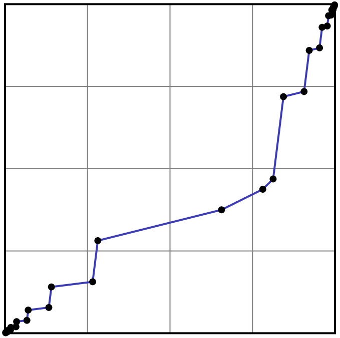
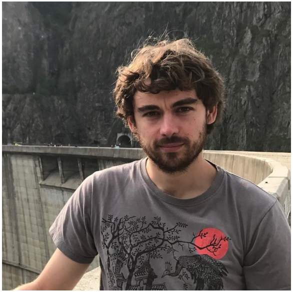
Finite Germ Extensions



Jim Belk, University of Glasgow

Groups and Topology, Lancaster University, 28 April 2023

Collaborators



James Hyde

University of Copenhagen

Francesco Matucci

University of Milan–Bicocca

Homeomorphisms with Singularities

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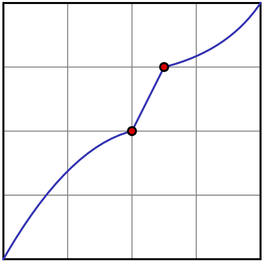
Example

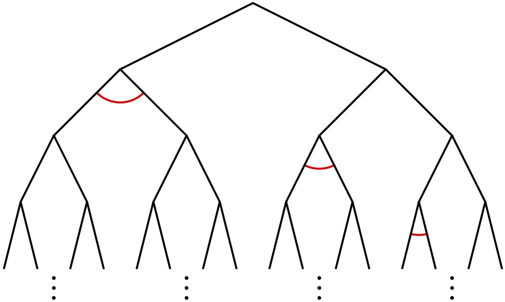
Let $X = [0, 1]$, and let h be a piecewise-smooth homeomorphism. Then we could regard the breakpoints of h as singularities.



Homeomorphisms with Singularities

Example 2

Let h be the following automorphism of the tree $\mathcal{T}_2$:



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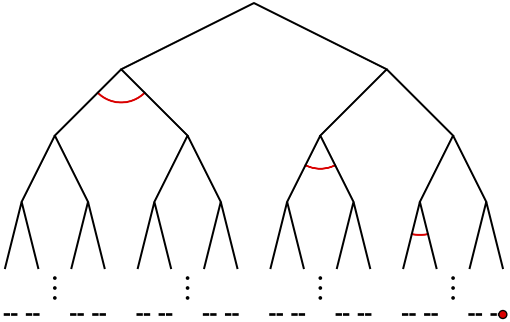
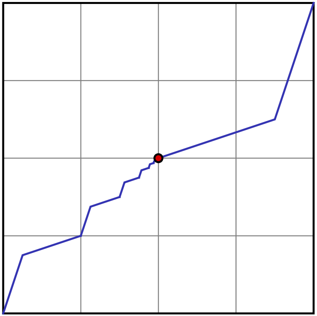


Then $h \in \text{Homeo}(\partial\mathcal{T}_2)$, and we could regard h as having a singularity at the rightmost point.

Homeomorphisms with Singularities

Example 3

Let h be a homeomorphism of $[0, 1]$ with infinitely many linear pieces:



Then we could regard the accumulation points of the breakpoints as singularities.

Making this Precise

Let X be a space, and fix a ***base group*** $B \leq \text{Homeo}(X)$.

We say $h \in \text{Homeo}(X)$ has a ***singularity*** at a point $p \in X$ if there is no neighborhood of p on which h agrees with an element of B.

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Examples

1. If $X = [0, 1]$ and $B = \text{Diff}^1(X)$, then any breakpoint (or critical point) is a singularity.
2. If $X = \partial\mathcal{T}_2$ and B is the group of piecewise-translations, then any point with complicated local behavior is a singularity.
3. If $X = [0, 1]$ and $B = \text{PL}(X)$, then any accumulation point of breakpoints is a singularity.

Finite Germ Extensions

Let X be a space, and let $B \leq G \leq \text{Homeo}(X)$.

We say that G is a ***finite germ extension*** of B if:

1. Every element of G has finitely many singularities.
2. Every element of G without singularities lies in B.
3. If $g \in G$ has a singularity at p, then there exists an $h \in G$ that agrees with g near p and has no other singularities.

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Example

Let $B = \text{PL}([0, 1])$, and let G be the group of all homeomorphisms with countably many linear pieces that accumulate at a finite set of points.

Main Simplicity Result

Let $G \leq \mathrm{Homeo}(X)$ be a finite germ extension of B.

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Suppose:

1. *B is simple, locally moving, and has no global fixed points,*
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By weakening these hypotheses, we can sometimes prove that G' is simple and describe the isomorphism type of G/G'.

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has type F_∞ for every pair $M_0 \subseteq M$ of finite subsets of $\mathrm{sing}(G)$.

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Thompson's Groups and Finiteness Properties

Thompson's Groups

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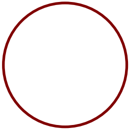

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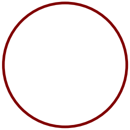

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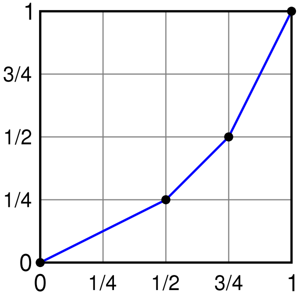
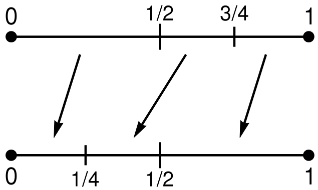
Thompson's Group F

Thompson's group F is the group of all piecewise-linear homeomorphisms of $[0, 1]$ for which:

- Each segment has slope 2^n ($n \in \mathbb{Z}$), and
- Each breakpoint has dyadic rational coordinates.





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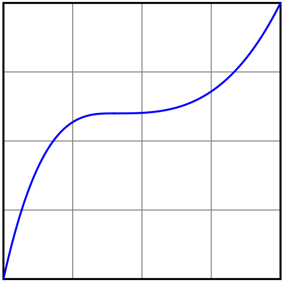

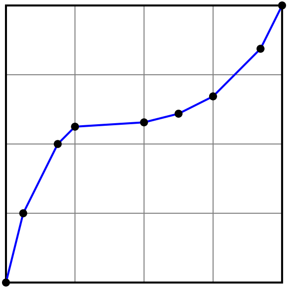

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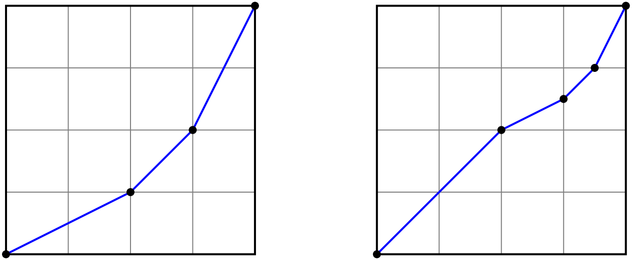

x_0 x_1

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- F is finitely presented.

$$F = \langle x_0, x_1 \mid x_2^{x_1} = x_3,\ x_3^{x_1} = x_4 \rangle$$

where

$$x_2 = x_1^{x_0}, \qquad x_3 = x_2^{x_0}, \qquad x_4 = x_3^{x_0}.$$

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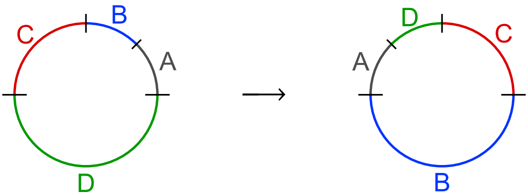

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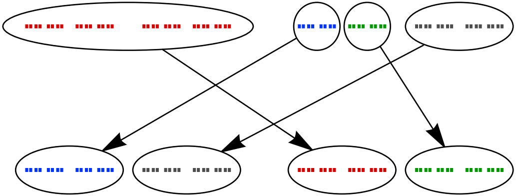

Type F_∞

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Ken Brown

Ross Geoghegan

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Brown (1987) later generalized this to T and V, using a method now known as ***Brown's criterion***.

Type F_n Using Actions

Proposition

Let G be a group acting rigidly on a CW complex K, and let $n \geq 1$. Suppose that:

1. *K is $(n-1)$-connected, and G acts cocompactly on K.*
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For example, in many cases there is a nice contractible K which is infinite-dimensional, but then G can't act cocompactly.

Brown's Criterion

Brown's idea is to use a *chain* of complexes:

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

We make sure that:

1. G acts cocompactly on each K_i, and
2. The union $K = \bigcup_{i=1}^{\infty} K_i$ is contractible.

Since the K_i are "converging" to a contractible space, they ought to be highly connected when i is large. We can prove this by showing that for each n the sequence

$$\pi_n(K_1) \to \pi_n(K_2) \to \pi_n(K_3) \to \cdots$$

eventually stabilizes.

Discrete Morse Theory

Bestvina and Brady (1996) introduced powerful methods for analysing the homomorphisms $\pi_n(K_i) \to \pi_n(K_{i+1})$.

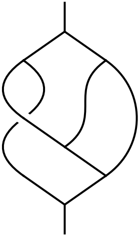
Mladen Bestvina

Noel Brady

They showed how to understand such homomorphisms by considering the connectivity of the ***descending links***.

Thompson-like groups

Using Brown's criterion and Bestvina–Brady Morse theory, many "Thompson-like" groups have been shown to have type F_∞

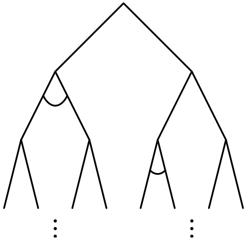


braided V

Bux et al. 2016

Brin's nV

Fluch et al. 2013



Röver's $V\mathcal{G}$

B–Matucci 2016

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Application to the Boone–Higman Conjecture

The Boone–Higman Conjecture

Let G be a group with a countable generating set S, and let W be the set of all words for the identity.

- G is ***computably presented*** if W is computably enumerable.
- G has ***solvable word problem*** if W is computable.

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Higman's Embedding Theorem (1961)

Every computably presented group embeds into a finitely presented group.

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In particular, $\text{Aut}(F) = \mathcal{A} \rtimes \mathbb{Z}_2$ for some $\mathcal{A} \leq \text{Homeo}_+([0,1])$.

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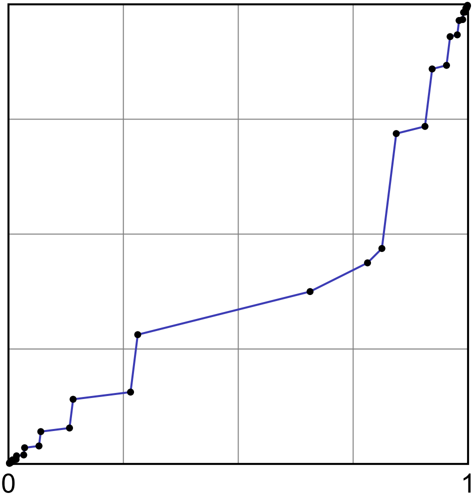

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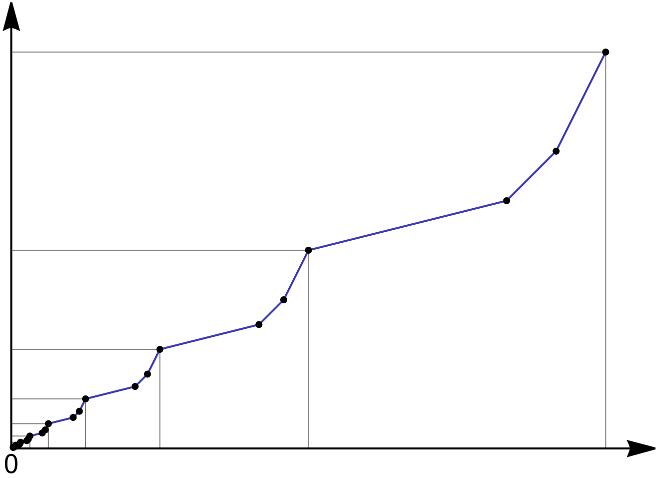

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Theorem (B–Hyde–Matucci 2022)

$\mathcal{A}$ has a subgroup isomorphic to $\mathbb{Q}$.

This was the first explicit example of a finitely presented group that contains $\mathbb{Q}$.

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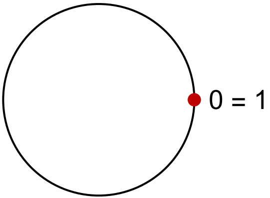

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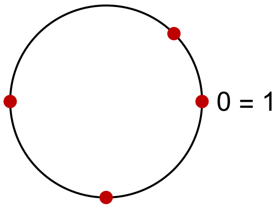

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$T\mathcal{A}$ contains $\mathcal{A}$ and hence $\mathbb{Q}$.

Some Boone–Higman Embeddings

Theorem (B–Hyde–Matucci 2023)

The group $\mathbb{Q}$ embeds into a finitely presented simple group.

Sketch of Proof:

1. Identify 0 and 1 to get an action of $\mathcal{A}$ on the circle S^1.
2. Let $T\mathcal{A} \leq \text{Homeo}(S^1)$ be the group generated by $\mathcal{A}$ and Thompson's group T.
3. Then $T\mathcal{A}$ is a finite germ extension of T. It follows from our main theorems that $T\mathcal{A}$ is simple and has type F_∞.

$T\mathcal{A}$ contains $\mathcal{A}$ and hence $\mathbb{Q}$. In fact, $T\mathcal{A}$ contains $\mathbb{Q}^\infty$, and hence contains every countable, torsion-free abelian group. □

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Every countable abelian group embeds into a finitely presented simple group.

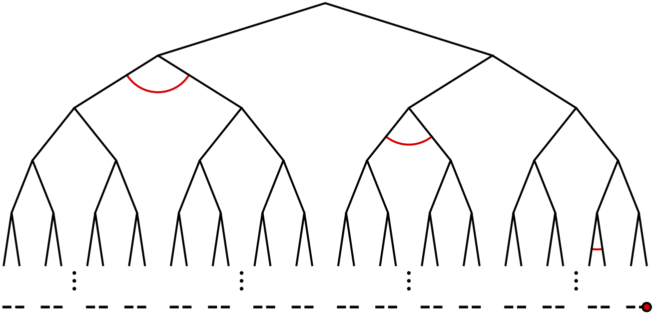
Sketch of Proof:

1. Define an action of $\mathcal{A}$ on the Cantor set.
2. Make a group $V\mathcal{A}$ which is a finite germ extension of V.
3. Then $V\mathcal{A}$ is simple and has type F_∞. It contains $\mathbb{Q}^\infty \oplus (\mathbb{Q}/\mathbb{Z})^\infty$ and hence every countable abelian group. □

Application to Röver–Nekrashevych groups

Grigorchuk's group

Grigorchuk's group $\mathcal{G}$ is a certain group of automorphisms of $\mathcal{T}_2$.



It is generated by four elements a, b, c, d, three of which have singularities at the rightmost point of $\partial\mathcal{T}_2$.

Röver's group

In 1999, Röver consider the group $V\mathcal{G}$ of homeomorphisms of a Cantor set generated by $\mathcal{G}$ and Thompson's group V.

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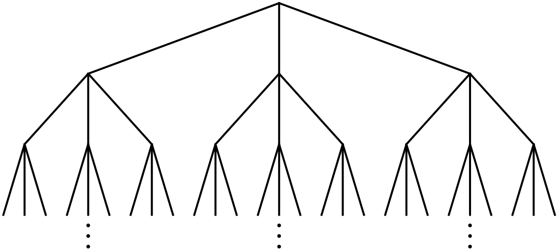
Theorem (B–Matucci 2016)

Röver's group $V\mathcal{G}$ has type F_∞.

Note: Röver's group is a finite germ extension of V. This gives an easier proof that it's simple and has type F_∞.

Röver–Nekrashevych groups

Grigorchuk's group generalizes to the class of ***self-similar groups*** $G \leq \mathrm{Aut}(\mathcal{T}_d)$.



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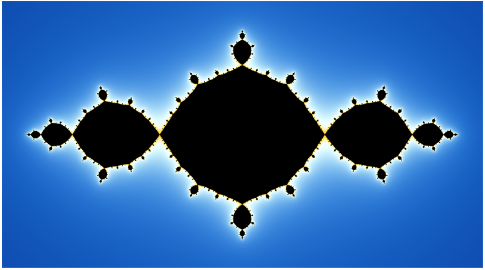

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Nekrashevych gave conditions under which $V_d G$ is simple.

Röver–Nekrashevych groups

Skipper and Zaremsky determined the finiteness properties for two infinite classes of Röver–Nekrashevych groups $V_d G$.

Theorem (Skipper–Witzel–Zaremsky 2019)

For every $n \geq 1$, there exists a simple group that has type F_n but not F_{n+1}.

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Conjecture (Nekrashevych)

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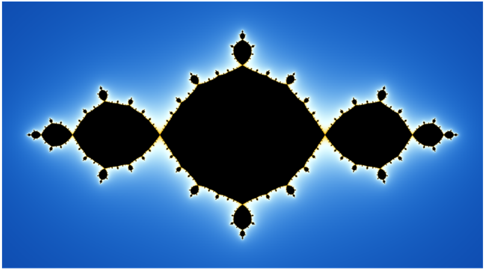
Gupta–Sidki groups, iterated monodromy groups for polynomials (e.g. the basilica group)



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Bondarenko (2007) proved that bounded automata groups are contracting.

Theorem (B–Hyde–Matucci 2023)

If G is a bounded automata group, then $V_d G$ has type F_∞.

The Complex

Main Result on Finiteness Properties

Let $G \leq \text{Homeo}(X)$ be a finite germ extension of B, and let

$$\text{sing}(G) = \{p \in X \mid p \text{ is a singular point of some } g \in G\}.$$

Theorem (B–Hyde–Matucci 2023)

Suppose that:

1. *B has finitely many orbits in $\text{sing}(G)^n$ for all $n \geq 1$.*
2. *The subgroup*

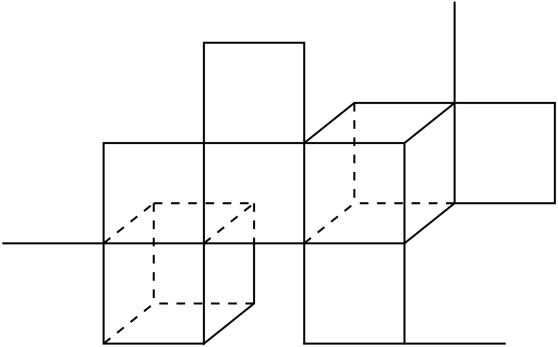
$$\{g \in G \mid g \text{ fixes } M \text{ and has singularities only on } M_0\}$$

has type F_∞ for every pair $M_0 \subseteq M$ of finite subsets of $\text{sing}(G)$.

Then G has type F_∞.

The Complex

To prove this theorem, we need a contractible complex K to which we can apply Brown's criterion and Bestvina–Brady discrete Morse theory.



Types of Singularities

Let $G \leq \text{Homeo}(X)$ be a finite germ extension of B

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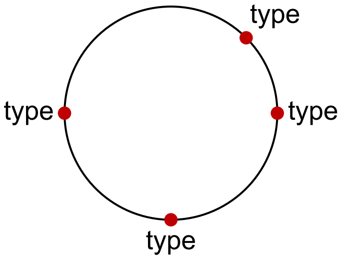
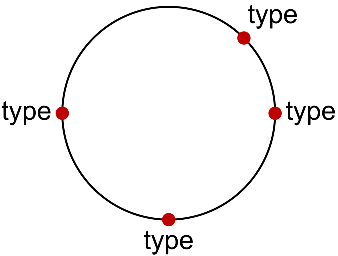
The ***portrait*** of an element $g \in G$ specifies the locations and types of the singularities.



Action on Portraits

There is a natural (right) action of G on the set of portraits.



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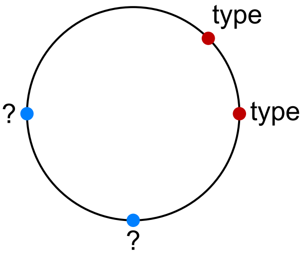
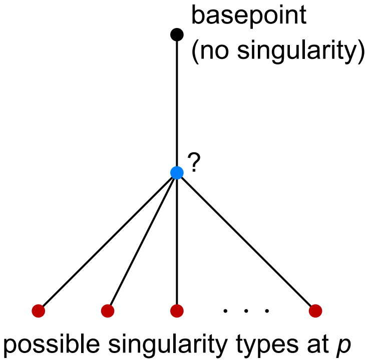


G also acts on the set of ***partial portraits***, which have finitely many "hidden points".

The Complex

For each $p \in \mathrm{sing}(G)$, let $T(p)$ be the following tree.



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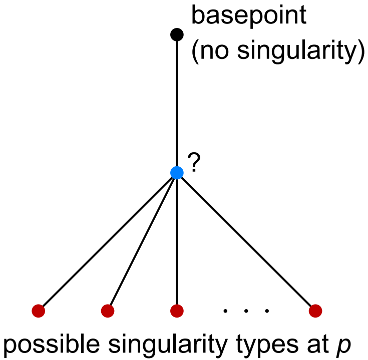


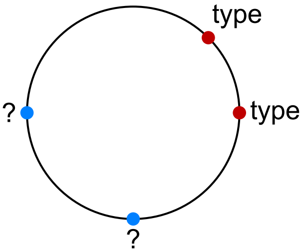
Then the infinite product $K = \prod_{p \in \text{sing}(G)} T(p)$ is a contractible cubical complex whose vertices are the partial portraits.

Using Brown's Criterion

Consider the subcomplexes

$$K_0 \subset K_1 \subset K_2 \subset \cdots$$

where the vertices of K_n are all partial portraits with $\leq n$ hidden points.



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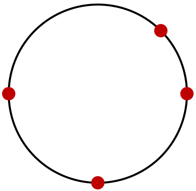
- K_0 is a disjoint union of vertices (one for each portrait).



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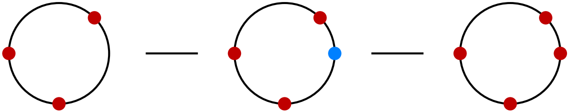

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We use Bestvina–Brady Morse theory to prove that each K_n is $(n-1)$-connected.

Main Result on Finiteness Properties

Theorem (B–Hyde–Matucci 2023)

Suppose that:

1. *B has finitely many orbits in $\mathrm{sing}(G)^n$ for all $n \geq 1$.*

2. *The subgroup*

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   Guarantees cocompactness for each K_n.

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2. *The subgroup* ← These subgroups are the cell stabilizers.

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Then G has type F_∞*.*

The End