottom pointer points at a trivial caret will be referred to as "*empty*″. Clearly $E$ does not depend on any finite number of moves, and thus Theorem 5.8 tells us that assigning the probability of such an event happening to each vertex will induce a harmonic function on $\Gamma(F)$. Thus $\sigma$ is harmonic. Determining $\sigma$ exactly for elements of $F$ is not an easy task. Our original goal in exploring harmonic functions was to find a formula for computing $\sigma$. Unfortunately, we were unable to find such a formula. However, we were able to obtain some lower bounds for what values such a function could take. To explain how these bounds were obtained, we first introduce another concept, that of automata.

**Definition 3.2.3.** An *automaton* is a directed graph with probabilities assigned to each edge, such that the sum of the probabilities of the edges directed out of each vertex equals 1. △

In an automaton, vertices are often called "states", and edges "transitions". If there is a directed edge leading from one state $s$ to another $t$, we say that $s$ transitions to $t$. We will adopt this convention as well, to distinguish our discussion of automaton from one concerning graphs.

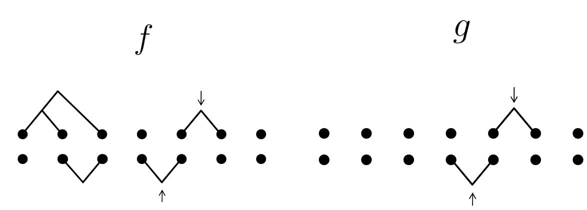
**Definition 3.2.4.** Let $A$ be an automaton, with states $S$. A function $\phi\colon S \to \mathbb{R}$ is *harmonic* if

$$\phi(s) = p_1\phi(s_1) + p_2\phi(s_2) + \cdots + p_n\phi(s_n)$$

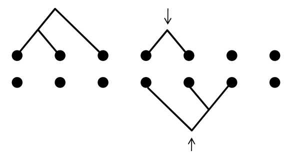
where $s_1, s_2, \ldots, s_n$ are the states which $s$ transitions to, and $p_i$ is the probability assigned to the transition from $s$ to $s_i$. △

This definition of a harmonic function closely resembles our previous definition for Cayley graphs. The difference is that, for any given vertex $v$, a harmonic function on a graph gives the unweighted average of its values on all vertices adjacent to $v$. On an automaton, the average provided by a harmonic function is weighted according to the probabilities assigned to each transition. To use automata in the context of $F$, we create an automaton in which each state represents a set of vertices of $\Gamma(F)$ on which $\sigma$ is constant. Note that $\sigma$ can remain constant on elements of $F$ in a few ways. One such way is through the addition or removal of "inessential" carets, or carets which are not part of a tree which is in either the top or bottom space pointed to by the bottom pointer.

**Example 3.2.5.** Consider the elements $f, g \in F$ shown as forest diagrams below. Note that $g$ is obtained from $f$ precisely by removing the trees to the left of the top and bottom pointer, neither of which are in the top or bottom space pointed to by the bottom pointer. Thus $g$ is obtained from $f$ by the removal of inessential carets, and $\sigma(f) = \sigma(g)$. It is clear that the addition and removal of such spurious carets to $f$ will not change $\sigma(f)$. In our automaton, $f$ and $g$ are represented by the same state.



$h$

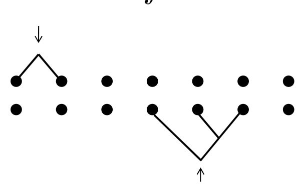


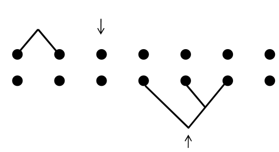
The element $h \in F$, however, is not represented by the same state as $f$ or $g$. To see this, note that the configuration of the trees pointed to by the bottom pointer in $h$ is totally different from the configurations in $f$ and $g$. The only way to move from $h$ to $f$, for example, is by the deletion and addition of essential carets. Thus we have no reason to expect $\sigma(h) = \sigma(f)$.. $\Diamond$

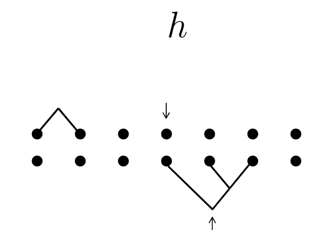
Another way in which $\sigma$ can remain constant on elements of $f$ pertains to the location of the top pointer. Specifically, if $f \in F$ is an element for which the top pointer is in a position where it can only add or remove a "spurious" caret, then moving it to either side will not change $\sigma(f)$. Also, if moving the top pointer to the left (or right) will result in an element for which the top pointer can only add or remove a spurious caret, then $\sigma(f) = \sigma(x_0^{-1}f)$ (or $\sigma(x_0 f) = \sigma(f)$).

**Example 3.2.6.** Consider the elements $f, g \in F$ shown as forest diagrams below. It is obvious that multiplication of $f$ by any generator will only add or remove a "spurious" caret, and thus will not affect $\sigma(f)$. Note that $g$ is obtained from $f$ by moving the top pointer one space to the right. As such, we see that $\sigma(f) = \sigma(g)$. Both $f$ and $g$ are then represented by the same state in our automaton.

$f$ $g$



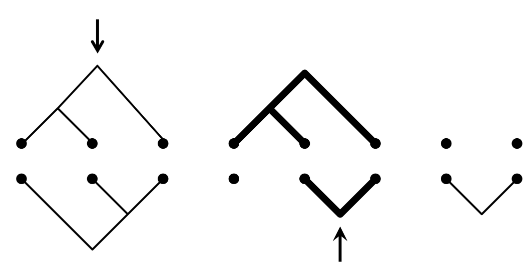




Now consider the element $h \in F$. We see that the top pointer of $g$ is in a position where it can create an non-spurious caret (via multiplication by $x_1$). Furthermore, we see that moving the top pointer of $g$ one space to the right will not result in an element for which the top pointer can only create spurious carets. As $h$ is obtained from $g$ by moving the top pointer one space to the right, we then have no reason to expect $\sigma(g) = \sigma(h)$. The elements $h$ and $g$ are represented by different states in our automaton. $\Diamond$

We provide a final example to highlight the difference between essential and inessential carets.

**Example 3.2.7.** Consider the element $f \in F$ shown below. We have bolded the essential carets while reducing the size of the inessential carets.



$\Diamond$

Our automata will be made up of states, each of which represent sets of elements of $F$ as described above. Note that each of these states will have a unique element $f$ for which

- $f$ has a minimal number of carets, and
- the top pointer of $f$ is in a position where it can make non-spurious carets.

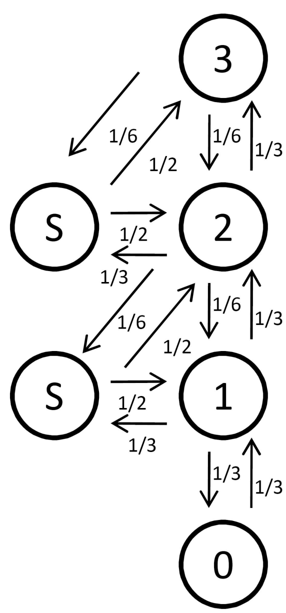
We represent our states by these elements. For each of these states $s$ we create a transition to another state $t$ if the element used to describe $t$ can be reached from the element used to describe $s$ via multiplication through one generator. We then assign a probability of $\frac{1}{n}$ to each transition, where $n$ is the total number of transitions leaving the state.

Although each state represents an infinite number of elements of $F$, it is nonetheless clear that there are an infinite number of states — for example, there will be a unique state represented by $x_1^{-n}$ for each $n \in \mathbb{N}$. To make the task of finding a harmonic function on our automata easier, we therefore consider only a finite subgraph of our automata.

**Definition 3.2.8.** Let $\gamma$ be a subgraph of a graph $\Gamma$ with vertices $V_\gamma$. Then the boundary $\partial V_\gamma$ is the set of all vertices of $\gamma$ which are adjacent to some vertex $v \notin V_\gamma$. △

We next introduce a theorem which enabled us to find our lower bounds.

**Theorem 3.2.9.** $\sigma(i) \leq .5$




**Proof.**

Figure 1

We prove the theorem by looking at automata, as described above. First consider Figure 1, which shows an automaton. The circled numbers in this figure represent states with that number of carets on top of the bottom pointer. For example, the "1" state represents elements of $F$ with one caret on the top space pointed to by the bottom pointer, and no carets in the bottom space. The "0" state represents the identity. The "S" state to the side of the "1" state represents a state where there is one caret on top of the bottom pointer, and the top pointer is to the side of that caret. It is evident that, starting at the 1 state, either $x_0$ or $x_0^{-1}$ will take you to the $S$ state. Thus, on a random walk, you would have a $\frac{1}{3}$ probability of moving from 1 to $S$, which is denoted by transition from 1 to $S$ with $\frac{1}{3}$ next to it. All of the transitions in the diagram are likewise labeled. For reasons described above, we only consider states from which essential carets can be added or removed by multiplication by a generator.

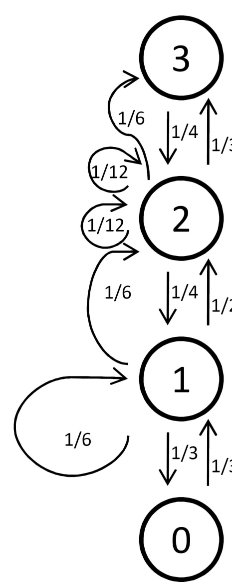



Figure 2

It is then clear that Figure 2 can be obtained from Figure 1 by combining the probabilities $S$ states into the probabilities for the numbered states and normalizing. In the same manner, Figure 3 can be obtained from Figure 2 by removing the redundant transitions and recalculating the probabilities for the numbered states.

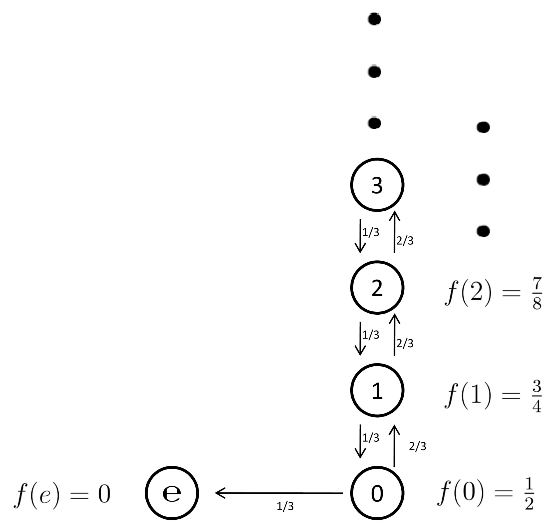



Figure 3

The probability that a random walk starting at the zero state will never build a lower caret is easy to find on Figure 3 — note that a random walk on a given numbered state will have a $\frac{2}{3}$ probability of moving up to the next higher number, and a $\frac{1}{3}$ chance of moving down to a lower number. Then, for any state with number $n$, we have $\mu(n) = \dfrac{2^{n+1}-1}{2^{n+1}}$. Clearly this function is harmonic, and thus $\alpha = \mu(0) = \frac{1}{2}$. Note that this is the probability that a random walk starting at the identity will never build a lower caret. As it is possible for a random walk to build a lower caret and later remove it, we must have $\mu(i) \leq \sigma(i)$, and thus $\sigma(i) \geq \frac{1}{2}$. $\square$

**Definition 3.2.10.** Let $V$ be the set of vertices of some finite subgraph $\Gamma$, and let $\partial V$ be the set of boundary vertices. Then a function $\phi$ is *harmonic* on $\Gamma$ if it harmonic on all vertices $v \in V$ such that $v \notin \partial V$, that is, if it is harmonic on all interior vertices. $\triangle$

Since an automaton is a directed graph, we can consider subgraphs of it the same way we would subgraphs of any other graph. Thus a harmonic function on our sub-automaton would be a function that is harmonic on all non-boundary states. We now define explicitly the elements of our automaton. Let $A$ be our automaton. Then the states of $A$ are all sets of elements of $f$ represented by elements with length less than or equal to 5, and such that if $e$ is the element of $F$ representing a state, then either $e = i$ (where $i$ is the identity) or $e$ is reached from $i$ by first multiplying by $x_1^{-1}$. Our boundary consists of all states represented by elements of length 5, together with the state represented by the identity element.

To see how such an automaton produces lower bounds for our harmonic function $\sigma$, we introduce the following theorem.

**Theorem 3.2.11.** *Let $V$ be the set of vertices of some finite graph $\Gamma$ with boundary $\partial V$. Then every function $f\colon \partial V \to \mathbb{R}$ extends to a unique harmonic function $\overline{f}$ on $V$. Furthermore, if $f, g\colon \partial V \to \mathbb{R}$ are two such functions such that $f(v) \leq g(v)$ for all $v \in \partial V$, then $\overline{f}(v) \leq \overline{g}(v)$ for all $v \in V$.*

In other words, Theorem 3.2.11 states that, given a finite graph with a boundary, the values that a harmonic function takes on the boundary vertices determine the values that the function takes on all vertices. Also, given two such functions defined on the boundary of a graph, if one function always outputs lower values than the other, the extension of that function to the entire graph will likewise always output lower values than the extension of the other.

To use Theorem 3.2.11 with respect to $A$, we define a new harmonic function $\mu$ by assigning to all boundary states represented by elements of length 5 the value of zero. Intuitively, these elements are of distance 5 from the identity, and thus would require multiplication by 5 generators to reach the identity. Since each of these elements also has at least one caret in the bottom space pointed to by the bottom pointer, it is reasonable that the probability of having the bottom pointer point always at a trivial caret for a transient random walk is relatively low. However, note that $\sigma$ represents a probability, and thus cannot give negative values. Thus we see that for all such boundary states $s$, we have $\mu(s) = 0 \leq \sigma(s)$. The only other element of our boundary is the identity.

We prove in Theorem 3.2.9 that the value of the actual harmonic function at the identity is greater than 0.5, that is $\sigma(i) \geq 0.5$. Thus we set $\mu(i) = 0.5 \leq \sigma(i)$. Theorem 3.2.11 then tells us that extending $\mu$ to the entire sub-automaton will give us a harmonic function such that for all $s \in S$, $\mu(s) \leq (s)$. This function $\mu$ will therefore provide lower bounds for our real harmonic function $\sigma$.

To obtain these lower bounds, we use the definition of a harmonic function, as well as the boundary values, to obtain a system of $n$ linear equations in $n$ unknowns, where $n$ is the number of non-boundary states of our automaton. We then use linear algebra to solve these equations. The results of these bounds are listed in the Appendix.

We now consider the state of our automaton represented by $yx_1^{-1}$. Note that, since $\sigma(i) \geq 0.5$, the probability that the top caret will not exist after a random walk on $F$ must be less than 0.5. Thus 0.5 provides an upper bound for $\sigma(yx_1^{-1})$. We also see from the Appendix that 0.5679 is a lower bound for the state represented by $x_0$. Thus we have a lower bound for $\sigma$ which is greater than an upper bound for $\sigma$, which gives us the following theorem.

**Theorem 3.2.12.** *There exists a non-constant bounded harmonic function on $F$.*

# 4
# Appendix

| Element | Lower Bound |
|---|---|
| $identity$ | 0.459901 |
| $x_0$ | 0.567921 |
| $x_1^{-1}$ | 0.135841 |
| $yx_1^{-1}$ | 0.052054 |
| $x_0^{-1}x_1^{-1}$ | 0.0939477 |
| $x_1^{-2}$ | 0.0383416 |
| $y^{-1}x_1^{-1}$ | 0.0349629 |
| $x_0^{-1}yx_1^{-1}$ | 0.0353425 |
| $y^2x_1^{-1}$ | 0.018631 |
| $x_0yx_1^{-1}$ | 0.0143956 |
| $x_1yx_1^{-1}$ | 0.0141656 |
| $x_0^{-1}x_1^{-2}$ | 0.0260323 |
| $yx_1^{-2}$ | 0.0137231 |
| $x_1^{-3}$ | 0.010162 |
| $y^{-1}x_1^{-2}$ | 0.00932762 |
| $x_1^{-1}y^{-1}x_1^{-1}$ | 0.0083245 |
| $y^{-2}x_1^{-1}$ | 0.0083245 |
| $x_0y^{-1}x_1^{-1}$ | 0.00947271 |
| $x_1y^{-1}x_1^{-1}$ | 0.00947271 |

| Element | Lower Bound | Element | Lower Bound |
|---|---|---|---|
| $x_0y^2x_1^{-1}$ | 0.0037262 | $x_1^{-1}y^{-1}x_1^{-2}$ | 0.00186552 |
| $x_1y^2x_1^{-1}$ | 0.0037262 | $y^{-2}x_1^{-2}$ | 0.00186552 |
| $x_0^{-1}y^2x_1^{-1}$ | 0.0118561 | $x_0y^{-1}x_1^{-2}$ | 0.00186552 |
| $y^3x_1^{-1}$ | 0.00508119 | $x_1y^{-1}x_1^{-2}$ | 0.00186552 |
| $x_1x_0yx_1^{-1}$ | 0.00479852 | $yx_1^{-1}y^{-1}x_1^{-1}$ | 0.0016649 |
| $x_0^2yx_1^{-1}$ | 0.00959704 | $x_0^{-1}x_1^{-1}y^{-1}x_1^{-1}$ | 0.0016649 |
| $x_1^{-1}x_0yx_1^{-1}$ | 0.00287911 | $x_1^{-2}y^{-1}x_1^{-1}$ | 0.0016649 |
| $y^{-1}x_0yx_1^{-1}$ | 0.00287911 | $y^{-1}x_1^{-1}y^{-1}x_1^{-1}$ | 0.0016649 |
| $yx_1yx_1^{-1}$ | 0.00386334 | $x_1y^{-2}x_1^{-1}$ | 0.0016649 |
| $x_1^2yx_1^{-1}$ | 0.00283312 | $x_0y^{-2}x_1^{-1}$ | 0.0016649 |
| $x_0x_1yx_1^{-1}$ | 0.00283312 | $x_1^{-1}y^{-2}x_1^{-1}$ | 0.0016649 |
| $x_0^{-1}x_1yx_1^{-1}$ | 0.00901447 | $y^{-3}x_1^{-1}$ | 0.0016649 |
| $y^2x_1^{-2}$ | 0.00374267 | $y^{-1}x_0y^{-1}x_1^{-1}$ | 0.00189454 |
| $x_0^{-1}yx_1^{-2}$ | 0.00873289 | $x_1^{-1}x_0y^{-1}x_1^{-1}$ | 0.00189454 |
| $x_0yx_1^{-2}$ | 0.00274462 | $x_0^2y^{-1}x_1^{-1}$ | 0.00602809 |
| $x_1yx_1^{-2}$ | 0.00274462 | $x_1x_0y^{-1}x_1^{-1}$ | 0.00258347 |
| $x_0^{-1}x_1^{-3}$ | 0.00646675 | $yx_1y^{-1}x_1^{-1}$ | 0.00189454 |
| $yx_1^{-3}$ | 0.00277146 | $x_0^{-1}x_1y^{-1}x_1^{-1}$ | 0.00189454 |
| $x_1^{-4}$ | 0.00203241 | $x_1^2y^{-1}x_1^{-1}$ | 0.00258347 |
| $y^{-1}x_1^{-3}$ | 0.00203241 | $x_0x_1y^{-1}x_1^{-1}$ | 0.00602809 |

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