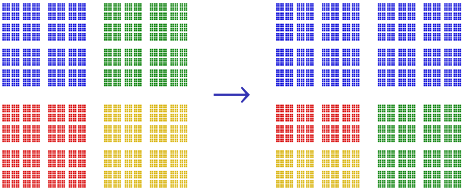
Embedding Right-Angled Artin Groups into Brin-Thompson Groups



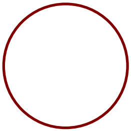
Jim Belk, Bard College

joint with Collin Bleak and Francesco Matucci

Thompson's Groups

There are three ***Thompson groups***:

F acts on the interval.

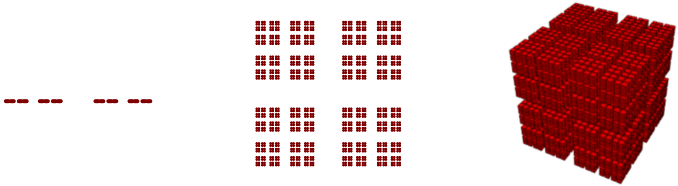


T acts on the circle.

V acts on the Cantor set.

Brin-Thompson Groups

The ***Brin-Thompson groups*** nV were defined by Matt Brin in 2004:



V $\quad$ $2V$ $\quad$ $3V$ $\quad \dots$

They are “higher-dimensional” versions of Thompson’s group V.

Definitions of V and nV

Definition of V

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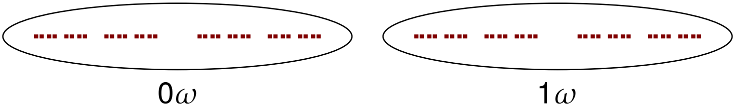

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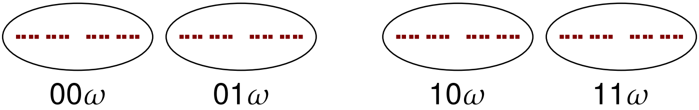

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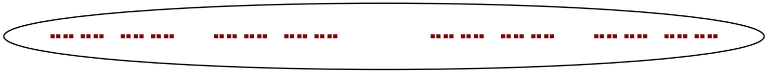

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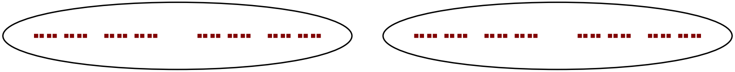

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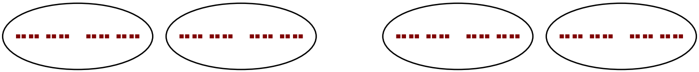

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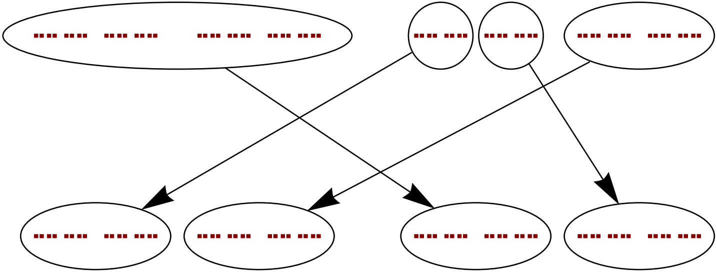
Definition of V

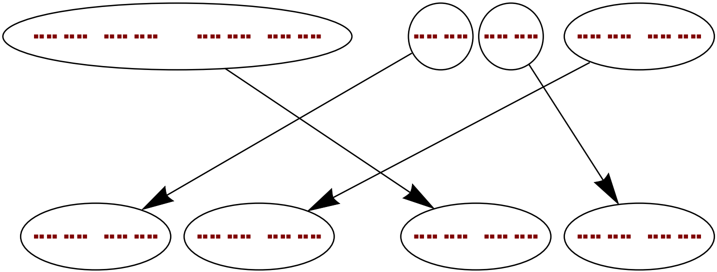
A ***dyadic rearrangement*** of C is a homeomorphism that maps "linearly" between the pieces of two dyadic subdivisions.



The group of all such homeomorphisms is ***Thompson's group V***.

Definition of V

Each piece maps by a ***prefix replacement***.



$0\omega \mapsto 10\omega$ $\qquad$ $100\omega \mapsto 00\omega$

$101\omega \mapsto 11\omega$ $\qquad$ $11\omega \mapsto 01\omega$

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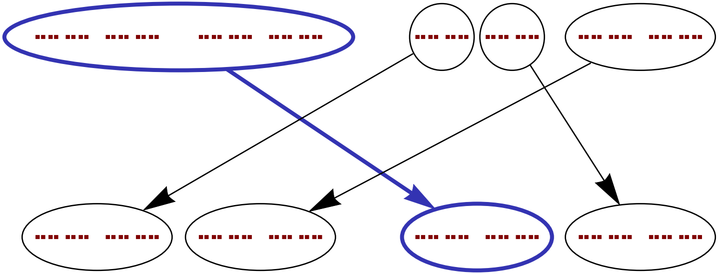

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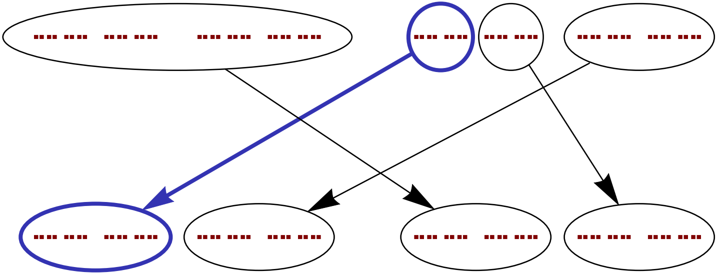

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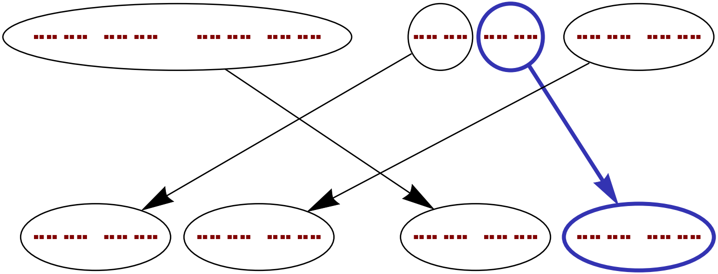

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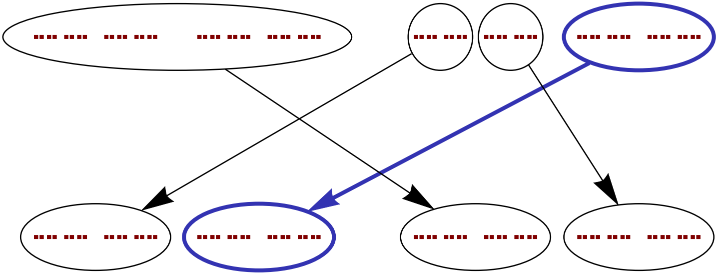

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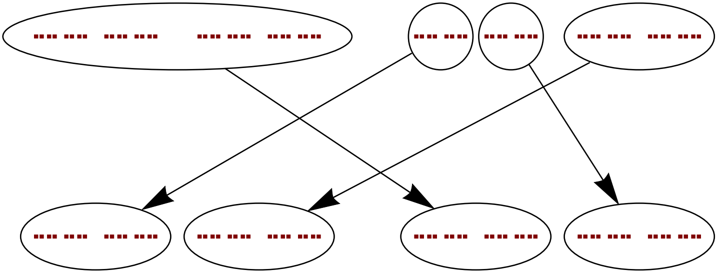

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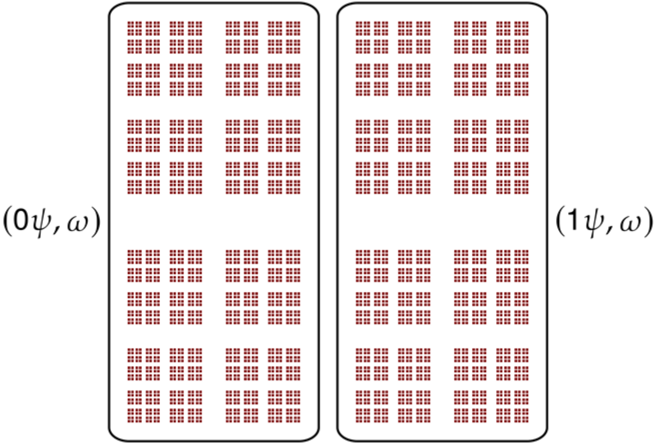

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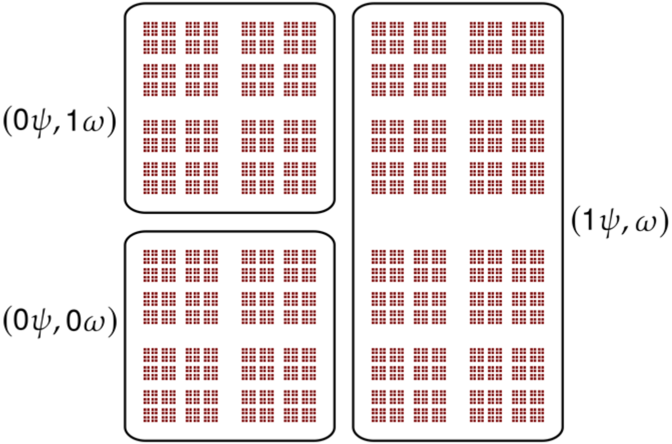

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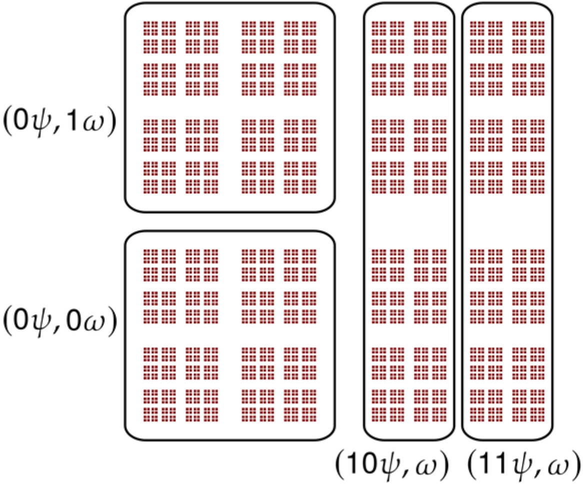

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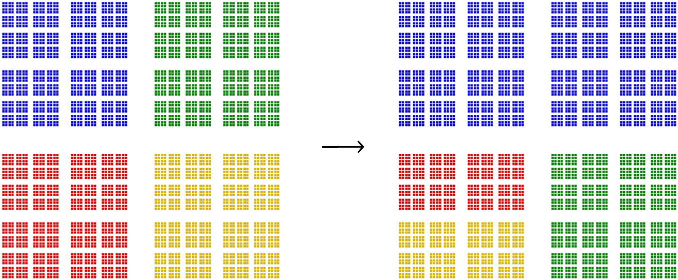
Homeomorphisms act piecewise by prefix pair replacements:



Definition of nV

In general, nV acts on the space C^n:

$$\begin{array}{cccccccccccccccc} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \cdots \end{array}$$

Elements of nV act piecewise by prefix tuple replacements:

$$\begin{array}{lcl} 0\ 1\ 0 & & 1 \\ 1 & \mapsto & 0\ 1 \\ 0\ 0 & & 0\ 1\ 1 \end{array}$$

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Properties of nV

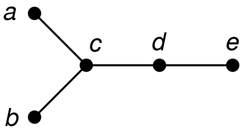
The groups nV

- Are finitely presented and simple,
  (Brin 2005 and 2010)
- Are non-isomorphic for different values of n,
  (Bleak, Lanoue 2010)
- Have type F_∞,
  (Kochloukova et al. 2013 and Fluch et al. 2013)
- Have the Haagerup property and Serre's property FA, and
  (Kato 2015)
- Have unsolvable torsion problem for $n \geq 2$.
  (Belk, Bleak 2017)

Main Results

Right-Angled Artin Groups

Let Γ be a finite graph.



The corresponding ***right-angled Artin group*** A_Γ has

- One generator for each vertex of Γ, where
- Generators commute if they are connected by an edge.

For the graph above,

$$A_\Gamma = \left\langle a, b, c, d, e \,\middle|\, [a, c], [b, c], [c, d], [d, e] \right\rangle.$$

Main Theorem

Theorem (Belk, Bleak, Matucci 2017)

For every right-angled Artin group A_Γ*, there exists an* $n \geq 1$ *so that* A_Γ *embeds into* nV*.*

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For every right-angled Artin group A_Γ, there exists an $n \geq 1$ so that A_Γ embeds into nV.

This is quite different from the situation for V.

Theorem (Bleak, Salazar-Díaz 2013)

$\mathbb{Z}^2 * \mathbb{Z}$ does not embed into Thompson's group V.

It follows that the only right-angled Artin groups that embed into V are direct products of free groups (Corwin, Haymaker 2016).

Stronger Version

Our embedding is "demonstrative" in the sense of (Bleak, Salazar-Díaz 2013). Combined with their work, this gives:

Theorem (Belk, Bleak, Matucci 2017)

For every right-angled Artin group A_Γ, there exists an $n \geq 1$ so that:

1. *$nV \wr A_\Gamma$ embeds into nV,*
2. *Every finite extension of A_Γ embeds into nV, and*
3. *Every group that virtually embeds into A_Γ embeds into nV.*

Consequences for the Subgroup Structure

Corollary (Belk, Bleak, Matucci 2017)

All of the following groups embed into nV for sufficiently large n:

1. *Finitely generated Coxeter groups.*
2. *Surface groups.*
3. *Graph braid groups.*
4. *Limit groups.*
5. *Many 3-manifold groups.*
6. *Many hyperbolic groups.*

Essentially none of these groups are known to embed into V.

Sketch of Proof

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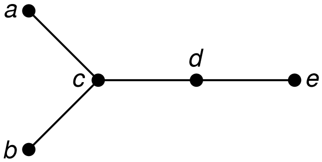
Our proof uses the ***complement*** Γ^c of the graph Γ:



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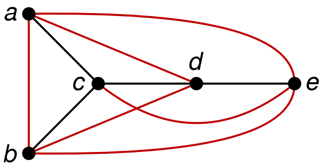

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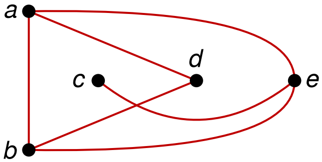

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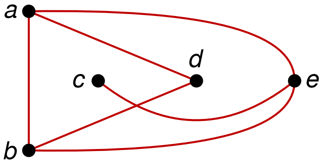

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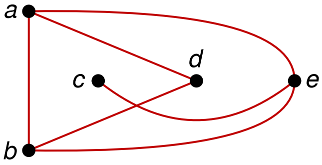

Edges in Γ^c correspond to generators that *don't* commute.

If Γ^c has v vertices and e edges, we embed A_Γ into $(v+e)V$.

Note: Kato has recently improved on our method, constructing an embedding of A_Γ into eV (Kato 2017).

Sketch of Proof

Let's embed A_Γ into $7V$ for the following graph Γ^c.

So $A_\Gamma = \left\langle a, b, c, d \;\middle|\; [a,c], [a,d], [b,d] \right\rangle$.

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Representing Elements of A_Γ

Most elements of A_Γ have several different minimum-length words, e.g.

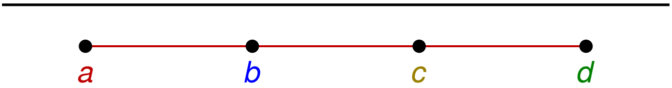
$$a\,c\,d \;=\; c\,a\,d \;=\; c\,d\,a.$$

Is there a good way of representing elements uniquely?

Sketch of Proof

We can represent each element of A_Γ using a ***stack of blocks***.

$a \quad c \quad b \quad c \quad d \quad b \quad a \quad b \quad c$



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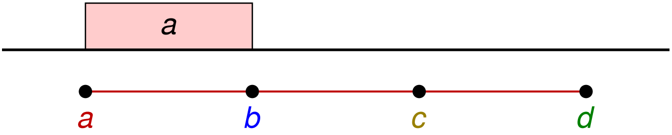

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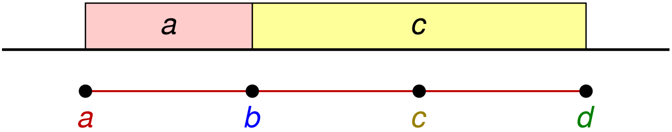

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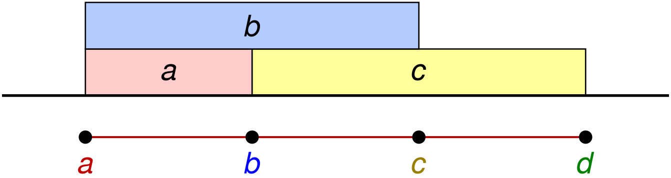

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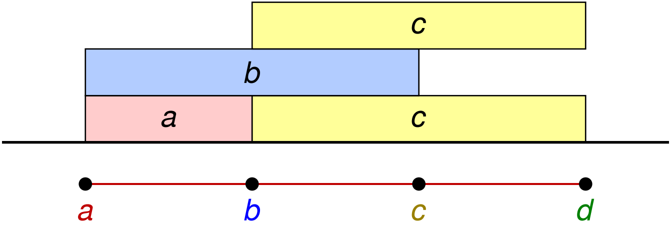

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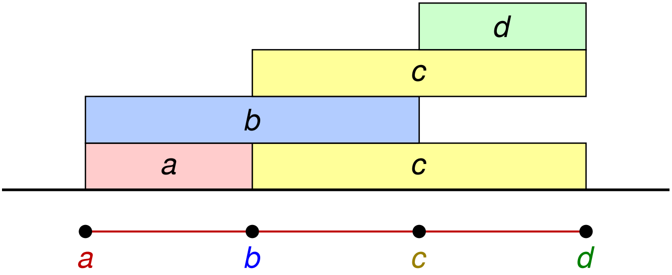

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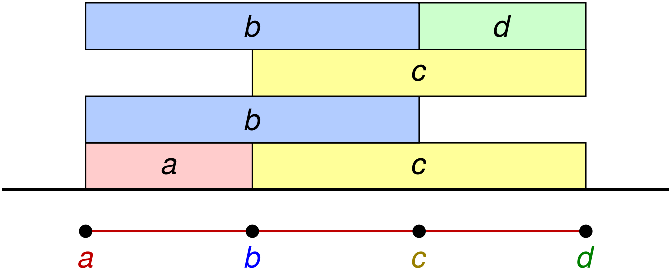

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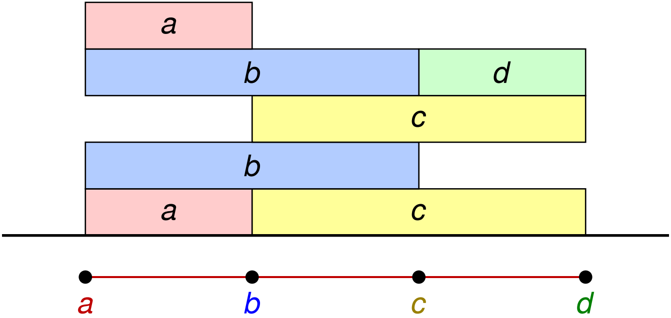

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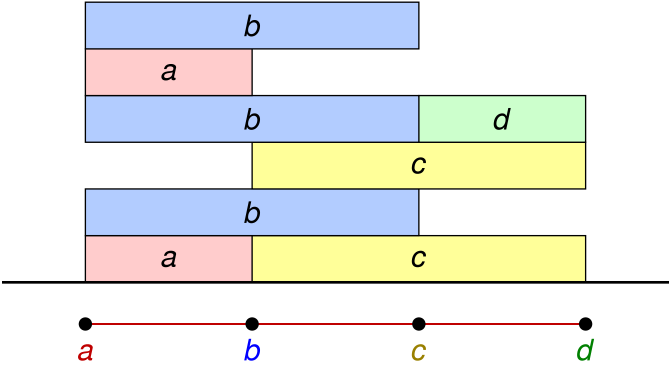

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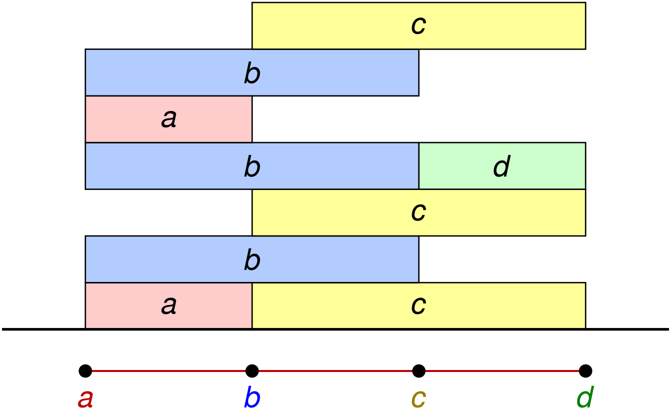

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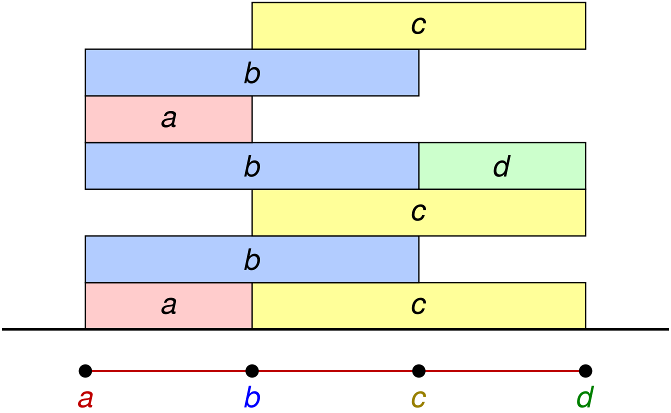
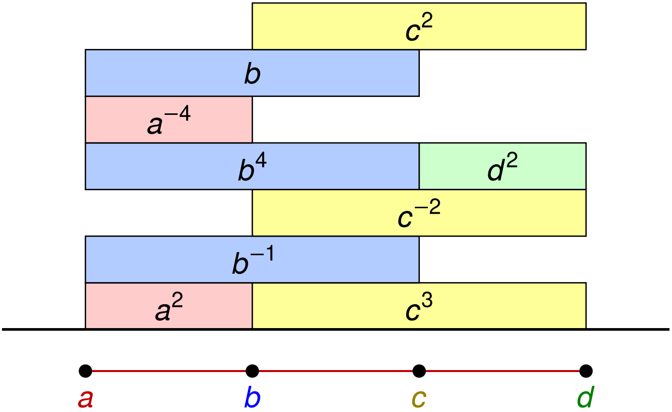


Sketch of Proof

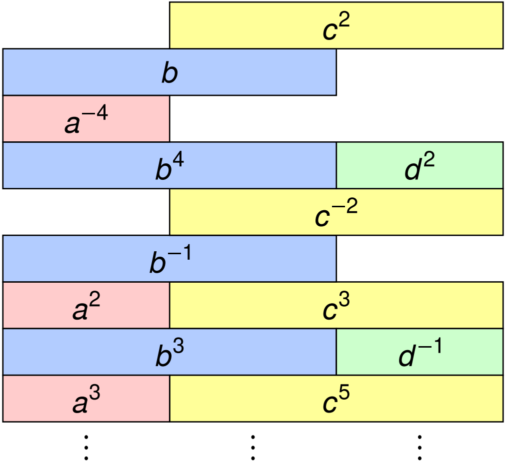
Each power of a generator corresponds to a single block.

$$a^2 \; c^3 \; b^{-1} \, c^{-2} \, d^2 \; b^4 \; a^{-4} \, b \; c^2$$



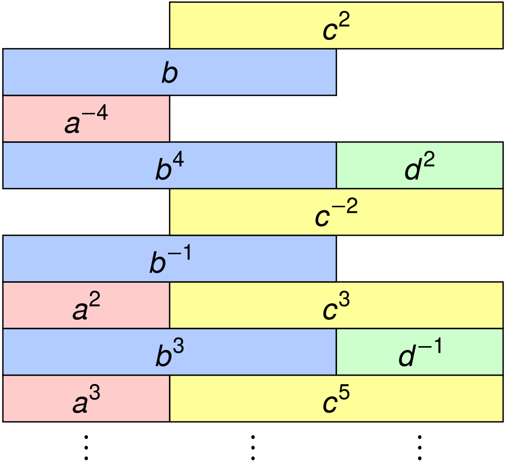
Sketch of Proof

An ***infinite block stack*** is a stack of blocks with no bottom.



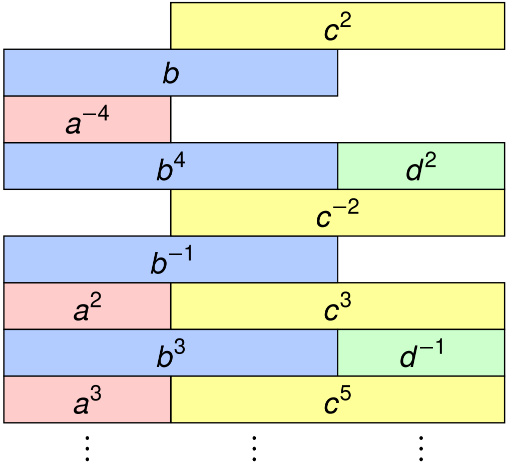
Sketch of Proof

The group A_Γ acts on the set of all infinite block stacks.



Sketch of Proof

Main Idea: Encode an infinite block stack using binary sequences.



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Main Idea: Encode an infinite block stack using binary sequences.

$$\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

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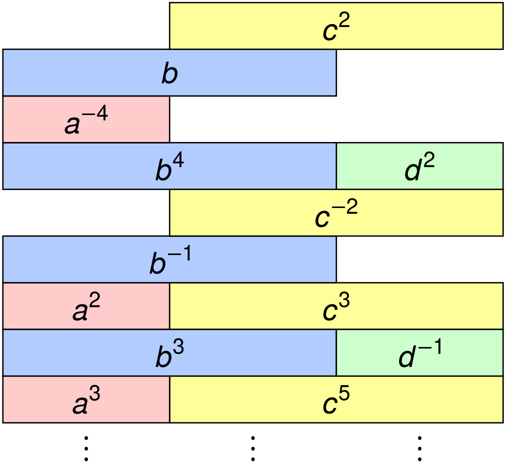

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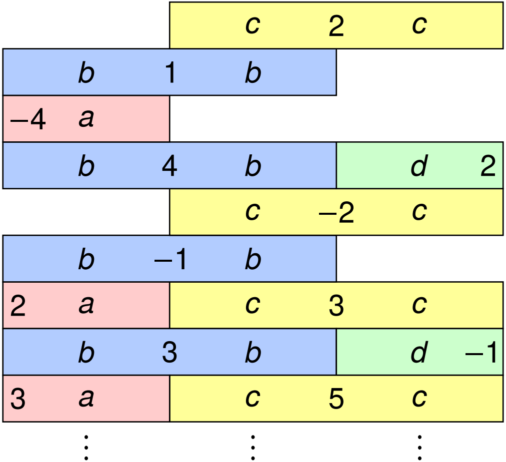

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$$
\begin{array}{ccccccc}
 & & & c & 2 & c & \\
 & b & 1 & b & & & \\
-4 & a & & & & & \\
 & b & 4 & b & & d & 2 \\
 & & & c & -2 & c & \\
 & b & -1 & b & & & \\
2 & a & & c & 3 & c & \\
 & b & 3 & b & & d & -1 \\
3 & a & & c & 5 & c & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
$$

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1	1	0	1	0	0	0
1	0	1	0	0	1	0
1	1	0	0	1	0	1
1	1	0	1	1	0	1
0	0	0	0	1	0	0
0	1	0	1	0	0	1
0	0	1	0	0	1	1
1	0	1	1	0	0	1
	⋮		⋮		⋮	

Questions

1. What is the minimum n for which a given A_Γ embeds into nV?
2. Do surface groups embed into V? Do they embed into $2V$? What about Coxeter groups, graph braid groups, etc.?
3. Do all hyperbolic groups embed into nV for sufficiently large n?
4. For $n \geq 2$ does nV act properly by isometries on any $\mathrm{CAT}(0)$ cubical complex?
5. For $n \geq 2$, is the conjugacy problem in nV solvable?