# Graph Replacement Systems for Julia Sets of Quadratic Polynomials 

A Senior Project submitted to The Division of Science, Mathematics, and Computing of

Bard College

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May, 2018

## Abstract

Belk and Forrest construct a specific class of graph replacement systems that give sequences of graphs that converge to fractals. Given a polynomial, we have an algorithm that gives a replacement system that leads to a graph sequence which we conjecture converges to the Julia set. We prove the conjecture for the quadratic polynomial $z^{2}+c$ where $c$ is a real number and the critical point is in a three cycle. We present some additional results and observations on replacement systems obtained from certain polynomials.

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## Dedication

For my grandfather.

## Acknowledgments

I would like to thank Lauren for all her support over the years.
Thanks and sorry to all the professors who have seen me cry.
Thanks to the following friends for putting up with me and believing in me:
Adrian, Alexis, Anna, Arti, Dani, Darren, Diljit, Justin, Kara, Leanna,Raina, Sarah, and Victoria.
You guys were the best part of college.
And thanks to my other friends whom I also love and appreciate.
This project would not have been possible with Jim's excellent and enthusiastic guidance and understanding.

## Introduction

Thompson's groups are three groups $F, T$, and $V$ are finitely presented infinite groups that are of interest to group theorists because of their unusual properties. These groups act as piece-wise linear homeomorphisms on the circle, the interval, and the Cantor set. In [4], Belk and Forrest presented a Thompson-like group that acts by homeomorphisms on the Basilica, which is the Julia set of the quadratic polynomial $z^{2}-1$. In their paper, they mention that their eventual goal is to associate a Thompson-like group to the Julia set of every quadratic polynomial, in a way such that the original three Thompson's groups are associated to the quadratic polynomials with Julia sets that are homeomorphic to the circle, the interval, and the Cantor set. Two previous senior projects at Bard, [8] [10], have been written to further this program. Jasper WeinhartBurd developed a Thompson-like group for the Julia set of the post-critically finite quadratic rational function with both critical points in the same three cycle, and Will Smith developed a Thompson-like group for the function $z^{2}+i$.

Belk and Forrest's subsequent paper, [5], acts as a more general intermediate step towards the goal of associating a Thompson-like group to every quadratic polynomial. In it, they define rearrangement groups of fractals, which are a generalization of Thompson's groups. The construction works by approximating each fractal with a sequence of finite graphs, called the expansion sequence, such that the fractal is homeomorphic to the limit space. The sequence
is given by the first graph of the sequence, the base graph, and graph rewriting rules called replacement rules. Each graph in the sequence is constructed by applying the replacement rules to the previous graph. Each element of the rearrangement group associated to a given fractal acts as as a piecewise homeomorphism of one of the graphs in the expansion sequence.

Now to associate a rearrangment group to an arbitrary polynomial, we need some way of finding the expansion sequence that has a limit space that is homeomorphic the correct Julia set. The essential information about the structure and dynamics of a post-critically finite polynomial is captured by a certain tree connecting the post-critical points within the Julia set, called the Hubbard tree. This construction is described in 7. Belk and Forrest have developed an unpublished algorithm to obtain a base graph and replacement rules starting with just the Hubbard tree, which they conjecture gives a limit space that is homeomorphic to the starting Julia set. In chapter 2 we give a proof of this conjecture for the Airplane Julia set, the Julia set for the quadratic polynomial with a critical point that is in a three-cycle where all three points are on the real line.

Additionally, in chapter 3 we present finite state automata that give some progress towards how one might go about defining rearrangement groups for matings, which are rational functions that can be constructed from two quadratic polynomials.

## 1

## Background

### 1.1 Introduction to Dynamical Systems

Most of the material of this section is taken from [1]. The theory of dynamical systems is the study of how physical systems evolve over time. It has its origins in Newton's study of planetary motions, and has since found applications in virtually every scientific field. Mathematically, a dynamical system refers to a set of possible states, together with a rule that determines the present state in terms of past states [??]. For example, population models in biology are generally modeled as dynamical systems, such that a state corresponds to a population size, and the rule is something like $f(x)=2 x$ (if the population size doubles each generation).

Until the second half of the 20th century, scientists were generally under the impression a dynamical system starting from an initial state will eventually settle down into a steady state or end up oscillating among a subset of possible states, like the motion of the planets. In the 1970s, a third option came to the world's attention: "chaos". In Maxwell's investigation of gas laws, he showed that if two particles with random positions and equal but opposite velocities collide, then all direction of travels will be equally likely after the collision. This behavior is distinct from that of systems that simply have quasi-periodic behavior with a large number of periods, in that chaotic systems exhibit sensitive dependence on initial conditions, meaning that


Figure 1.1.1. The Mandelbrot Set
an arbitrarily small change of the initial state will have large, essentially unpredictable affects on the long term behavior of the system.

The introduction of "chaos" ${ }^{\square}$ into the vernacular of dynamical systems theorists coincided with the introduction of computer visualization to the field. Edward Lorenz, the meteorologist who gave the famous talk "Predictability: does the flap of a butterfly's wing in Brazil set off a tornado in Texas?", in the same talk presented a computer graphic of a strange attractor, a subset of states the dynamical system tends to and in which the system behaves chaotically. ${ }^{2}$ Strange attractors have fractal structures, meaning that they exhibit self similarity at increasingly small scales.

One of the most well known mathematical structures in the popular consciousness is the Mandelbrot set, a fractal first visualized by Benoit Mandelbrot on an IBM computer in 1980. The Mandelbrot set is the set of $c$ values for which the critical point of the function $z^{2}+c$ does not diverge, and so in this sense it indexes the set of complex quadratic polynomials, which will be the objects of focus in this project.

[^0]

Figure 1.2.1. The Basilica Julia set

### 1.2 Dynamics on the Complex Plane

We begin by presenting some basic definitions from complex dynamics, taken from 3 .

Definition 1.2.1. The orbit of a point $p$ on the complex plane under a function $f: \mathbb{C} \rightarrow \mathbb{C}$ refers to the set

$$
\{p, f(p), f(f(p)), f(f(f(p))), \ldots\}
$$

The filled Julia set of $f$ is the set of points that have a bounded orbit.

Example 1.2.2. Let $f=z^{2}$. The orbits of points under iteration of $f$ include:

$$
\{i,-1,1,1, \ldots\},\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\},\{2 i,-4,16,256, \ldots\}
$$

It general, if $|p| \leq 1$, then the orbit of $p$ is bounded, and if $|p|>1$, then the orbit of $p$ is unbounded. Thus the filled Julia set of $z^{2}$ is the closed unit disc.

Definition 1.2.3. The Julia set of a polynomial is defined as the boundary of the closed Julia set. (Equivalently, it is the closure of the set of repelling periodic points.)

### 1.3 Hubbard Trees

Most of the material in this section is taken from [7] and many of the pictures are from Belk.


Figure 1.2.2. The Douady Rabbit Julia set


Figure 1.2.3. $z^{2}+i$

Recall that a critical point of a polynomial $f$ is a point $z$ such that $f^{\prime}(z)=0$. If a point $p$ is in the orbit of a critical point, we call $p$ a postcritical point. If $f$ has a finite number of postcritical points, we say that $f$ is postcritically finite. In other words,

Definition 1.3.1. A polynomial function $p$ is postcritically finite if every critical point of $p$ is periodic or pre-periodic.

In this project we will only be concerned with post-critically finite polynomials. The Julia sets and filled Julia sets of post-critically finite polynomials are compact, connected, locally connected. Moreover, there is a simple criterion for when the Julia set has interior.

Theorem 1.3.2. Let $f$ be a postcritically finite polynomial. If $f$ has no periodic critical points, then the Julia set has empty interior. If $f$ has at least one periodic critical point, then the interior is non-empty.

When the interior of $f$ is empty, then the Julia set is called a dendrite. One example is the Julia set for $z^{2}+i$. When the interior of $f$ is nonempty, then the interior of the Julia set is the union of all points whose orbits converge to a cycle containing a critical point. Properties of interior components are summarized in the following theorem:

Theorem 1.3.3. Suppose $f$ is postcritically finite and has Julia set $K_{f}$ with non-empty interior. Then:

1. Each interior component of $K_{f}$ is homeomorphic to an open disk.
2. If $U$ is an interior component of $K_{f}$, then so is $f(U)$. If $U$ does not contain a critical point, then $f$ maps $U$ homeomorphically to $f(U)$.
3. For each interior component $U \subset K_{f}$, there exists a unique point $p \in U$ whose forward orbit a periodic critical point. We call $p \in U$ the center point of $U$.

Definition 1.3.4. Let $K_{f}$ be a Julia set with interior, and let $U$ be an interior component of $K_{f}$. Then there is a family of Riemann maps $\phi: D^{2} \rightarrow U$ that maps 0 to the center point of $U$.

The internal rays of $U$ are the images of radial lines under the Riemann maps. Note that the definition does not depend on which map you choose.

Example 1.3.5. Some internal rays of the Basilica.


Definition 1.3.6. An arc $A \subset K_{f}$ is regulated if for each interior component $U$ of $K_{f}, A \bigcap U$ is contained in at most two internal rays of $U$.

Theorem 1.3.7. Let $K_{f}$ be the filled Julia set of a post-critically finite polynomial. For any two points $p$ and $q$ in $K_{f}$, there is a unique regulated arc with $p$ and $q$ as endpoints.

Definition 1.3.8. Let $K_{f}$ be a filled Julia set. The restricted hull of a set of points $P \in K_{f}$ is the union of all regulated arcs with endpoints in $P$.

Example 1.3.9. The restricted hull for some points in the Basilica.


Definition 1.3.10. Let $f$ be a post-critically finite polynomial. The Hubbard tree $H^{f}$ is the restricted hull of the postcritical points of $f$.

Example 1.3.11. The Basilica together with its Hubbard tree (the regulated hull of 0 and -1 ).


We will remember on a Hubbard tree $H$ the following information:

Definition 1.3.12. Let $H$ be the Hubbard tree of a post-critically finite quadratic polynomial $f$. We will remember on $H$ the structure given by a set of data called the primary structure, consisting of:

1. the topology
2. the cyclic order of the branches at branch points
3. the set $P$ of post-critical points $\bigcup_{n \geq 0} f^{\circ n}(0)$
4. the dynamics on $P$, i.e. $f_{\mid} P: P \rightarrow P$.
5. The degree of elements of $P$ under the mapping $f$ (i.e. 0 has degree 2 and all other postcritical points have degree 1)

Theorem 1.3.13 (Corollary 6.2 of the Orsay notes). Let $f: z \mapsto z^{2}+c_{1}$ and $g: z \mapsto z^{2}+c_{2}$ be two quadratic polynomials. If there exists a homeomorphism from $H_{f}$ to $H_{g}$ preserving the primary structure, then we have $c_{1}=c_{2}$.

Thus quardratic polynomials are uniquely determined by the primary structure of their Hubbard trees.

### 1.4 Replacement Systems

Using graph replacement systems, we can recover the topology of a Julia set from the combinatorial information given by the Hubbard tree. Informally, a graph replacement system can be thought of an algorithm to "draw" a sequence of representations of fractals that have increasing levels of accuracy. We start by drawing a base graph. Given an graph $G_{n}$ in the graph replacement sequence, we obtain $G_{n+1}$ by replacing edges of $G_{n}$ with other graphs, adding "detail" to the representation. This is formalized in the following definition:

Definition 1.4.1. An (edge) replacement rule is a pair of the form $e \rightarrow R$, where

1. $e$ is a single edge with two vertices, with one marked as the initial vertex and the other marked as the terminal vertex, and
2. $R$ is a finite directed graph with two distinct vertices marked as the initial vertex and the terminal vertex.

Given a directed graph $G$ and a replacement rule $e \rightarrow R$, we can expand any edge $\epsilon$ of $G$ by removing $\epsilon$ and pasting in a copy of $R$, such that the initial vertex of $R$ is attached to the initial vertex of $\epsilon$ and the terminal vertex of $R$ is attached to the terminal vertex of $\epsilon$.

Definition 1.4.2. A colored graph replacement system $\mathcal{R}$ consists of the following data:

1. A finite set of colors $C$.
2. A base graph $G_{0}$, which is a finite directed graph with all edges colored by elements of $C$.
3. For every $c \in C$, a directed graph $R_{c}$ with two distinguished vertices (the initial vertex and the terminal vertex).

From the initial data, we construct a sequence of directed graphs called the expansion sequence:

Definition 1.4.3. Given a colored replacement system $R$, the full expansion of the base graph $G_{0}$ is the graph $G_{1}$ obtained by expanding each edge $\epsilon$ of $G_{0}$ with a replacement rule $e \rightarrow R_{c}$,
such that $c$ is the color of $\epsilon$.
Iterating this process gives us the full expansion sequence $\left\{G_{n}\right\}_{n=0}^{\infty}$, where $G_{n}$ is the full expansion of $G_{n-1}$ for every $n$.

Note that each edge of every graph $G_{n}$ of the expansion can be written as a finite sequence of edges

$$
\epsilon_{0} \epsilon_{1} \epsilon_{2} \ldots \epsilon_{n}
$$

where $\epsilon_{0}$ is an edge of the base graph and if $\epsilon_{k}$ has color $c$ then $\epsilon_{k+1}$ is an edge of $R_{c}$.

The graphs in the replacement sequence converge to the limit space, which we will define as a quotient of an infinite product of edges.

Example 1.4.4. The replacement system for the Vicsek snowflake, along with some graphs from its expansion sequence.


Definition 1.4.5. Let $R$ be a colored replacement system. The symbol space $\Omega$ of $R$ is the space of infinite sequences
where $\epsilon_{0}$ is an edge of the base graph and if $\epsilon_{k}$ has color $c$ then $\epsilon_{k+1}$ is an edge of $R_{c}$. Alternatively, it is the space of infinite sequences with the property that every finite sub-sequence $\epsilon_{0} \ldots \epsilon_{n}$ is an edge in some graph $G_{n}$ of the expansion sequence.

Definition 1.4.6. The gluing relation on $\Omega$ is the equivalence relation $\sim$ such that two sequences $\epsilon_{0} \epsilon_{1} \epsilon_{2} \ldots$ and $\epsilon_{0}^{\prime} \epsilon_{1}^{\prime} \epsilon_{2}^{\prime} \ldots$ are equivalent if for all $n$ the edges $\epsilon_{0} \epsilon_{1} \ldots \epsilon_{n}$ and $\epsilon_{0}^{\prime} \epsilon_{1}^{\prime} \epsilon_{2}^{\prime} \ldots \epsilon_{n}^{\prime}$ share a vertex in $G_{n}$.

The limit space is the quotient $\Omega / \sim$.
In order for the gluing relation to be an equivalence relation, the following restrictions must be placed on the replacement system:

Definition 1.4.7. A colored graph replacement system $R$ is said to be expanding if the following conditions are satisfied:

1. Neither $G_{0}$ or any of the $R_{c}$ have any isolated vertices.
2. None of the $R_{c}$ have an edge connecting the initial and terminal vertices.
3. Each $R_{c}$ has at least three vertices and two edges.

Theorem 1.4.8. If the colored graph replacement system $R$ is expanding, then the gluing relation is an equivalence relation and the limit space $\Omega / \sim$ is compact and metrizable.

Given a polynomial $f$, we construct a replacement system from the Hubbard tree $H$. This construction is due to Belk and Forrest (unpublished).

Definition 1.4.9. The replacement system derived from the post-critically finite polynomial $f$ is obtained according the following procedure:

Step 1. Compute the Hubbard tree $H$ for $f$.
Step 2. If $T$ is a tree, then the blow up of $T$, denoted $\bar{T}$, is the graph obtained by replacing every vertex of $T$ with a counterclockwise directed cycle.

Construct the blow up of $H$, denoted $\bar{H}$. This is the base graph. Step 3. Let $H_{1}$ be the pre-image of $H$ under $f$. Construct the blowup of $H_{1}, \overline{H_{1}}$.

Step 4. Orient the edges of the blowup $\bar{H}$. Choose a distinct color for each edge.
Step 5. The branched covering map from $H_{1} \rightarrow H$ induces a covering map $\overline{H_{1}} \rightarrow H$. Use this covering to lift the orientations and colors of the edges of $H$ to the edges of $H_{1}$, so that the covering map preserves orientations and colors.

Step 6. The set obtained by cutting $\bar{H}$ at $F$ is the disjoing union of the closures of the connected components of $\bar{H} \backslash F$. Since every vertex of $\bar{H}$ is a vertex of $\overline{H_{1}}$, removing the vertices of $\bar{H}$ from $\bar{H}$ and $\overline{H_{1}}$ cuts them into corresponding pairs of components, which are the replacement rules.

Example 1.4.10. Using Definition 1.4.9, we compute a replacement system for the Basilica.
Step 1: The Basilica Hubbard tree:


Step 2: The blow-up of the Hubbard tree


Step 3: The pre-image of the Basilica Hubbard tree, and the blow up:


Step 5: The edges are already oriented, and the edge labels are the colors.
Step 6: We obtain the following replacement rules:


Note that the subscripts are necessary in the first replacement rule to differentiate between the two edges labeled with $c$.

In general, the gluing relation we originally defined is not an equivalence relation on replacement systems obtained from the above procedures. The following definition for the gluing relation turns out to be an equivalence relation for this class of replacement systems, even though they are not necessarily expanding:

Definition 1.4.11. The gluing relation for replacement systems obtained from Julia sets is the relation $\sim$ such that two edge addresses $\epsilon_{0} \epsilon_{1} \epsilon_{2} \ldots$ and $\epsilon_{0}^{\prime} \epsilon_{1}^{\prime} \epsilon_{2}^{\prime} \ldots$ are related if there is a finite upper bound on the distance between $\epsilon_{0} \epsilon_{1} \epsilon_{2} \ldots \epsilon_{n}$ and $\epsilon_{0}^{\prime} \epsilon_{1}^{\prime} \epsilon_{2}^{\prime} \ldots \epsilon_{n}^{\prime}$ for all $n$.

We can make the following simplifications to the replacement system.

Step 1: If for any replacement rule $e \rightarrow R_{c}, R_{c}$ is a single edge with color $c^{\prime}$ (where $c \neq c^{\prime}$ ), recolor every edge in the replacement system that was colored with $c$ with $c^{\prime}$ instead.

Step 2: Once all the redundant colors from Step 1 are removed, if there are trivial replacements where $R_{c}$ is a single edge with color $c$, then contract all edges colored with $c$.

Lemma 1.4.12. For any replacement system obtained from a Julia set according to Definition 1.4.9. procedure outlined in Step 1 and Step 2 above gives an expanding replacement system with limit space homeomorphic to the limit space of the original replacement system.

### 1.5 Laminations

The material in this section is adapted from [9] and [2]. A more topological perspective on Julia sets is to study them as quotients of a circle. This idea was introduced by Thurston. Suppose we
have a monic polynomial $p$ with degree $d \geq 2$, such that its Julia set $K_{p}$ is connected ${ }^{3}$. The trick is to study the dynamics on the Julia set by associating points of the Julia set with points on $S^{1}$, the unit circle a way such that the map conjugates the dynamics of $p$ on $K_{p}$ to to $z^{d}$ on $S^{1}$. This map is known as the Carathéodory loop, and in general it is defined as the extension of a Riemann map from the open disc to the interior of $K_{p}$. The existence of such a Riemann map follows from the Riemann mapping theorem:

Theorem 1.5.1. If $U$ is a non-empty simply connected open subset of the complex plane that is not all of $C$, then there exists a biholomorphic mapping from $U$ onto the open disc, known as a Riemann mapping.

Then since the complement of $K_{p}$ is locally connected, Carathéodory's theorem lets us extend the inverse of the Riemann map onto the boundary of the disc. This extension maps the unit circle to $K_{p}$.

Theorem 1.5.2. Let $f$ be a Riemann map from an open subset $U$ to the open disc $D^{2}$, and let $g=f^{-1}$. The map $g$ extends to a continuous map $\bar{g}$ defined on the closed disc $\overline{D^{2}}$ if and only if the complement $S^{2}-U$ is locally connected.

Now let's suppose we have a Carathéodory loop $\sigma$ that maps the unit circle to $\bar{U}$, where $\bar{U}$ is the boundary of some open subset $U$. Unless $\bar{U}$ is topologically equivalent to the unit circle, $\sigma$ will not be injective, and some points in the unit circle will be identified with one another under the mapping. It turns out that the structure of the set of identifications cannot be too complicated - if $x$ and $y$ are two points in $\bar{U}$, then the convex hull of the preimages of $x$ is disjoint from the convex hull of the preimages of $y$. Given this fact, it is convenient to draw a picture of the identifications by drawing the convex hulls (called leaves) of each pair of points that get identified under the mapping $\sigma$. The set of leaves is called the lamination.

[^1]If the Carthéodory loop mapping $\mathbb{R} / \mathbb{Z}$ to $K_{p}$ maps $\alpha$ to a point $z \in K_{p}$, then we say that $\alpha$ is an external ray that lands at $K_{p}$.

If a point in $K_{p}$ has two external angles $\alpha_{1}$ and $\alpha_{2}$, then we stipulate that there is a leaf in the lamination connecting $e^{i \alpha_{1}}$ and $e^{i \alpha_{2}}$. If a point in $K_{p}$ has more than two external angles, then we draw leaves connecting points on the circle that correspond to adjacent angles, so that all external angles corresponding to the point are the vertices of a single polygonal gap in the lamination.

From the lamination, we obtain a model of the Julia set as a quotient of the circle by identifying points that are connected by leaves. This is called the pinched disc model of the Julia set.

Example 1.5.3. Below we have a picture of the Basilica with external rays marked, as well as a lamination for the pinched disc model of the Basilica.



A natural question to ask is which quotients of the circle can be realized by Julia sets. Thurston approached this problem by first defining laminations more abstractly as disjoint sets of chords on the unit disc, and then classifying the set of laminations that are forwards and backwards invariant under the function $z \mapsto z^{2}$. The definitions and results below are from Thurston's manuscript on invariant laminations:

Definition 1.5.4. A geodesic lamination $L$ is a closed set of chords, called leaves, on the closed unit disc $\overline{D^{2}}$ that are disjoint, except possibly at the endpoints.

A gap of a lamination is the closure of a component of the complement of $\bigcup L$. Any gap is the convex hull of its intersection with a boundary of the disc.

Definition 1.5.5. A geodesic lamination $L$ is quadratic invariant, i.e. invariant under the map $f(z)=z^{2}$, if it satisfies the following conditions:

1. Forward invariance: If a leaf $\overline{p q}$ is in $L$, then either $f(p)=f(q)$ or $\overline{f(p) f(q)}$ is a leaf in $L$.
2. Backwards invariance: If $\overline{p q}$ is a leaf in $L$, then there are two disjoint leaves connecting a preimage of $p$ with a preimage of $q$.
3. Gap invariance: For any gap $G$, the convex hull of $G \bigcap S^{1}$ is either
(a) a gap,
(b) a leaf,
(c) or a single point

If the image of $G$ is a gap, then the image of a point moving clockwise around $G$ must move clockwise around the image of $G$. In other words, the boundary of $G$ must map locally monotonically to the boundary of the image gap with positive orientation.

Definition 1.5.6. If the two endpoints of a leaf map to the same point, then that leaf is a critical leaf. If the degree of the map from a gap to its image gap is not 1 , then that gap is a critical gap

Definition 1.5.7. The length of a leaf in a lamination is the length of the smaller arc between the two endpoints on the circle. The perimeter of the circle is defined to have length 1 , so the length of a leaf is at most $1 / 2$, and a leaf with length $1 / 2$ is a critical leaf.

The following definitions are from Definition II.6.2 of Thurston's manuscript.
Definition 1.5.8. Let $L$ be a quadratic invariant lamination, and let $M$ be the longest length of a leaf of $L$. Unless $M=1 / 2$, there are two leafs with length $M$. We will call any leaf of length $M$ a major leaf. If there are two major leaves, then their images coincide. We will call the image of a major leaf the minor leaf of the lamination.

The following is from Definition II.6.3 of Thurston's manuscript.
Definition 1.5.9. A lamination is clean if no three leaves have a common endpoint.
The following theorem is part (a) of Proposition II.6.7 of Thurston.
Theorem 1.5.10. Any invariant lamination which is clean is the minimal lamination with its given minor leaf.

Corollary 1.5.11. If a clean invariant lamination has the same minor leaf as the lamination that gives the pinched disc model of a Julia set, then the two laminations are the same.

Bandt and Keller take a more abstract view of the pinched disc model, defining it more directly as an equivalence relation on the circle, given by identifying itineraries of points on the circle under the angle doubling map. The notation is summarized in the following definition:

Definition 1.5.12. $T=\mathbb{R} / \mathbb{Z}$ is the unit circle, and $h: \beta \mapsto(2 \beta \bmod 1)$ is the angle doubling map.

Fix $\alpha \in T$. The diameter (critical leaf) $\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}\right)$ divides $T$ into the open semi-circles $T_{1}^{\alpha}$ and $T_{0}^{\alpha}$ such that $\alpha \in T_{1}^{\alpha}$.

Definition 1.5.13. An equivalence relation on $\{0,1\} *$ is forward invariant under the shift $\sigma$ if $s \sim t$ implies $\sigma(s) \sim \sigma(t)$.

Theorem 1.5.14. Let $\alpha$ be a periodic angle. Among all forward invariant equivalence relations on $T$ that do not identify $T_{0}^{\alpha}$ with $T_{1}^{\alpha}$, there is a largest lamination $\sim_{\alpha}$.

Let $f$ be a periodic quadratic polynomial of period $p$ with critical value $c$ and Julia set $J_{f}$. Suppose that the fixed point under $f^{p}$ on the basin containing $c$ has external angle $\alpha$. Then $\left(J_{f}, f\right)$ is conjugate to $\left(T / \sim_{\alpha}, h\right)$.

Corollary 1.5.15. Let $f$ be a periodic quadratic polynomial. The lamination of the pinched disc model for $f$ is the largest clean lamination that does not have a leaf connecting a point from $T_{0}^{\alpha}$ to a point from $T_{1}^{\alpha}$. The points $\frac{\alpha}{2}$ and $\frac{\alpha+1}{2}$ are each an endpoint of a major leaf.

## 2

## The Airplane

### 2.1 The Replacement System

The Airplane is the Julia set of the polynomial $z^{2}+c$, where $c$ is the real root of $c^{3}+2 c^{2}+c+1$ (which turns out to be around -1.75488 ).

The critical point is periodic with period three. All three points in the orbit are on the real axis, and the Hubbard tree is a path.


The pre-image of the Hubbard tree $H^{1}$ is also a path.


Below we have the blow-ups of $H_{0}$ and $H_{1}$.


We construct $H_{1}$ by gluing together two copies of $H_{0}$. Then we use the vertices of $H_{0}$ to cut $H_{0}$ and $H_{1}$ into corresponding pairs of components, and assign addresses to $H_{1}$ accordingly.



From the cuts we get our first set of replacement rules:


Our function $f$ maps every edge in $H_{1}$ to an edge in $H$. Our second set of replacement rules are of the form $e_{1} \rightarrow e_{2}$ if $e_{1} e_{2}$ is an address of $H_{1}$. The set of possible edge addresses in the symbol space correspond to paths in the following graph:


There are four gluing vertices on the base graph, which have addresses:

$$
\begin{gathered}
\overline{e b e}=\overline{f a c}=\overline{f a d} \\
\overline{e e b}=\overline{c f a}=\overline{d f a} \\
\overline{b e e}=\overline{a c f}=\overline{a d f} \\
\overline{b e b e}=d \overline{f a c}=c \overline{f a d}
\end{gathered}
$$

Note that the replacement system can actually be simplified to just two colors: $a$ and $e$, making it into an expanding replacement system.

### 2.2 The Lamination

The airplane is in the component of the Mandelbrot set that has an external ray of $\alpha=\frac{3}{7}$. We construct the corresponding lamination $S_{0}^{\alpha}$ according to the methods of Bandt and Keller.

The diameter of the circle that maps to $\alpha$ under the angle doubling map has endpoints at $\frac{5}{7}$ and $\frac{3}{14}$. This diameter divides the circle $T=\mathbb{R} / \mathbb{Z}$ into two semicircles $T_{0}$ and $T_{1}$, where $0=1$ is in $T_{1}$. The itinerary of a point $\beta \in T$ with respect to $\alpha$ is defined as

$$
I(\beta)=s_{1} s_{2} s_{3} \ldots \text { with } s_{i}= \begin{cases}0 & h^{i-1}(\beta) \in T_{0} \\ 1 & h^{i-1}(\beta) \in T_{1} \\ * & h^{-1}(\beta) \in\left\{\frac{3}{14}, \frac{5}{7}\right\}\end{cases}
$$

Note that our $\alpha$ is periodic with period three, where $h(\alpha)=\frac{6}{7}$ and $h^{2}(\alpha)=\frac{5}{7}$. Our kneading sequence, which is the itinerary of $\alpha$, is $\overline{01 *}$. The characteristic symbol of $\alpha$ is 1 , so we consider the pre-image of $\alpha$ which is periodic to be in $T_{1}$ and the pre-image of $\alpha$ which is pre-periodic to be in $T_{0}$. I.e., $\frac{5}{7} \in T_{1}$ and $\frac{3}{14} \in T_{0}$.

Now all points of $T$ have 01 itineraries. Now we say that two 01 sequences $\beta$ and $\gamma$ are equivalent if and only if either $I(\beta)=I(\gamma)$, or $\{I(\beta), I(\gamma)\}=\{w 101 \overline{001}, w 001 \overline{101}\}$ for any 01 word $w$, or $\{I(\beta), I(\gamma)\}=\{w \overline{001}, w \overline{101}\}$ for any 01 word $w$ that does not end in 001 or 101.


Below is the lamination of the airplane, $L$ :


The minor leaf $m$ of the airplane is the leaf connecting $3 / 7$ and $4 / 7$, and the major leaves are $(2 / 7,5 / 7)$ and $(3 / 14,11 / 14)$.

Let $L_{0}$ be the lamination containing the major leaves and all forward images of the major leaves. $L_{0}$ is shown below.


Note that from Thurston, we know that the airplane lamination has the property that any other lamination having minor leaf $(3 / 7,4 / 7)$ contains the airplane lamination. From Bandt and

Keller, we know that the airplane lamination is the largest forward invariant lamination that does not have a leaf connecting a point of $T_{0}$ with a point of $T_{1}$.

Thus to check that a lamination is equal to the airplane lamination, it is enough to check that the lamination is invariant, has minor leaf $(3 / 7,4 / 7)$, and does not identify a point of $T_{0}$ with a point of $T_{1}$.

Here are some observations that are helpful in determining whether or not a lamination has minor leaf $(3 / 7,4 / 7)$. If a leaf is in a three-cycle, there are only three possibilities for what the three cycle looks like. Those possibilities are shown below:


Definition 2.2.1. Notice that the cycle containing the leaf $(3 / 7,4 / 7)$ is the only three cycle with disjoint leaves. Call the forward invariant lamination consisting of the three leaves in this cycle $L_{F}$.

### 2.3 Laminations of the Replacement System

Next, we construct a sequence of laminations that give the graphs of the expansion sequence defined in the previous section.

Definition 2.3.1. Let $L_{0}$ be the following lamination. We also show the base graph $\bar{H}$ of the airplane replacement system again for reference:


It will be important for us to distinguish between the lamination $L_{0}$ on $S^{1}$ and the graph consisting of an edge for each labeled arc on $S^{1}$ in $L_{0}$, where two edges are adjacent if the corresponding arcs are adjacent on $S^{1}$. Clearly this graph is homeomorphic to a circle. We will call it $C_{0}$.
$L_{0}$ is exactly the set of leaves that identifies each point on the closure of $e_{r}$ or $b_{r}$ with its conjugate on $e_{l}$ or $b_{l} . H_{0}$ is homeomorphic to the quotient of the circle obtained by the points connected by the leaves.
$C_{0}$ is the boundary of the external face of $\bar{H}$.

Similarly, the preimage of the base graph, $\overline{H_{1}}$, is given by identifying endpoints of leaves the lamination $\overline{L_{1}}$ shown below. Note that the degree 2 covering map from $\overline{H_{1}}$ to $\bar{H}$ extends to a degree 2 covering map from $C_{1}$, the boundary of the external face of $\overline{H_{1}}$, to $C_{0}$. That map
extends to a degree 2 covering map from $L_{1}$ to $L_{0}$. The labels in the below figure correspond to the labels that each arc maps to under the map.


Lemma 2.3.2. $\overline{L_{0}}$ contains $L_{F}$.

Proof. From Definition 1.4.9, we know that the map from $f H_{1}$ to $H$ extends to a covering map $\hat{f}$ from $\overline{H_{1}}$ to $\bar{H}$. Thus $\hat{f}$ maps the vertices of $\overline{H_{1}}$ to the vertices of $\bar{H}$. We can restrict the map to the vertices of $\bar{H}$. Since $\bar{f}$ was an extension of $f$, it must map the two vertices that are the endpoints of $c$ and $d$ to the vertex on $f$, the vertex on $f$ to the vertex on $a$, and the vertex on $a$ to one of the endpoints of $c$ and $d$. Each one of these vertices is an endpoint of $e$ or $b$, and thus corresponds a leaf of the lamination $L_{0}$ (marked in blue in the image). Also note that all the leaves of $L_{0}$ are disjoint. Thus this three-cycle must be $L_{F}$.

Definition 2.3.3. We define the following replacement system (which we will refer to as the circle replacement system:

Step 1: Let $C_{0}$ from Definition 2.3 .1 be the base graph.
Step 2: As we do with Julia sets in 1.4.9, use the vertices of $C_{0}$ to partition $C_{1}$ into paths. The corresponding pairs of components (where each component in $C_{0}$ is an edge and each component in $C_{1}$ is an edge or a path) are the replacement rules.

The limit space of this replacement system is homeomorphic to a circle, and we will refer to it as the circle limit space, $C_{\infty}$.

We would like to define an equivalence relation $\sim_{C}$ on the circle limit space such that $C_{\infty} / \sim_{C}$ is homeomorphic to the limit space of the airplane replacement system.

Definition 2.3.4. For each finite expansion graph $C_{n}$, there is a surjection $\sigma_{n}$ from the set of edge addresses of $C_{n}$ to the set of edge addresses graph $G_{n}$ of the airplane expansion sequence, such that two edge addresses map to the same edge addresses if they are equal if you drop all subscripts from edge labels in the edge addresses.

Similarly, we can define $\sigma_{\infty}$, such if $s$ is an edge address of $C \infty, \sigma(s)=t t$ is the sequence obtained by removing the subscripts from any letter of $s$ with a subscript.

This gives us an equivalence relation $\sim_{C}$ such that two edge labels are equivalent if they are the same edge label, and additionally,

$$
e_{r} \sim_{C} e_{l} \quad \text { and } \quad b_{r} \sim_{C} b_{l}
$$

Two edge addresses $\epsilon_{1} \epsilon_{2} \ldots$ and $\epsilon_{1}^{\prime} \epsilon_{2}^{\prime} \ldots$ are equivalent under $\sim_{C}$ if $\epsilon_{k} \sim_{C} \epsilon_{k}^{\prime}$ for all $k \in \mathbb{N}$.
In other words, two edge addresses are equivalent if they map to the same point under $\sigma$.

We state the following proposition without proof:

Proposition 2.3.5. The shift map on edge addresses of the circle limit space is a degree 2 covering map on the circle. Furthermore, it is conjugate to the angle doubling map $z \mapsto z^{2}$.

Definition 2.3.6. $\overline{L_{\infty}}$ be the lamination such that two points are connected by a leaf if is the image under the conjugation from the previous proposition of two points of $C \infty$ that are equivalent by $\sim_{C}$.

Lemma 2.3.7. All leaves of $\overline{L_{\infty}}$ are disjoint.

Proof. First we want to show that no two leaves cross within the disc. This is equivalent to the statement that if $a$ and $b$ are two edge addresses of $C_{\infty}$ such that there are two other edge addresses $a^{\prime}, b^{\prime}, a \sim_{C} a^{\prime}$ and $b \sim_{C} b^{\prime}, a$ and $a^{\prime}$ are on the same arc connecting $b$ and $b^{\prime}$. This
is true in the base graph $C_{0}$, and the expansion rules preserve this property, thus it is true for all $C_{n}$ by induction. Since the limit space is a Hausdorff space, we can find 4 disjoint cells that contain $a, b, a^{\prime}$, and $b^{\prime}$, so it is enough to prove the statement for the finite expansions.

If two points are connected by a leaf, then they are in the same equivalence class under the relation $\sim_{C}$. Each equivalence class of points on the circle limit space under the equivalence relation from contains at most two points. Thus if two leaves share at least one endpoint, they must be the same leaf.

Lemma 2.3.8. $L_{F}$ is a subset of $\overline{L_{\infty}}$.

Proof. Under $\sigma_{0}$ from Definition 2.2.4, the endpoints of each leaf in $L_{f}$ map to vertices of $G_{0}$ which in the limit space can be represented by an edge addresses containing only $e$ and $b$. Thus in $G_{\infty}$, the vertices in a three cycle must have two pre-images under $\sigma_{\infty}$, and they must correspond to leaves.

Lemma 2.3.9. $\overline{L_{\infty}}$ contains all pre-images of the leaves in $L_{F}$.

Proof. Each edge address in the symbol space of $G_{\infty}$ has two preimages (see the graph of edge addresses in Section 1 of this chapter). If an point in $G_{\infty}$ has two preimages under $\sigma_{\infty}$, then its pre-image also has two preimages under $\sigma_{\infty}$.

Lemma 2.3.10. $\overline{L_{\infty}}$ is closed.

Proof. Since the endpoints of $e$ and $b$ correspond to leaves, for each $L_{n}, L_{n}$ is closed in the disc and $\mathbb{D} / L_{n}$ is open in the disc. The complement of $L_{\infty}$ is the set in $S^{1} \times S^{1}$ for which a leaf connecting the two coordinates would cross some preimage of the critical leaf. This is an open set, so $\overline{L_{\infty}}$ is a closed lamination.

Lemma 2.3.11. $\overline{L_{\infty}}$ does not identify a point of $T_{0}^{3 / 7}$ with a point of $T_{1}^{3 / 7}$.
Proof. We know that $C_{0}$ does not identify any points of $T_{0}^{3} / 7$ with $T_{1}^{3 / 7}$, since we have a complete description of the identifications.Thus we can partition $C_{0}$ into $T_{0}^{3 / 7}$ and $T_{1}^{3 / 7}$. Now note that
the expansion rules of the circle replacement system preserve the property in the statement of the lemma.

Lemma 2.3.12. $\overline{L_{\infty}}$ has minor leaf $(3 / 7,4 / 7)$.
Proof. $\overline{L_{\infty}}$ contains the two pre-images of $(3 / 7,4 / / 7)$, and does not identify a point of $T_{0}^{3 / 7}$ with a point of $T_{1}^{3 / 7}$. Thus the two preimages must be the major leaves.

Theorem 2.3.13. $G_{\infty}$ is homeomorphic to the airplane.

Proof. Claim: The lamination $L_{\infty}$ is the lamination for the pinched disc model of the airplane. The lamination for the airplane is the closure of the union of the set of pre-images of $L_{F}$. We've proven that $L_{\infty}$ contains that union and $L_{\infty}$ is closed, so it must contain the airplane lamination. The airplane lamination is the largest clean lamination that does not have leaves that connect points from $T_{0}^{3 / 7}$ points from $T_{1}^{3 / 7}$, so it must contain $L_{\infty}$.

### 2.4 The Correspondence

We define a map $\sigma$ from the set of edge labels to $\{0,1\}$ by by

$$
\begin{aligned}
& \sigma(a)=\sigma(b)=\sigma(d)=0 \\
& \sigma(c)=\sigma(e)=\sigma(c)=1
\end{aligned}
$$

This extends to a map from the set of address in the symbol space to $\{0,1\}^{\infty}$ where for $x_{1} x_{2} x_{3} \cdots \in \Omega$

$$
\sigma\left(x_{1} x_{2} x_{3} \ldots\right)=\sigma\left(x_{1}\right) \sigma\left(x_{2}\right) \sigma\left(x_{3}\right) \ldots
$$

Note that the gluing vertex $[\overline{e b e}]$ maps to $\alpha$.

## 3

## Polynomial Matings

### 3.1 Background

Quadratic rational functions are functions of the form $p(x) / q(x)$, where $p(x)$ and $q(x)$ have degrees of at most 2 and at least one has degree 2. Rational functions act on the Riemann sphere $\hat{C}$, which can be thought of as the complex plane together with a point at infinity.

In general the structure of Julia sets of rational functions are not well understood. However, when a post-critically finite rational map has two critical points and the two critical points are attracted to disjoint cycles, the dynamics of the rational map can be understood as the dynamics of two post-critically finite polynomials combined in a specific way, and the rational map is then called the mating of the two polynomials.

Definition 3.1.1. Let $f, g$ be two post-critically finite quadratic polynomials with pinched disc models given by laminations $L_{f}$ and $L_{g}$.

Let $\sim_{l a m}$ be the smallest equivalence relation on $S^{1}$ such that two points $p$ and $q$ are identified if they are connected by a leaf in $L_{f}$ or $1-p$ and $1-q$ are connected by a leaf in $L_{g}$. If the quotient $S^{1} / \sim_{l a m}$ is a topological two sphere then we say that $f$ and $g$ are topologically matable, and the action of the angle doubling map on $L_{f}$ and $L_{g}$ extends to a map on the quotient, which we call the topological mating.


Figure 3.1.1. A Julia set of a quadratic rational function on the Riemann sphere.

If there is a orientation preserving homeomorphism that conjugates the topological mating to a quadratic rational function $R$ on $\hat{\mathbb{C}}$, then we say that $f$ and $g$ are geometrically matable and $R$ is the geometric mating of $f$ and $g$.

The original intuition for a mating comes from thinking of putting one Julia set on the top hemisphere of the sphere and putting one Julia set on the bottom, and then stretching both towards the equator. This is why the definition of $\sim_{l a m}$ identifies $p$ and $q$ if $1-p$ and $1-q$ are identified in $L_{g}$. When we attach the two laminations, $L_{g}$ is in a sense "upside down".

Matings of quadratic polynomials are well-understood by the following theorem:

Theorem 3.1.2. Suppose $f$ and $g$ are post-critically finite quadratic polynomials. Then the following are equivalent:

1. $f$ and $g$ are topologically matable.
2. $f$ and $g$ are geometrically matable.
3. $f$ and $g$ do not lie in conjugate limbs of the Julia set.

Now if we have graph replacement systems for quadratic polynomial Julia sets, its natural to try to extend the construction to quadratic matings, which would hopefully lead to rearrangement groups on matings. In the following two sections we present, without proof, finite state automata for the Rabbit Julia set and the Basilica Julia set that describe how the limit spaces should be attached to give the Julia set of a mating. The finite state automata we present here were conjectured after extensive experimentation in Mathematica.

### 3.2 Replacement Systems for the Rabbit and the Basilica

The replacement system for the Basilica is given in the background section on replacement systems. Here we present the simplified version, as well as the simplified version of the replacement system for the Rabbit Julia set. The replacement systems given in this section are similar to ones to the ones in Rearrangement Groups of Fractals. However, we do not do all of the simplifications to get to the ones in that paper, as the "delay" in the expansions of some edges is essential information.

Definition 3.2.1. The components of the replacement system for the basilica are given below:
The base graph:


Edges labeled with 1 get replaced with the first graph below, and edges labeled with 2 get replaced with an edge labeled with 1 :


Definition 3.2.2. The components of the replacement system for the rabbit are given below:

The base graph:


Edges labeled with 1 get replaced with the first graph below, edges labeled with 2 get replaced with an edge labeled with 1 , and edges labeled with 3 get replaced with edges labeled with 2 :




### 3.3 Mating of the Rabbit and the Basilica

The goal of this section is to present a relation from the set of points in the Basilica limit space to the set of points in the Rabbit limit space, in a way such that gluing together the two limit spaces at points that are related gives the Julia set of the topological mating of the Rabbit and the Basilica, and the shift map on the glued together limit spaces is conjugate to the action of the topological mating on its Julia set.

Below are two pictures of the mating of the Basilica and the Rabbit. In the first one the critical point in the two cycle is 0 , and the critical point in the three cycle is $\infty$, and in the second one the critical point in the three cycle is $\infty$ and the critical point in the first cycle is 0 . We can see in these pictures that the Julia set of the mating of the Rabbit and the Basilica looks like the Julia sets for the polynomials stuck together. Indeed, one can construct the mating by gluing together two laminations such that points with the same binary angle are glued together, and then identifying points connected by leaves as usual.


The finite state automata works below as follows. For a pair of edge addresses $e_{1} e_{2} e_{3} \ldots$ and $e_{1}^{\prime} e_{2}^{\prime} e_{3}^{\prime} \ldots$ where $e_{1} e_{2} e_{3} \ldots$ represents a point in the Basilica limit space and $e_{1}^{\prime} e_{2}^{\prime} e_{3}^{\prime} \ldots$ represents
a point in the Rabbit limit space, the two points of the limit spaces are identified if and only if $\left(e_{1}, e_{1}^{\prime}\right),\left(e_{2}, e_{2}^{\prime}\right),\left(e_{3}, e_{3}^{\prime}\right) \ldots$ is a valid sequence of state transitions in the diagram.


### 3.4 Mating of Two Rabbits

The automata for the mating of two rabbits works the same as the automata from the section above.


## Bibliography

[1] Kathleen Alligood et al, Chaos: An Introduction to Dynamical Systems, Textbooks in Mathematical Science, Springer, New York, 2000.
[2] Cristoph Bandt and Karsten Keller, Symbol Dynamics for Angle Doubling on the Circle: I. The Topology of Locally Connected Julia Sets, Ergodic Theory and Related Topics III (1992), 1-23.
[3] Alan Beardon, Iteration of Rational Functions, Graduate Texts in Mathematics, Springer, New York, 1991.
[4] James Belk and Bradley Forrest, A Thompson Group for the Basilica, Groups, Geometry, and Dynamics 9(4) (2015), 975 - 1000.
[5] , Rearrangement Groups of Fractals. https://arxiv.org/abs/1510.03133v2.
[6] Xavier Buff et al, Questions about Polynomial Matings, Annales de la Faculté des sciences de Toulouse : Mathématiques, Série 621 (2012), 1149-1176.
[7] Adrien Douady and John Hubbard, Etude dynamique des polynômes complexes, Publications Mathématiques d'Orsay [Mathematical Publications of Orsay] 84 (1984).
[8] Will Smith, Thompson-like Groups for Dendrite Julia Sets, Senior project.
[9] William Thurston, On the Geometry and Dynamics of Iterated Rational Maps, Complex Dynamics: Families and Friends (2009), 3-109.
[10] Jasper Weinrich-Burd, A Thompson-like group for the Bubble Bath Julia Set, Senior project.


[^0]:    ${ }^{1}$ The term "chaos theory" was coined by James Yorke, one of the authors of 1
    ${ }^{2}$ The existence of the Lorenz attractor would not be proved analytically until the year 2000 .

[^1]:    ${ }^{3}$ Recall that $K_{p}$ is connected if and only if the forward orbits of the critical points (aside from infinity) are bounded.

