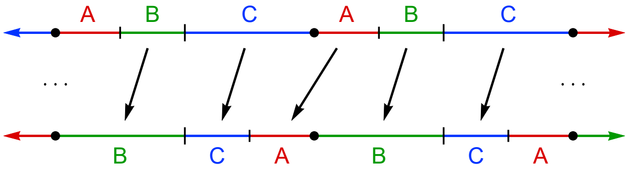
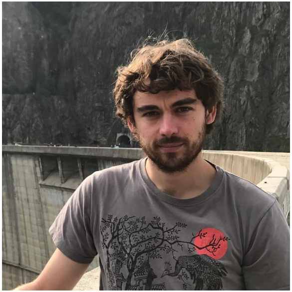
Finitely Presented Groups that Contain $\mathbb{Q}$



Jim Belk

University of St Andrews

Collaborators



James Hyde

Cornell University

Francesco Matucci

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Higman's Embedding Theorem

Consider a countable group presentation

$$\langle s_1, s_2, s_3, \dots \mid r_1, r_2, r_3, \dots \rangle.$$

Such a presentation is ***computable*** if there exists an algorithm that outputs the list of relations $r_1, r_2, r_3, \dots$ as words in the generators.

A group G is ***computably presented*** if there exists a computable presentation for G.

Examples

1. Any finitely presented group is computably presented.
2. Any finitely generated subgroup of a finitely presented group is computably presented.

Higman's Embedding Theorem (1961)

Every computably presented group can be embedded into a finitely presented group.

Graham Higman, 1960

Example: The Rational Numbers

Let $\mathbb{Q}$ be the group of rational numbers under addition.

Then $\mathbb{Q}$ is computably presented:

$$\mathbb{Q} \cong \left\langle s_1, s_2, s_3, \ldots \;\middle|\; (s_n)^n = s_{n-1} \text{ for all } n \geq 2 \right\rangle.$$

Corollary

The group $\mathbb{Q}$ can be embedded into a finitely presented group.

Higman's proof is only partially constructive, but can be carried out explicitly for $\mathbb{Q}$ (Mikaelian 2020). The resulting presentation is very large.

Other Abelian Groups

The situation for other countable abelian groups is similar. It is known that any countable abelian group embeds into

$$\bigoplus_{k=1}^{\infty} \mathbb{Q} \;\oplus\; \bigoplus_{k=1}^{\infty} \mathbb{Q}/\mathbb{Z}.$$

But this group is computably presented, so by Higman's theorem it embeds into some finitely presented group.

Corollary

There exists a finitely presented group G that contains all countable abelian groups.

The Problem

In 1999, Bridson and de la Harpe submitted the following "well-known problem" to the Kourovka Notebook.

Problem (Kourovka Notebook 14.10)

(a) *Find an explicit and "natural" finitely presented group G and an embedding of the additive group of the rationals $\mathbb{Q}$ in G.*

There is an analogous question for a group G_n and an embedding of $\mathrm{GL}_n(\mathbb{Q})$ in G_n.

(b) *Find an explicit embedding of $\mathbb{Q}$ in a finitely generated group.*

Mikaelian’s Examples

In 2005, Mikaelian gave two examples of finitely generated groups that contain $\mathbb{Q}$, solving part (b) of the problem.

Vahagn Mikaelian

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Theorem (Mikaelian 2005)

The wreath product $(\mathbb{Q} \wr \mathbb{Z}) \wr \mathbb{Z}$ has a two-generated subgroup which contains $\mathbb{Q}$.

Theorem (Mikaelian 2005)

The free product $\mathbb{Q} * F_2$ has an HNN extension which can be generated by two elements.

Neither of Mikaelian's examples are finitely presented.

Main Results

We give five explicit examples of finitely presented groups that contain $\mathbb{Q}$.

All of them are related to the ***Thompson groups*** F, T, and V.

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All of them are related to the ***Thompson groups*** F, T, and V.

Theorem (BHM 2020)

The following well-known finitely presented groups contain $\mathbb{Q}$:

1. *The lift $\overline{T}$ of Thompson's group T to the real line.*
2. *The braided Thompson group BV.*
3. *The automorphism group $\mathrm{Aut}(F)$ of Thompson's group F.*

Main Results: Two More Examples

We also give two **simple** examples $T\mathcal{A}$ and $V\mathcal{A}$.

These act on the circle and the Cantor set, respectively.

Theorem (BHM 2020)

The groups $T\mathcal{A}$ and $V\mathcal{A}$ are two-generated, finitely presented simple groups. Moreover:

1. *$T\mathcal{A}$ contains $\mathbb{Q}$, and*
2. *$V\mathcal{A}$ contains every countable abelian group.*

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We actually prove something stronger than finite presentability:

Theorem (BHM 2020)

Both $T\mathcal{A}$ and $V\mathcal{A}$ have type F_∞.

Thompson’s Groups

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In the 1960's, Richard J. Thompson defined three infinite groups.

Richard Thompson, 2004

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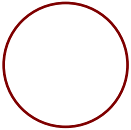
F acts on the interval.



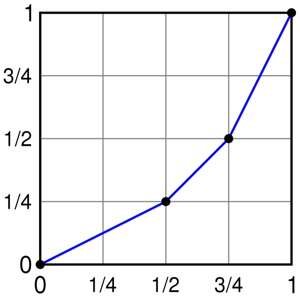
T acts on the circle.

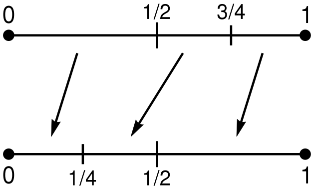
V acts on the Cantor set.

Thompson's Group F

Thompson's group F is the group of all piecewise-linear homeomorphisms of $[0, 1]$ for which:

- Each segment has slope 2^n ($n \in \mathbb{Z}$), and
- Each breakpoint has dyadic rational coordinates.





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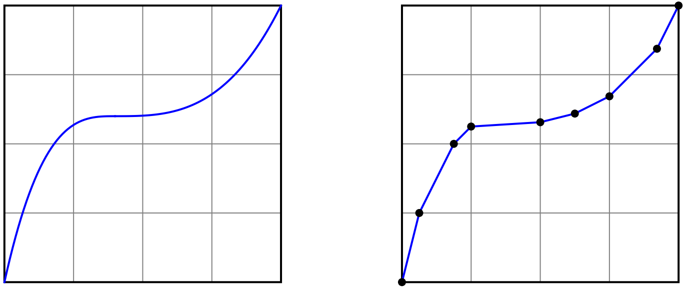

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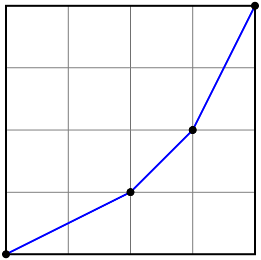
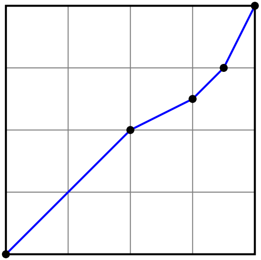


x_0



x_1

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- F is generated by two elements.
- F is finitely presented.

$$F = \langle x_0, x_1 \mid x_2^{x_1} = x_3,\ x_3^{x_1} = x_4 \rangle$$

where

$$x_2 = x_1^{x_0}, \qquad x_3 = x_2^{x_0}, \qquad x_4 = x_3^{x_0}.$$

Type F_∞

In 1983, Brown and Geoghegan generalized this result by proving that F has ***type*** $\mathbf{F}_\infty$.

Kenneth Brown

Ross Geoghegan

Type F_∞

Let G be a group.

A ***$K(G,1)$-complex*** is a connected CW complex X such that:

1. $\pi_1(X) \cong G$, and
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$$\begin{aligned} \text{type } F_1 &= \text{finitely generated} \\ \text{type } F_2 &= \text{finitely presented} \\ &\vdots \end{aligned}$$

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We say that G has ***type F_∞*** if it has type F_n for all n.

Type F_∞

Brown and Geoghegan proved that Thompson's group F has a $K(F,1)$-complex with exactly two cells in each dimension.

Kenneth Brown

Ross Geoghegan

Thus F has type F_∞.

Thompson's Group T

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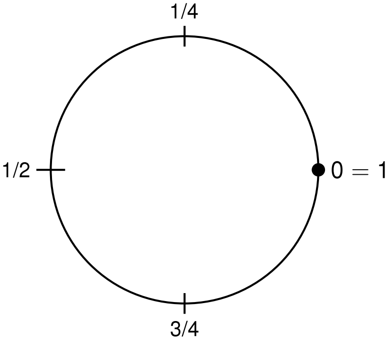
Let S^1 be the circle obtained from $[0, 1]$ by identifying 0 and 1.



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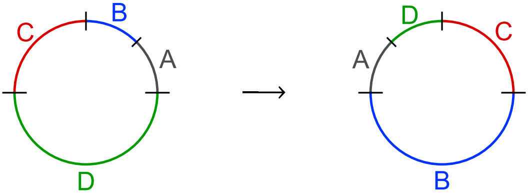
- $f(0)$ is dyadic.



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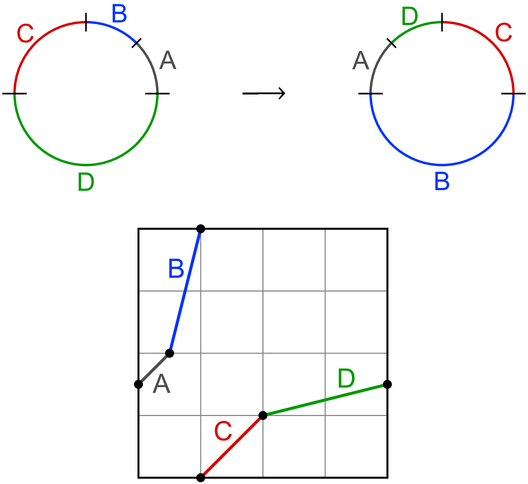

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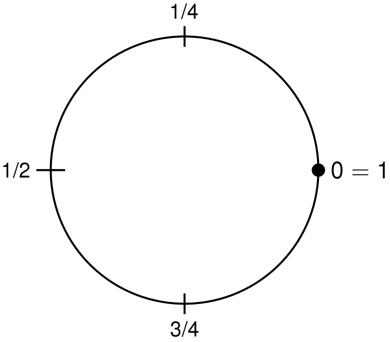

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Lochak–Schneps presentation (1997):

$$\left\langle a, b \,\middle|\, a^4, b^3, \left[bab, a^2baba^2\right], \left[bab, a^2b^2a^2baba^2ba^2\right], (ba)^5 \right\rangle$$

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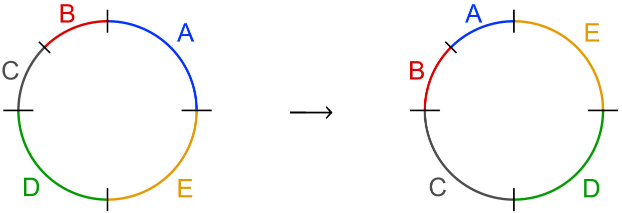

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- Indeed, T contains $\mathbb{Q}/\mathbb{Z}$ (Bleak, Kassabov, Matucci 2011).

The Group $\overline{T}$

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A ***lift*** of an element $g \in T$ is a homeomorphism $\overline{g} : \mathbb{R} \to \mathbb{R}$ that makes the following diagram commute:

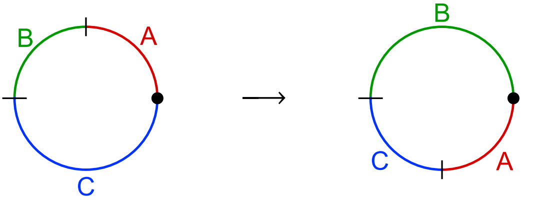
$$\begin{tikzcd} \mathbb{R} \arrow[r, "\overline{g}"] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "g"'] & S^1 \end{tikzcd}$$

Note: If $\overline{g}$ is a lift of g then so is $\overline{g} + n$ for any $n \in \mathbb{Z}$.

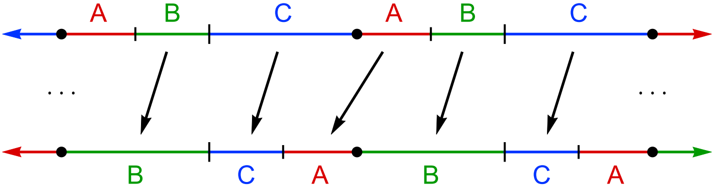
Let $\overline{T}$ be the group of all lifts of elements of T.

The Group $\overline{T}$

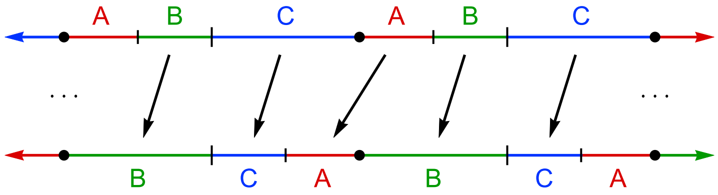
For example, here's an element of T:



and here's one possible lift in $\overline{T}$:



The Group $\overline{T}$



A PL homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ lies in $\overline{T}$ if and only if:

- Each segment of f has slope 2^n ($n \in \mathbb{Z}$),
- Each breakpoint of f has dyadic rational coordinates,
- $f(0)$ is dyadic, and
- $f(t+1) = f(t) + 1$ for all $t \in \mathbb{R}$.

Properties of $\overline{T}$

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- $\overline{T}$ is generated by x_0, x_1, and $s_2(t) = t + \frac{1}{2}$.
- The center of $\overline{T}$ is infinite cyclic, generated by the translation

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Indeed, we have a short exact sequence

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- $\overline{T}$ is finitely presented. Indeed, it has type F_∞.

Presentation for $\overline{T}$

Theorem (BHM 2020)

The group $\overline{T}$ has a presentation with two generators and four relators:

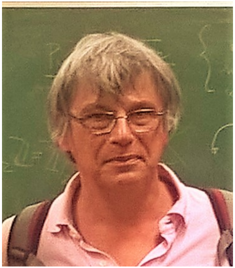
$$\left\langle a, b \,\middle|\, a^4b^{-3}, (ba)^5b^{-9}, \left[bab, a^2baba^2\right], \left[bab, a^2b^2a^2baba^2ba^2\right] \right\rangle$$

Note: This is obtained by "lifting" the Lochak–Schneps presentation for T and simplifying. Adding $a^4 = 1$ gives a presentation for T.

Properties of $\overline{T}$

The group $\overline{T}$ was first considered by Ghys and Sergiescu in 1987, as part of their work on the cohomology of T.

Étienne Ghys



Vlad Sergiescu

They proved that $\overline{T}$ is perfect and is a central extension of T (but not the universal central extension).

$\overline{T}$ Contains $\mathbb{Q}$

Theorem (BHM 2020)

The group $\overline{T}$ has uncountably many subgroups isomorphic to $\mathbb{Q}$.

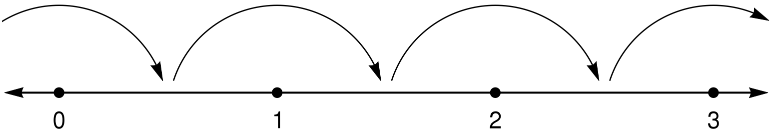
Note: Every such subgroup contains the center of $\overline{T}$.

Our strategy will be to find subgroups that realize the presentation

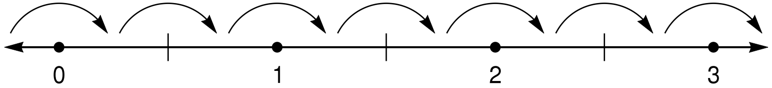
$$\left\langle s_1, s_2, s_3, \ldots \;\middle|\; s_n^n = s_{n-1} \text{ for all } n \geq 2 \right\rangle.$$

$\overline{T}$ Contains $\mathbb{Q}$

Let $s_1 \in \overline{T}$ be the map $s_1(t) = t + 1$.



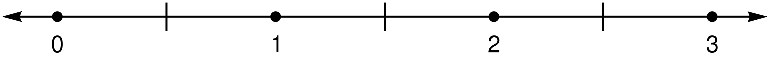
Then s_1 has a square root $s_2(t) = t + \frac{1}{2}$.



Does s_2 have a cube root?

$\overline{T}$ Contains $\mathbb{Q}$

Yes. We just need to cut each half-interval into three pieces of sizes 1/8, 1/8, and 1/4:



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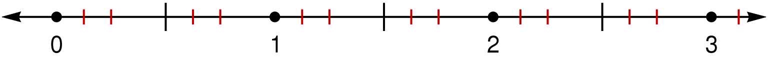
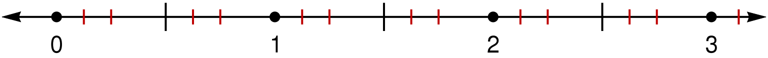


$\overline{T}$ Contains $\mathbb{Q}$

Yes. We just need to cut each half-interval into three pieces of sizes $1/8$, $1/8$, and $1/4$:



Each piece maps linearly to the next under s_3.

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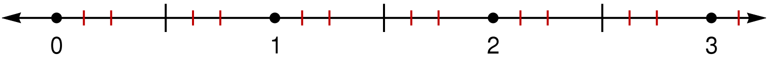
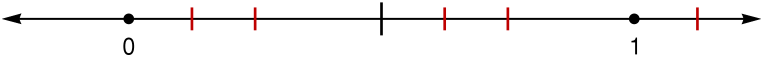


Each piece maps linearly to the next under s_3.

To get a fourth root of s_3, we cut each interval into four pieces.



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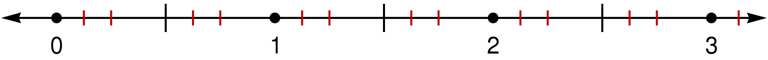

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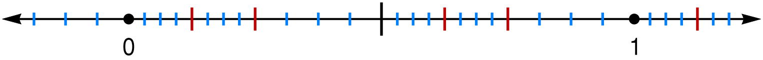

$\overline{T}$ Contains $\mathbb{Q}$

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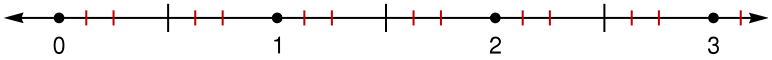

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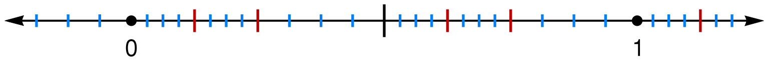
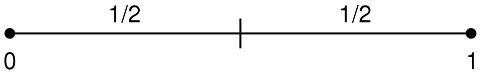


Again, each piece maps linearly to the next under s_4.

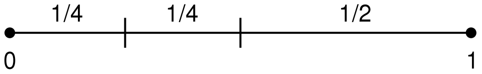
$\overline{T}$ Contains $\mathbb{Q}$

In general, for each $n \in \mathbb{N}$ we need a ***cut pattern*** that cuts $[0, 1]$ into n intervals whose widths are powers of $1/2$.

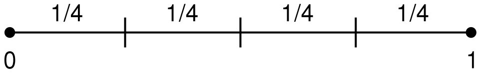
$n = 2$:



$n = 3$:



$n = 4$:



By iteratively cutting subintervals using the cut patterns, we can construct the desired sequence $\{s_n\}$ in $\overline{T}$.

The group BV

The group *BV*

The ***braided Thompson group BV*** was introduced independently by Brin and Dehornoy in 2004.

Matthew Brin

Patrick Dehornoy

The group BV

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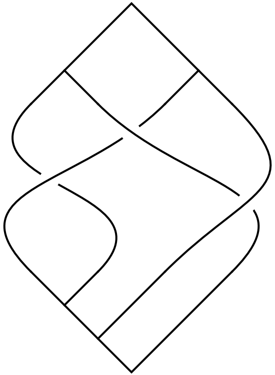
Elements of BV are "braided tree pair diagrams".



The group BV

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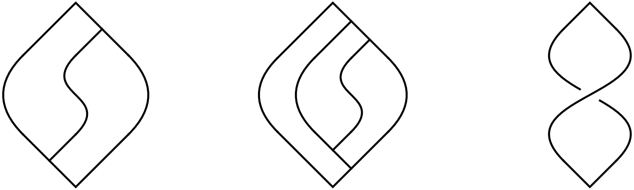
The following three elements generate a copy of $\overline{T}$ in BV.



x_0 x_1 s_2

Thus BV contains $\mathbb{Q}$.

Automorphisms of F

Automorphisms of F

The structure of $\mathrm{Aut}(F)$ was pinned down by Brin in 1996.
His methods are based on a theorem of Rubin.

Matthew Brin

Matatyahu Rubin

Rubin's Theorem

Given a space X and a subgroup $G \leq \mathrm{Homeo}(X)$, consider the normalizer

$$N(G) = \{n \in \mathrm{Homeo}(X) \mid n^{-1}Gn = G\}.$$

Each element $n \in N(G)$ induces an automorphism of G defined by

$$g \mapsto n^{-1}gn.$$

Theorem (Rubin 1996)

Suppose X is locally compact, Hausdorff, and has no isolated points. If G is "locally moving", then

$$\mathrm{Aut}(G) \cong N(G).$$

Structure of Aut(F)

Brin proved that the action of F on $(0,1)$ is locally moving. Thus:

Corollary

Aut(F) *is the normalizer of* F *in* Homeo($[0,1]$).

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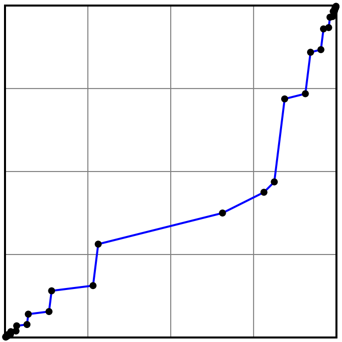
It follows easily that

$$\mathrm{Aut}(F) = \mathcal{A} \rtimes \mathbb{Z}_2$$

where $\mathcal{A}$ is the orientation-preserving subgroup of Aut(F).

Elements of $\mathcal{A}$

Brin showed that elements of $\mathcal{A}$ are piecewise-linear on $(0, 1)$, but breakpoints can accumulate near 0 and 1.



Elements of $\mathcal{A}$

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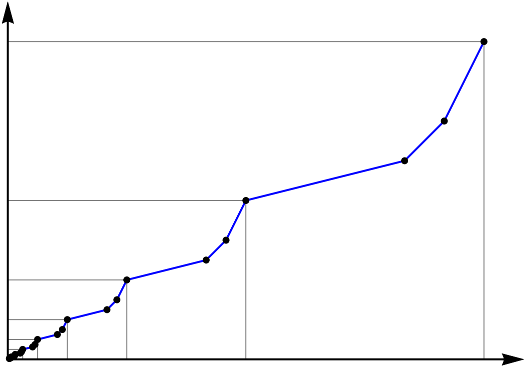
Theorem (Brin 1996)

A homeomorphism $f : [0, 1] \to [0, 1]$ lies in $\mathcal{A}$ if and only if it satisfies the following conditions:

1. *f is piecewise-linear, except perhaps at 0 and 1.*
2. *Each linear segment of f has slope 2^n ($n \in \mathbb{Z}$).*
3. *Each breakpoint of f has dyadic rational coordinates.*
4. *$f(2t) = 2f(t)$ for all t in a neighborhood of 0.*
5. *$f(2t - 1) = 2f(t) - 1$ for all t in a neighborhood of 1.*

Elements of $\mathcal{A}$

The condition that $f(2t) = 2f(t)$ for t near 0 means that the graph of f is self-similar near $(0,0)$.



Structure of $\mathcal{A}$

Brin constructed two homomorphisms

$$\varphi_0 : \mathcal{A} \to T, \qquad \varphi_1 : \mathcal{A} \to T$$

that describe the “bad part” of an element of $\mathcal{A}$ near 0 and 1.

Theorem (Brin 1996)

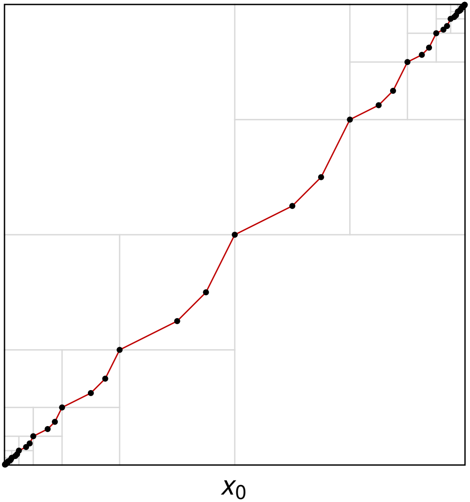
The group $\mathcal{A}$ fits into a short exact sequence

$$1 \longrightarrow F \longrightarrow \mathcal{A} \longrightarrow T \times T \longrightarrow 1$$

It follows that $\mathcal{A}$ is finitely presented, and indeed has type F_∞.

Embedding $\overline{T}$ into $\mathcal{A}$

It is not difficult to embed $\overline{T}$ into $\mathcal{A}$.



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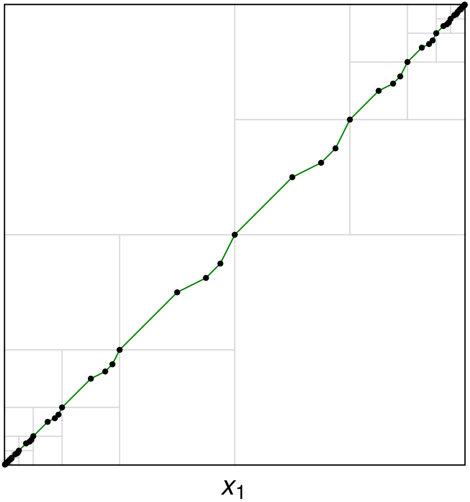

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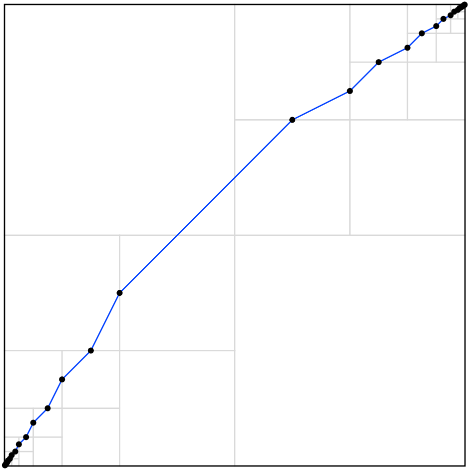

s_2

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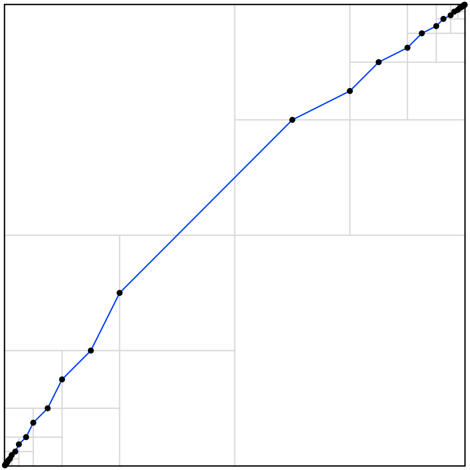
It is not difficult to embed $\bar{T}$ into $\mathcal{A}$. It follows that $\mathcal{A}$ contains $\mathbb{Q}$.

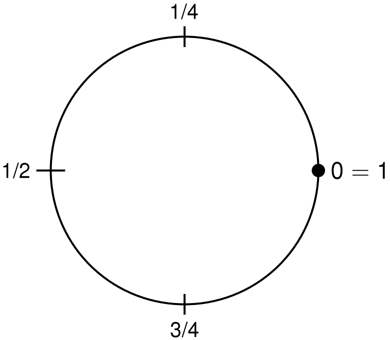


s_2

The Group $T\mathcal{A}$

The Group $T\mathcal{A}$

Identifying 0 and 1 gives an action of $\mathcal{A}$ on the circle.



Elements of $\mathcal{A}$ are piecewise-linear on the complement of the point $0 = 1$.

The Group $T\mathcal{A}$

Let $T\mathcal{A}$ be the group of circle homeomorphisms generated by T and $\mathcal{A}$.

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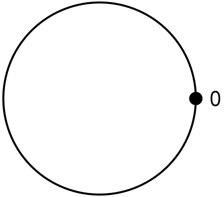
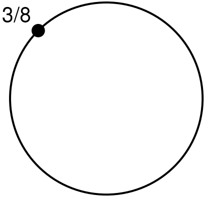
Conjugating an element $a \in \mathcal{A}$ by an element $t \in T$ moves the accumulation point.



a



$t^{-1}at$

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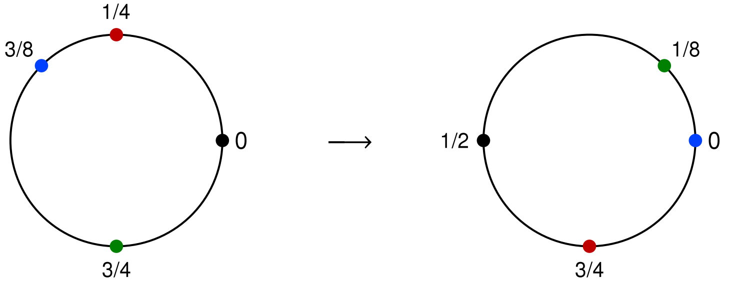
General elements of $T\mathcal{A}$ are piecewise-linear except at finitely many dyadic accumulation points.



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$T\mathcal{A}$ is simple and finitely presented. Indeed, it has type F_∞.

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Note: $T\mathcal{A}$ contains $\bigoplus_{n\in\mathbb{N}} \mathcal{A}$ and hence $\bigoplus_{n\in\mathbb{N}} \mathbb{Q}$.

Thus $T\mathcal{A}$ contains every countable, torsion-free abelian group.

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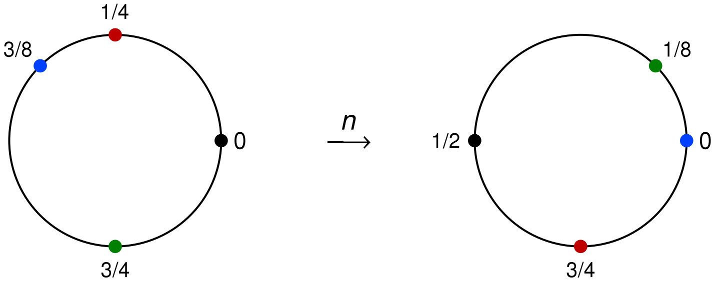

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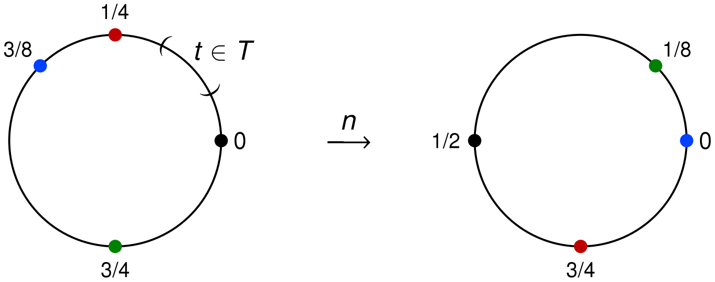

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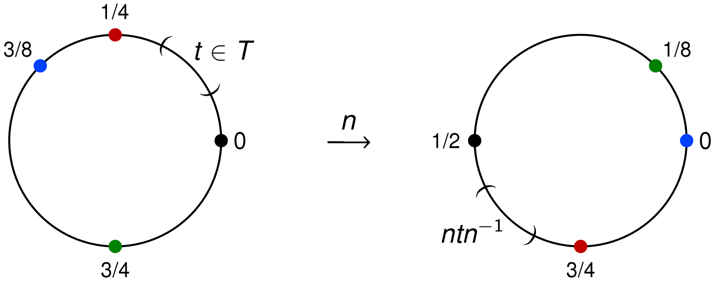

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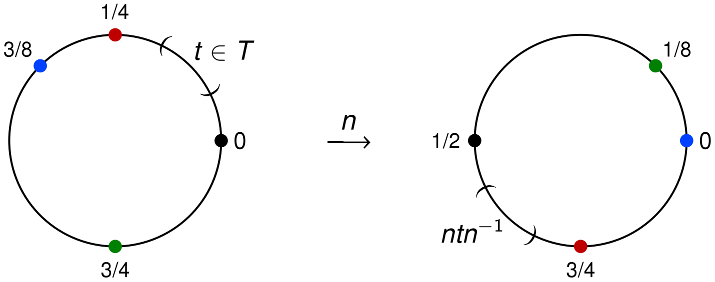

Then $ntn^{-1}t^{-1} \in N \cap T$

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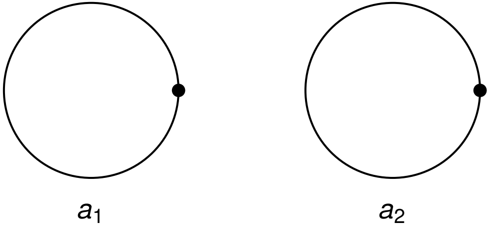

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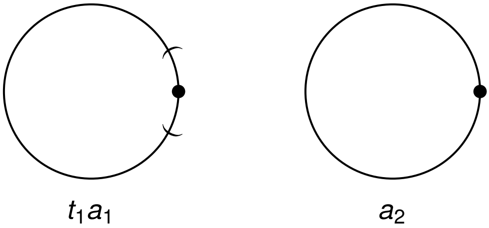

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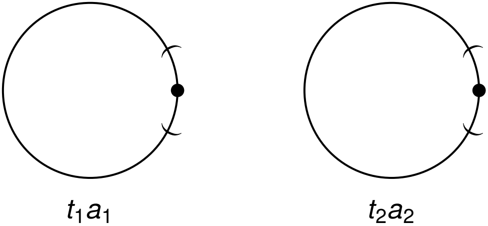

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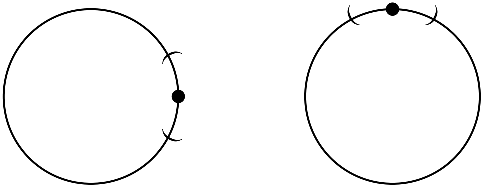

$t_1 a_1$ $\qquad$ $t_3^{-1} t_2 a_2 t_3$

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Step 4: But $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$, so N contains $\mathcal{A}$.

Thus $N = T\mathcal{A}$, so $T\mathcal{A}$ is simple.

Proof that $T\mathcal{A}$ has type F_∞

We use ***Brown's criterion*** (Brown 1987).

Step 1: Construct a contractible simplicial complex K on which $T\mathcal{A}$ acts by isometries, with simplex stabilizers having type F_∞.

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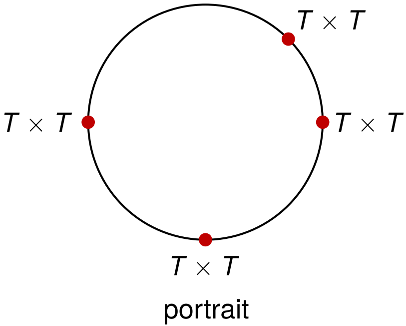

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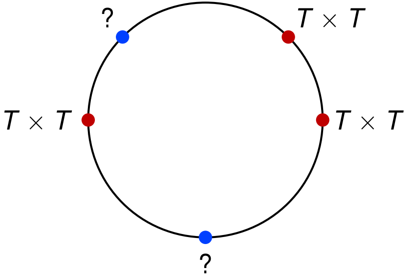

partial portrait

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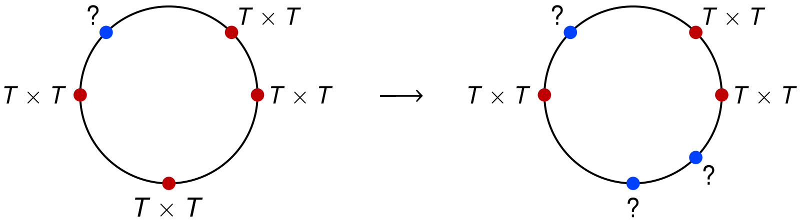

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Step 2: Filter K as a union of invariant subcomplexes

$$K_1 \leq K_2 \leq K_3 \leq \cdots .$$

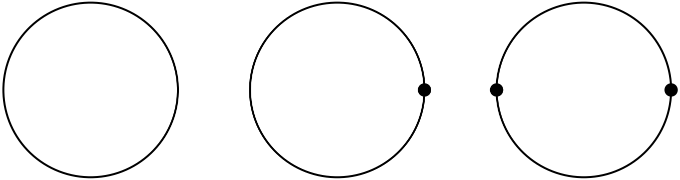
Each K_n has finitely many orbits of simplices.

Step 3: Use Bestvina–Brady discrete Morse theory to prove that the connectivity of K_n goes to ∞ as $n \to \infty$.

Presentation for $T\mathcal{A}$

Theorem (BHM 2020)

$T\mathcal{A}$ is the amalgam of three of its finitely presented subgroups:



T $\qquad$ $\mathcal{A}$ $\qquad$ $H = \mathcal{A} \wr \mathbb{Z}_2$

Note that the intersections

$$T \cap \mathcal{A} \cong F, \qquad T \cap H \cong F \wr \mathbb{Z}_2, \qquad \mathcal{A} \cap H \cong \mathrm{Stab}_{\mathcal{A}}(1/2)$$

are finitely generated.

Other Groups

The Group $V\mathcal{A}$

The action of $\mathcal{A}$ on $[0, 1]$ induces an action of $\mathcal{A}$ on the Cantor set.

Let $V\mathcal{A}$ be the group generated by $\mathcal{A}$ and Thompson's group V.

The Group $V\mathcal{A}$

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Note: $V\mathcal{A}$ contains $\bigoplus_{n\in\mathbb{N}} V\mathcal{A}$ and hence $\bigoplus_{n\in\mathbb{N}} (\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$.

It follows that $V\mathcal{A}$ contains every countable abelian group.

Nekrashevych groups

Our simplicity and F_∞ proofs apply to a large class of groups G that satisfy the following conditions:

1. G contains a generalized Thompson group F_n, T_n, or V_n.
2. Every element of G has finitely many "unusual" points.

For example, ***Röver's group*** is the group $V\mathcal{G}$ generated by Thompson's group V and Grigorchuk's group $\mathcal{G}$. Our methods give a new proof that $V\mathcal{G}$ is simple (Röver 1999) and has type F_∞ (BM 2014).

Indeed, we can prove simplicity and finiteness results for a large class of Nekrashevych groups.

The End