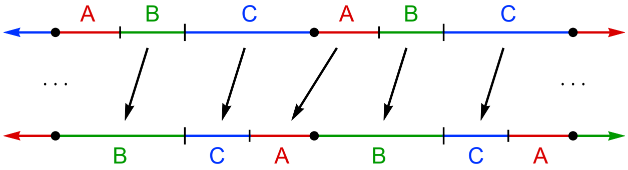
Embeddings into Finitely Presented Simple Groups



Jim Belk, University of Glasgow

Modern advances in geometric group theory

University of Manchester, September 2022

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

This conjecture remains open after nearly 50 years.

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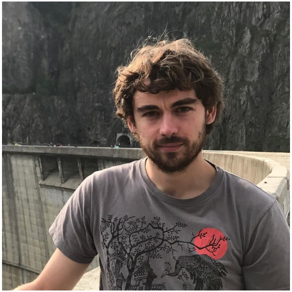
Recent progress: Many groups of interest embed into finitely presented simple groups.

Collaborators

Collin Bleak

University of St Andrews



James Hyde

University of Copenhagen

Collaborators

Francesco Matucci

University of Milano–Bicocca

Matthew Zaremsky

SUNY University at Albany

Higman's Embedding Theorem

Higman's Embedding Theorem

A countable group presentation

$$\langle s_1, s_2, s_3, \ldots \mid r_1, r_2, r_3, \ldots \rangle$$

is ***computable*** if there exists an algorithm that outputs the list of relations.

A group is ***computably presented*** if it admits such a presentation.

Examples

1. Any finitely presented group.
2. Any finitely generated subgroup of a finitely presented group.

Higman's Embedding Theorem (1961)

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Graham Higman, 1960

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1. *Countably generated groups with a computable presentations.*

   Follows from Higman–Neumann–Neumann 1949.

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1. *Countably generated groups with a computable presentations.*
2. *Countable abelian groups.*

   Since every such group embeds in $\bigoplus_\omega \mathbb{Q} \oplus \bigoplus_\omega \mathbb{Q}/\mathbb{Z}$.

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Problem (Higman): Find an explicit and natural example of a finitely presented group that contains $\mathbb{Q}$.

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This theorem has the form

G has a certain algorithmic property	$\Leftrightarrow$	G embeds into a certain kind of group

Question (Higman): Are there other theorems of this type?

The Boone–Higman Conjecture

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Richard J. Thompson, 2004

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Proof.

Given a presentation $\langle s_1, \dots s_m \mid r_1, \dots r_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1

Search for a proof that

$$w = 1$$

using the relations $r_1, \dots, r_n$.

Search #2

Search for a proof that

$$s_1 = \cdots = s_m = 1$$

using $w = 1$ and $r_1, \dots, r_n$.

Eventually one of the searches terminates. □

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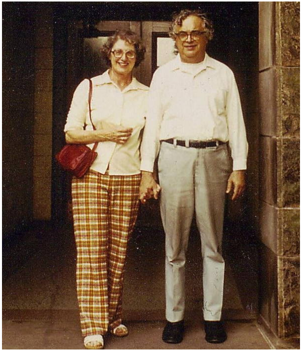
Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.



William and
Eileen Boone, 1979

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Every complete theory with finitely many axioms is decidable.

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inconsistent theory	trivial group
complete theory	simple group
decidable theory	decidable word problem

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Like Higman's embedding theorem, this statement has the form

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As a corollary, the following groups would also embed into finitely presented simple groups:

1. Any computably presented group with solvable word problem.
2. Any countable abelian group.

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Sketch of Proof. We want a simple group that contains G.

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Trick: Given words $u, v \neq_G 1$, consider the group

$$G' = \left\langle G, x, t \,\middle|\, (uu^x)^t = u^x v \right\rangle.$$

G' is an HNN extension of $G * \langle x \rangle$, so G embeds into G'.

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But now v lies in the normal closure of u.

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$$\sigma(G) = \left\langle G, x, t_1, t_2, \ldots \,\middle|\, (u_i u_i^x)^{t_i} = u_i^x v_i \right\rangle$$

where (u_i, v_i) is an enumeration of *all* pairs of non-identity words in G.

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Then G embeds into $\sigma(G)$, and the normal closure of any non-identity element of G contains G.

The desired simple group is the union of the sequence

$$G \;\leq\; \sigma(G) \;\leq\; \sigma^2(G) \;\leq\; \sigma^3(G) \;\leq\; \cdots . \qquad \square$$

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Theorem (Thompson 1980)

Let G be a finitely generated group. Then:

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Theorem (Sacerdote 1977)

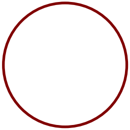
There are analogues of Boone and Higman's theorem for the order, conjugacy, power, and subgroup membership problems.

Finitely Presented Simple Groups

Thompson's Groups

In 1965, Richard J. Thompson defined three infinite groups.

F acts on the interval.



T acts on the circle.

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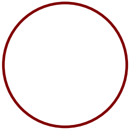

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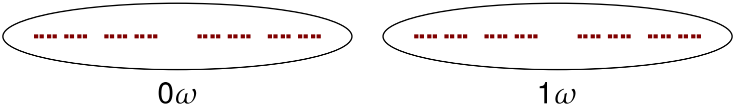

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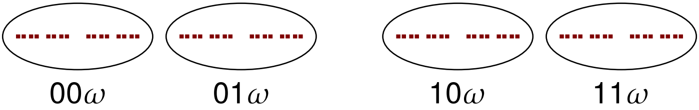

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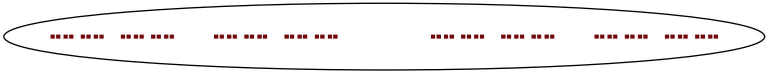

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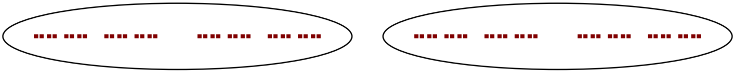

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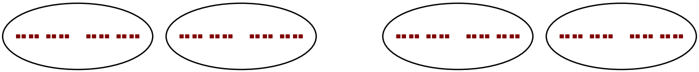

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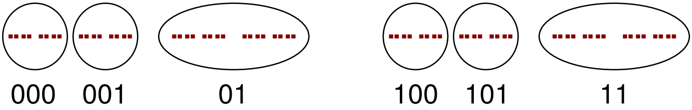

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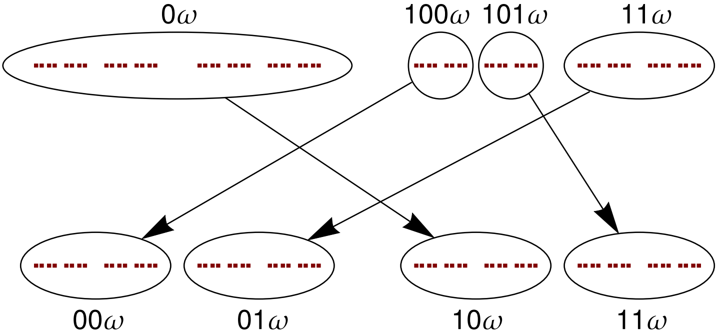
Definition of V

Thompson's group V is the group of all homeomorphisms that map "linearly" between the pieces of two dyadic subdivisions.



This group V is finitely presented and simple.

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V acts by homeomorphisms on the Cantor set.

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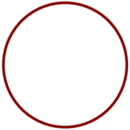

T is the subgroup of V that preserves the circular order.

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V acts by homeomorphisms on the Cantor set.

F and T are subgroups of V.

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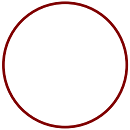

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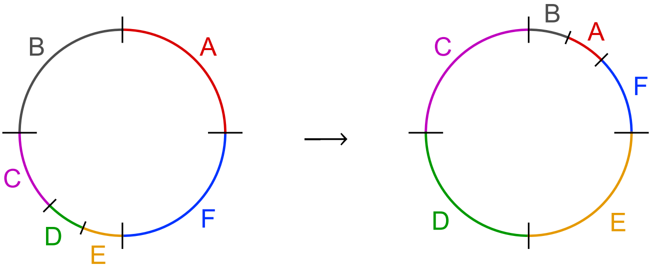
Thompson's Group T

For example, here is an element of Thompson's group T.



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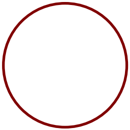

T acts on the circle.
finitely presented, simple

V acts on the Cantor set.
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Subgroups of V

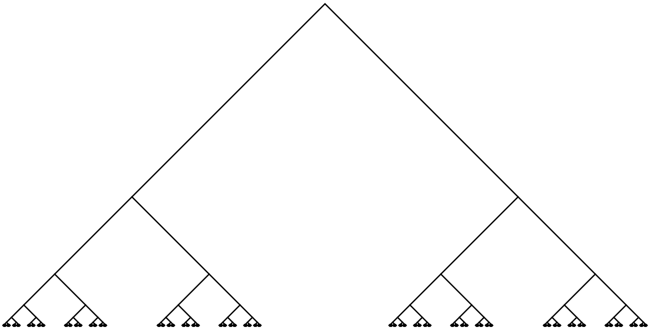
The following groups embed into V:

1. All finite groups, free groups, free abelian groups, $\bigoplus_{\omega} V$.
2. (Higman 1974, Brown 1987) Generalised Thompson groups F_n, T_n, and V_n.
3. (Röver 1999) The Houghton groups H_n, and free products of finitely many finite groups.
4. (Guba–Sapir 1999) $\mathbb{Z} \wr \mathbb{Z}$, $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$, $((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \mathbb{Z}$, $\ldots$
5. (Bleak–Kassabov–Matucci 2011) $\mathbb{Q}/\mathbb{Z}$.
6. (Bleak–Salazar-Díaz 2013) $V \wr A$ and $V * A$, where A is any finite group or $A \in \{\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\}$.

Automata Groups

Grigorchuk's Group

Grigorchuk's group $\mathcal{G}$ (of intermediate growth) is a certain group of automorphisms of the infinite rooted binary tree T_2.



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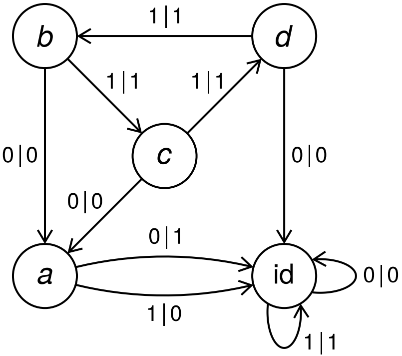
The boundary ∂T_2 is the Cantor set $\{0, 1\}^\omega$. $\mathcal{G}$ acts by homeomorphisms on this Cantor set.

Grigorchuk's Group

The action of $\mathcal{G}$ on binary sequences in $\{0, 1\}^{\omega}$ can be described by ***automata***.



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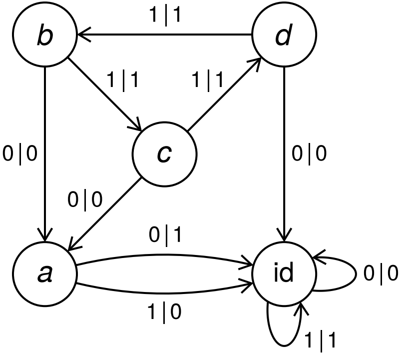

$c(1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$

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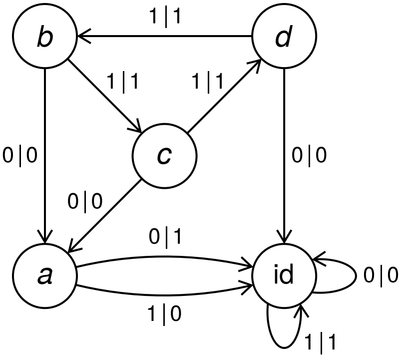

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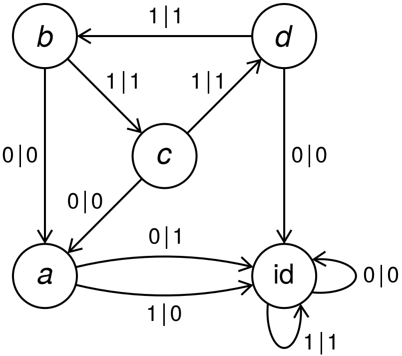

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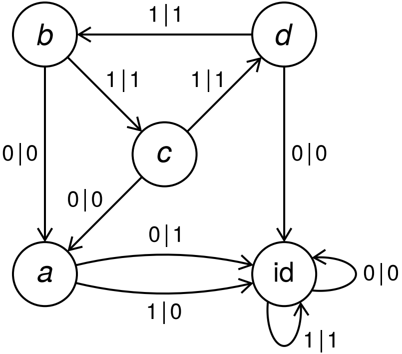

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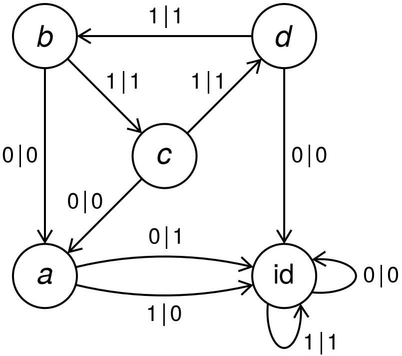

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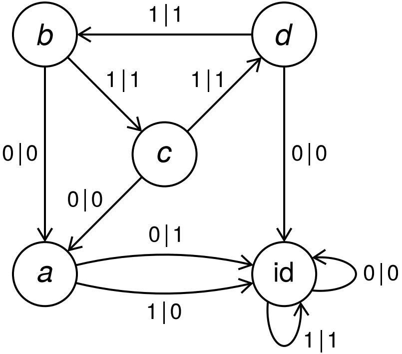
Grigorchuk's Group

The action of $\mathcal{G}$ on binary sequences in $\{0,1\}^\omega$ can be described by ***automata***.



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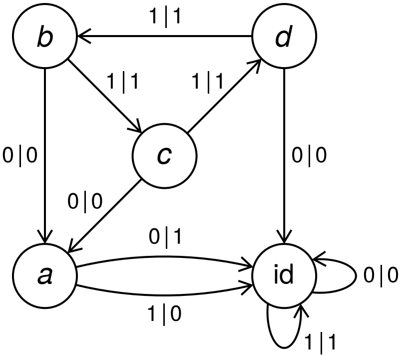

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Every element of $\mathcal{G}$ has such an automaton.

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Theorem (Grigorchuk 1979)

The group $\mathcal{G} = \langle a, b, c, d \rangle$ has intermediate growth, and every element has finite order.

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Automata Groups

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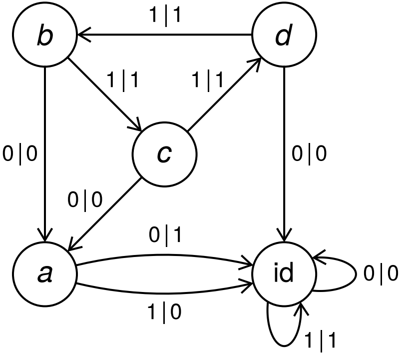

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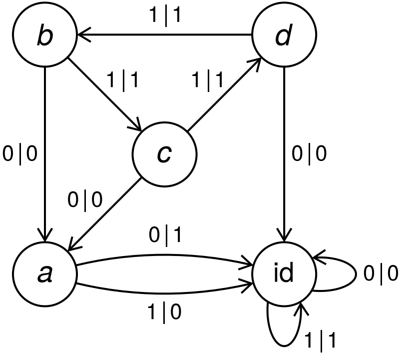
Write $s|_w$ for the state obtained by starting in state s and then taking input w.



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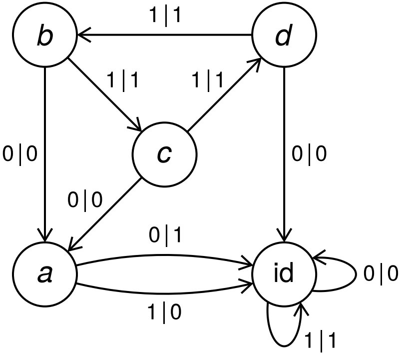

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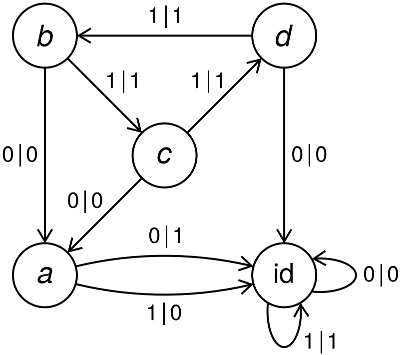

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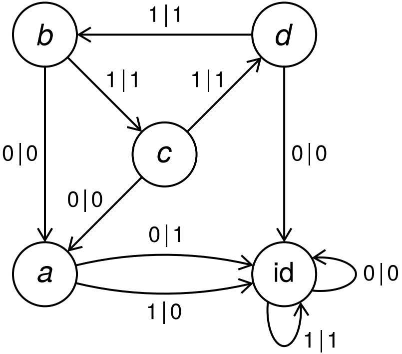

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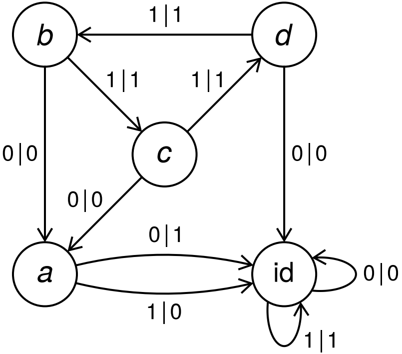

G is ***self-similar*** if $g|_w \in G$ for every $g \in G$ and every finite word w.

G is ***contracting*** if there exists a finite set $N \subset G$ such that $g|_w \in N$ for all $g \in G$ and all sufficiently long words w.

Automata Groups

Nekrashevych (2004) considered the group $V_d G$ generated by:

- The generalised Thompson group V_d, and
- A self-similar group G acting on $\{0, \ldots, d-1\}^\omega$.

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So this gives Boone–Higman embeddings for *some* contracting self-similar groups.

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Matthew Brin

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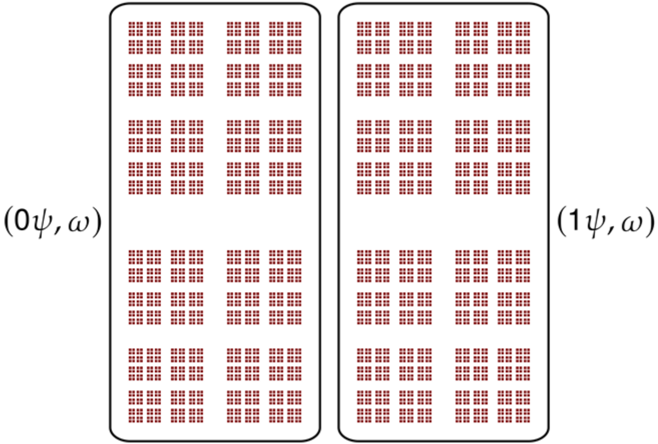

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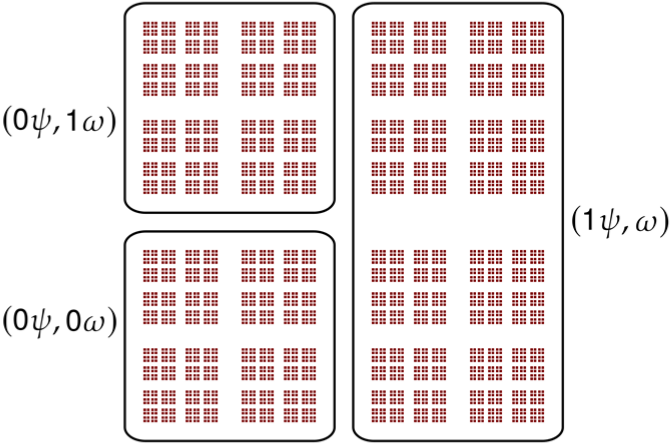

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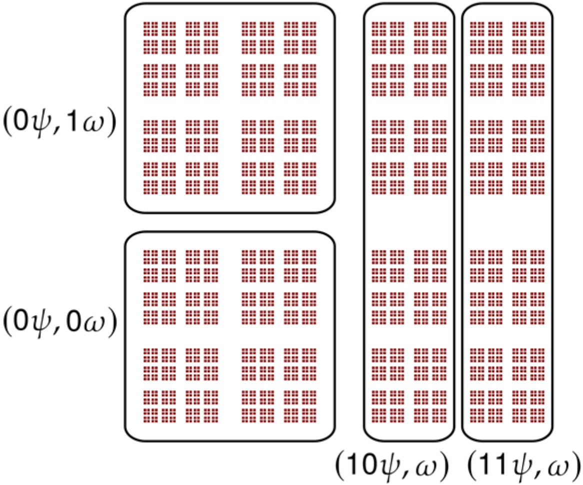
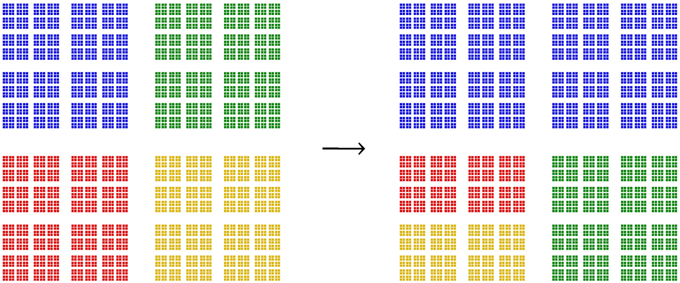


Brin's Groups

Elements of $2V$ map "linearly" between two subdivisions.



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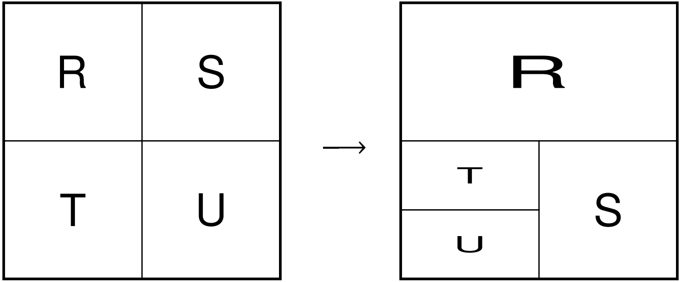

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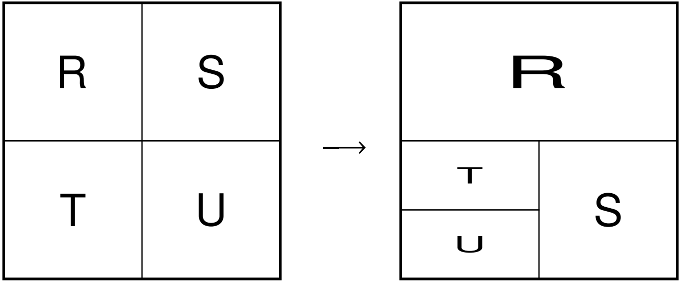


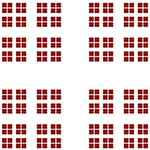
Theorem (Brin 2004)

The group $2V$ is finitely presented and simple.

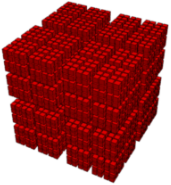
Brin's Groups

Brin defined a family of groups nV $(n \geq 1)$ similarly, with $1V = V$.

V



$2V$



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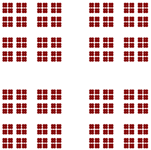

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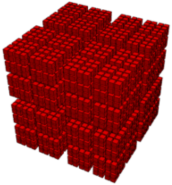


$3V$

$\cdots$

Theorem (Brin 2009)

The group nV is finitely presented and simple for all $n \geq 1$.

Brin's Groups

These groups have very interesting algorithmic properties.

Theorem (B–Bleak 2014)

The order problem in nV is unsolvable for $n \geq 2$

Theorem (B–Bleak–Matucci 2016)

The subgroup membership problem in nV is unsolvable for $n \geq 2$.

Theorem (Salo 2020)

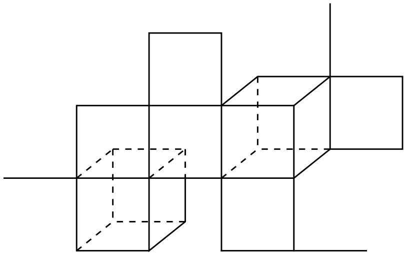
The conjugacy problem in nV is unsolvable for $n \geq 2$.

Virtually Special Groups

Virtually Special Groups

RAAG's are very interesting from an embeddings perspective.

Haglund and Wise (2008) have shown that the fundamental group of any (compact) ***special cube complex*** embeds into a RAAG.



Such groups are called ***special***. Many groups of interest are ***virtually special***.

Virtually Special Groups

The virtually special groups include:

1. (Wise 2009) All limit groups.
2. (Haglund–Wise 2010) All finitely generated Coxeter groups.
3. (Agol 2012) All cubulated hyperbolic groups.
4. (Przytycki–Wise 2012) Fundamental groups of Riemannian 3-manifolds of non-positive curvature.
5. (Groves–Manning 2020, Oregón-Reyes 2020) Certain cubulated relatively hyperbolic groups.

Virtually Special Groups

Theorem (B–Bleak–Matucci 2016)

Let G be a finitely generated group. If G has a finite-index subgroup that embeds into a RAAG, then G embeds into one of Brin's groups nV.

Corollary

Every virtually special group embeds into a finitely presented simple group.

Note: Scott (1984) had previously shown that each $\mathrm{GL}_n(\mathbb{Z})$ embeds into a finitely presented simple group, which covers RAAG's themselves.

Virtually Special Groups

Let G be a group that virtually embeds into the RAAG for the graph (V, E).

Theorem (B–Bleak–Matucci 2016)

G embeds into nV for $n = \binom{|V|+1}{2} - |E|$.

Theorem (Kato 2016)

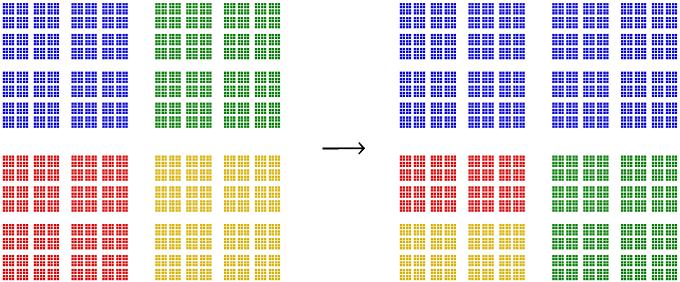
G embeds into nV for $n = \binom{|V|}{2} - |E|$.

Theorem (Salo 2021)

G embeds into $2V$.

Virtually Special Groups

So Brin's finitely presented simple group $2V$ contains all virtually special groups.



Countable Abelian Groups

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Problem (Higman): Find an explicit and natural example of a finitely presented group that contains $\mathbb{Q}$.

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In 1999, Martin Bridson and Pierre de la Harpe submitted this question to the Kourovka notebook as a “well-known” problem.

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In 1999, Martin Bridson and Pierre de la Harpe submitted this question to the Kourovka notebook as a “well-known” problem.

In 2020, James Hyde, Francesco Matucci, and I noticed an elementary solution.

Countable Abelian Groups

Recall that Thompson's group T acts on S^1.

A ***lift*** of an element $g \in T$ is a homeomorphism $\overline{g} \colon \mathbb{R} \to \mathbb{R}$ that makes the following diagram commute:

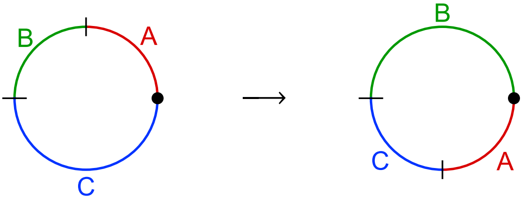
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\overline{g}} & \mathbb{R} \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow[g]{} & S^1 \end{array}$$

Note: If $\overline{g}$ is a lift of g then so is $\overline{g} + n$ for any $n \in \mathbb{Z}$.

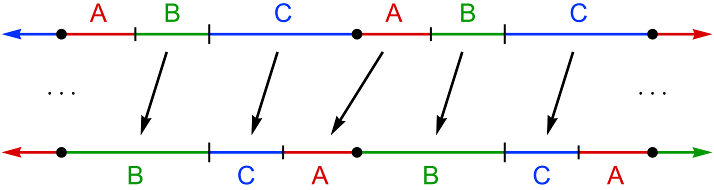
Let $\overline{T}$ be the group of all lifts of elements of T.

Countable Abelian Groups

For example, here's an element of T:



and here's one possible lift in $\overline{T}$:



Countable Abelian Groups

Theorem (B–Hyde–Matucci 2020)

The group $\overline{T}$ is finitely presented and contains $\mathbb{Q}$.

$$\overline{T} = \left\langle a, b \;\middle|\; a^4b^{-3}, (ba)^5b^{-9}, [bab, a^2baba^2], [bab, a^2b^2a^2baba^2ba^2] \right\rangle$$

Note: We did not introduce this group $\overline{T}$. It had previously appeared in the work of Ghys and Sergiescu (1987).

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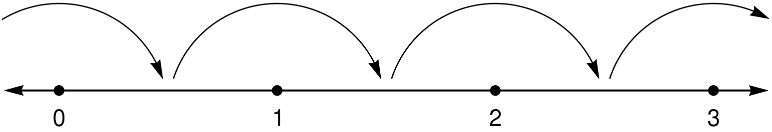
Proof. Start with the element $f_1(t) = t + 1$:



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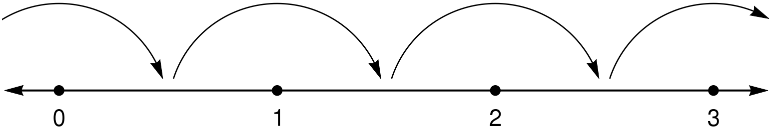
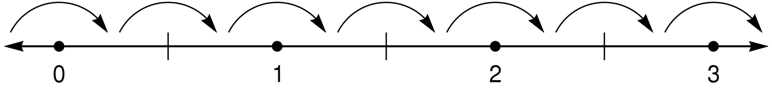


It's easy to find a square root f_2 of f_1:



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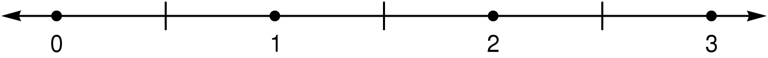

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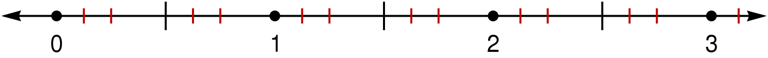

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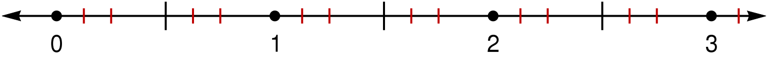
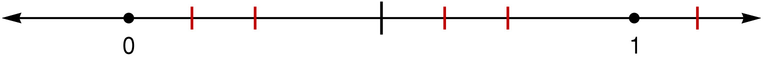


Next, construct a fourth root f_4 of f_3:



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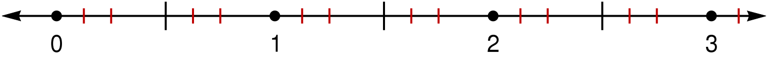

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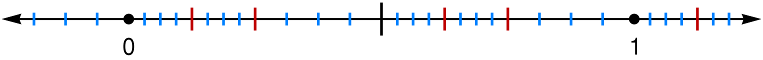

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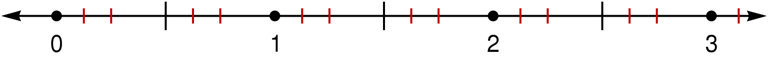

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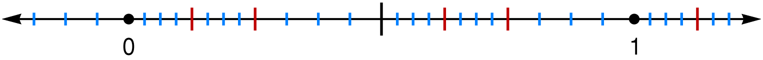

Then $\langle f_1, f_2, f_3, f_4, \ldots \rangle \cong \mathbb{Q}$. □

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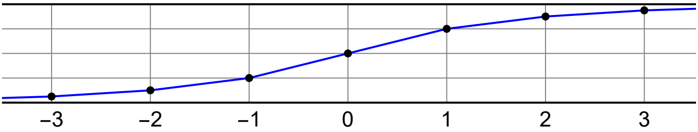

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We prove that the group $V\overline{T}$ generated by V and $\overline{T}$ is finitely presented, simple, and contains $\bigoplus_\omega \mathbb{Q} \oplus \bigoplus_\omega \mathbb{Q}/\mathbb{Z}$. □

Twisting Brin's Groups

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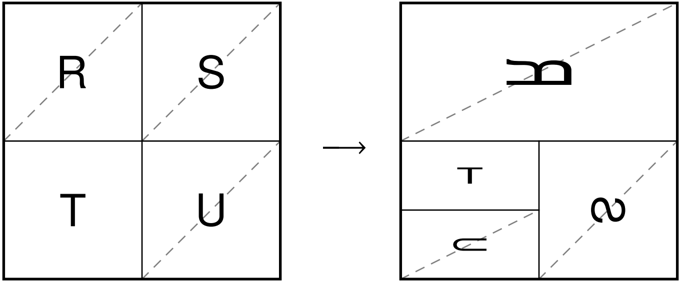
In 2020, Matthew Zaremsky and I considered a “twisted” version of Brin’s group $2V$.



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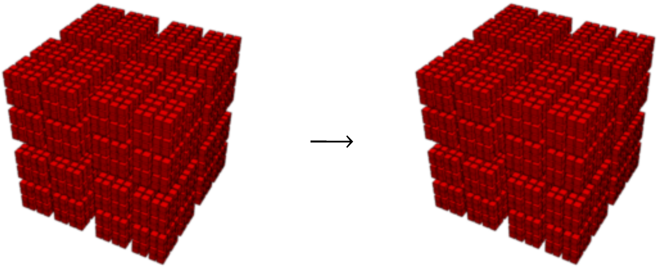
In general, you can twist nV by any group of permutations of $\{1, \ldots, n\}$.



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Corollary (B–Zaremsky 2020)

Any finitely generated group G embeds isometrically into a finitely generated simple group.

Twisting Brin's Groups

We can also get finitely presented simple groups.

Theorem (B–Zaremsky 2020)

Suppose:

1. *G is finitely presented,*
2. *G acts highly transitively on a set X, and*
3. *Stabilizers of finite subsets of X are finitely presented.*

Then the resulting twisted ωV is a finitely presented simple group that contains G.

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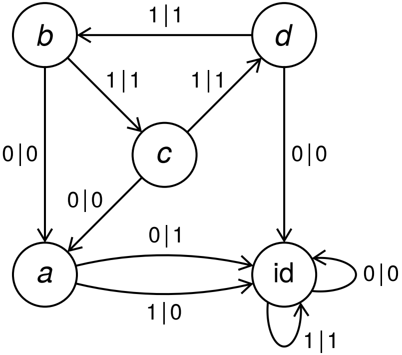
Zaremsky (2022) improves condition (3) to the stabilizers being finitely *generated*.

Twisting Brin's Groups

Corollary

Every self-similar, contracting group G of automata embeds into a finitely presented simple group.



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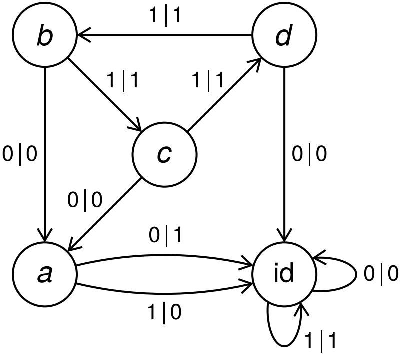

Sketch of Proof.

The Nekrashevych group $V_d G$ is finitely presented, highly transitive on any orbit, and has finitely generated stabilizers, so the resulting twisted ωV is finitely presented and simple.

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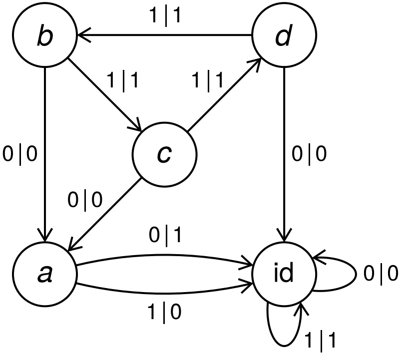

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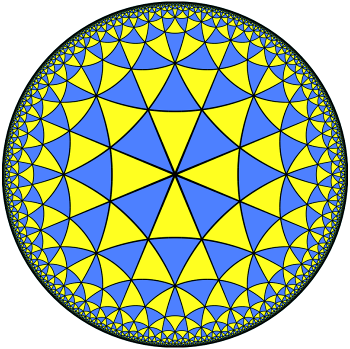
We can similarly handle many other “Thompson-like” groups.

Hyperbolic Groups

Hyperbolic Groups

Theorem (B–Bleak–Matucci–Zaremsky last week)

Every hyperbolic group G embeds into a finitely presented simple group.



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4. Conclude that $V[G]$ embeds into a twisted ωV which is finitely presented and simple.

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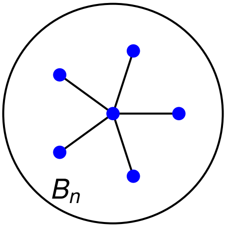

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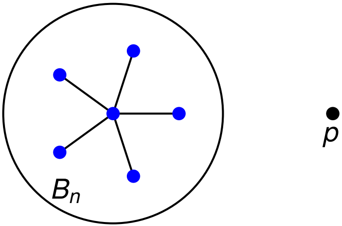

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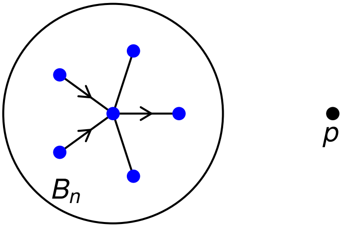

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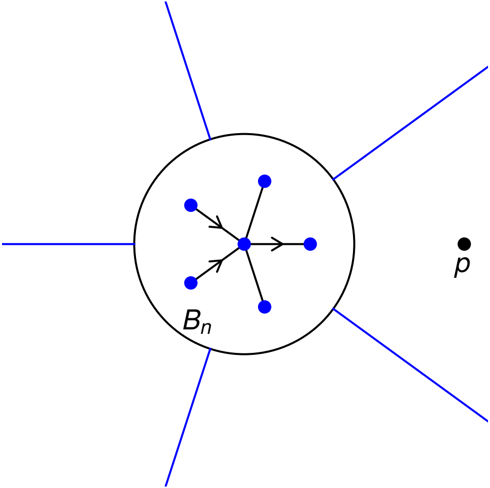

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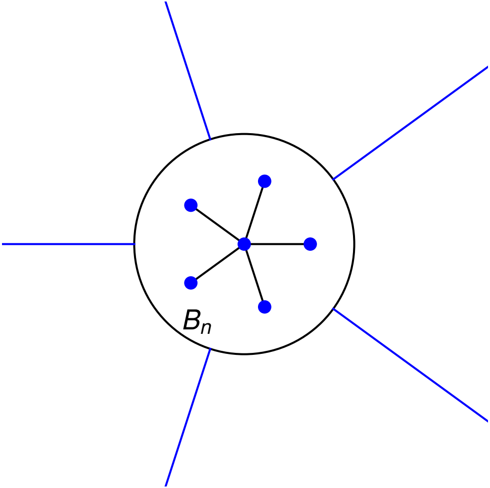

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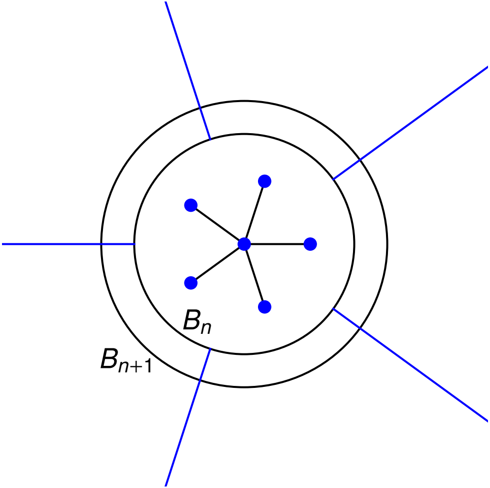

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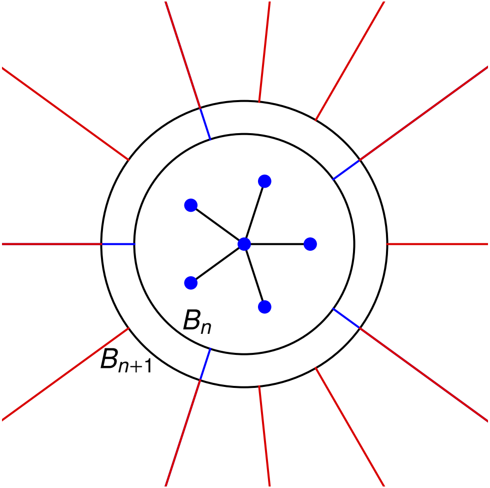

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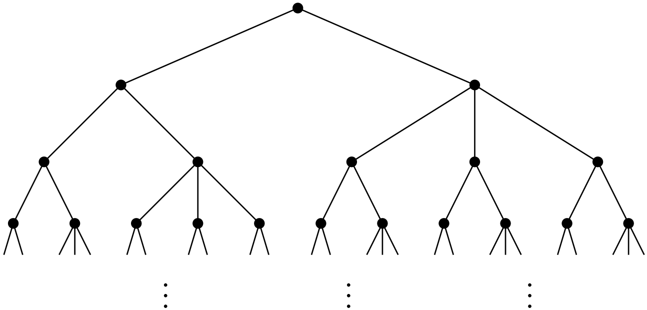


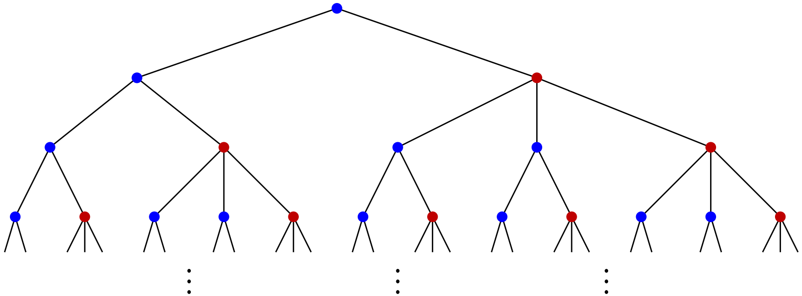
This is the ***tree of atoms***. Its space of ends is $\partial_h G$.

Hyperbolic Groups

Theorem (B–Bleak–Matucci 2018)

If G is a hyperbolic group, then:

1. *The tree of atoms has a self-similar structure, and*
2. *G acts on $\partial_h G$ by asynchronous automata.*



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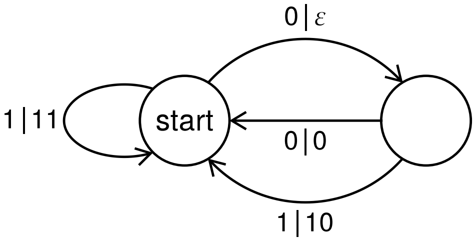

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Theorem (B–Bleak–Matucci–Zaremsky last week)

The action of G on $\partial_h G$ is contracting, and hence $V[G]$ is finitely presented.

In particular, you always arrive at a state in the nucleus after at most $2|g| + 39\delta + 13$ steps.

Open Questions

Which of the following groups embed into finitely presented simple groups?

1. Braid groups B_n for $n \geq 4$?
2. Mapping class groups?
3. $\mathrm{Out}(F_n)$?
4. Finitely generated nilpotent groups?
5. Finitely generated metabelian groups?
6. One relator groups?

Also, what is an explicit, natural example of a finitely presented group that contains $\mathrm{GL}_n(\mathbb{Q})$?

Mapping Class Groups

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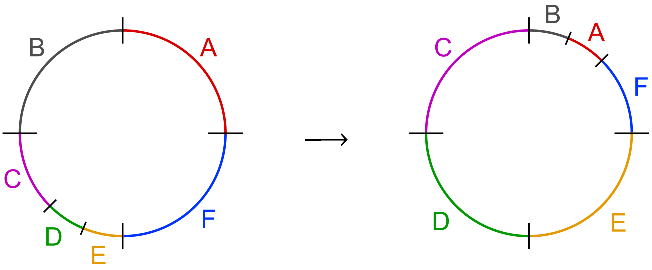
It is an open question whether mapping class groups and braid groups embed into finitely presented simple groups.

For one possible approach, consider the following element of Thompson's group T.



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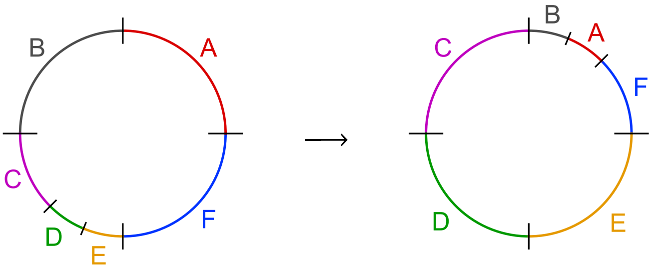

This looks just like the action of a pseudo-Anosov on $\mathcal{PMF}$!

Mapping Class Groups

Train tracks give $\mathcal{PMF}$ a ***piecewise-integral projective (PIP) structure***, with elements of $\mathrm{Mod}(S)$ acting as PIP maps.

Thurston observed that the group $\mathrm{PIP}(S^1)$ of PIP homeomorphisms of S^1 is isomorphic to Thompson's group T.

Open Question (Thurston): For $n \geq 2$, is the group $\mathrm{PIP}(S^n)$ finitely generated?

$\mathrm{Mod}(S_{g,n})$ embeds into $\mathrm{PIP}(S^{6g-7+2n})$ for $g \geq 3$. Is this a finitely presented simple group?

The End