Thompson-Like Groups Acting on Julia Sets

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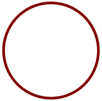
Groups St Andrews, August 2013

Thompson's Groups

In the 1960's, Richard J. Thompson defined three infinite groups:

F acts on the interval.

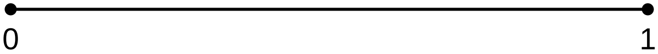


T acts on the circle.

V acts on the Cantor set.

Definition of F

A ***dyadic subdivision*** of $[0, 1]$ is any subdivision obtained by repeatedly cutting intervals in half:



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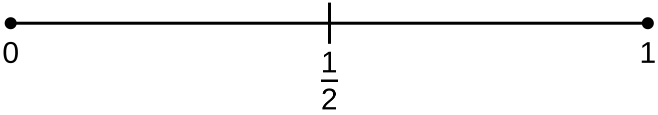

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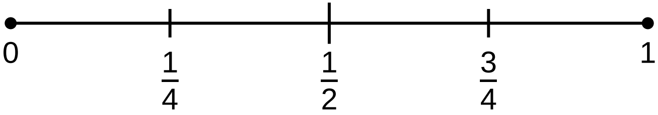

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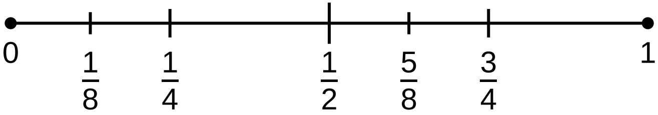

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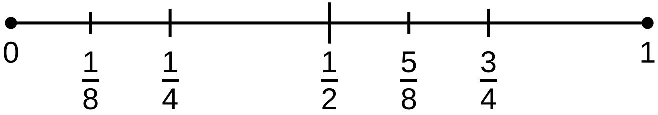
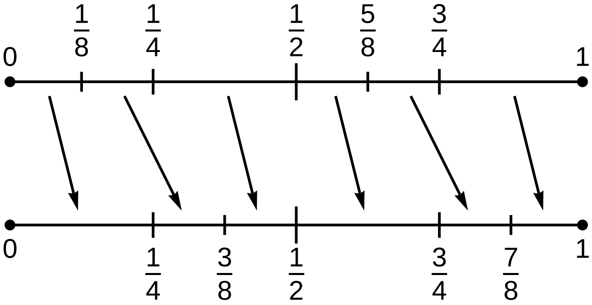


The partition points are always dyadic fractions.

Definition of F

A ***dyadic rearrangement*** of $[0,1]$ is a PL homeomorphism that maps linearly between the intervals of two dyadic subdivisions:



The set of all dyadic rearrangements of $[0,1]$ is ***Thompson's group F***.

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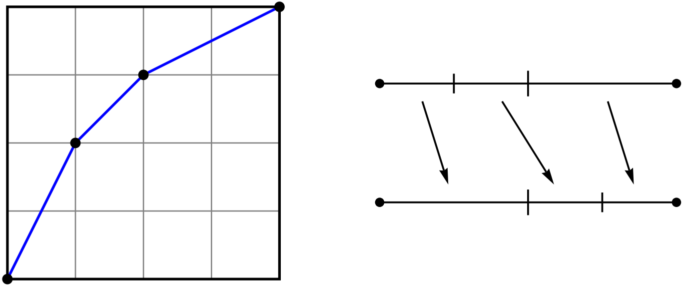
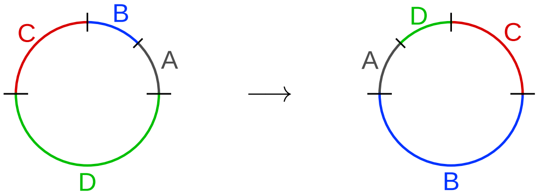
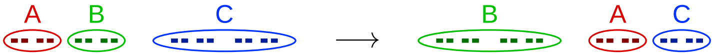


Definitions of T and V

Thompson's Group T acts on a circle.



Thompson's Group V acts on a Cantor set.



Properties of the Thompson Groups

- T and V are **infinite, finitely presented simple groups**.
- F is finitely presented but not simple.
- Finiteness properties: All three have **type $\boldsymbol{F_\infty}$**.
(Brown & Geoghegan, 1984)
- Geometry: All three act properly and isometrically on CAT(0) cubical complexes. (Farley, 2003)

Generalizations

Basic Question:

Why are there *three* Thompson groups?

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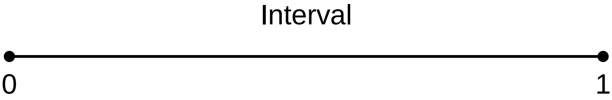
Generalizations:

- $F(n)$, $T(n)$, and $V(n)$ (Higman 1974, Brown 1987)
- Other PL groups (Bieri & Strebel 1985, Stein 1992)
- Diagram Groups (Guba & Sapir 1997)
- "Braided" V (Brin 2004, Dehornoy 2006)
- $2V, 3V, \ldots$ (Brin 2004)

Self-Similarity

Self-Similarity

F depends on the ***self-similarity*** of the interval:



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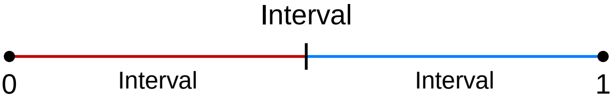

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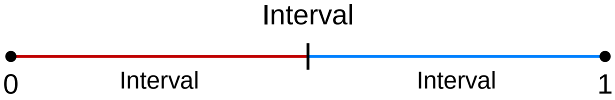
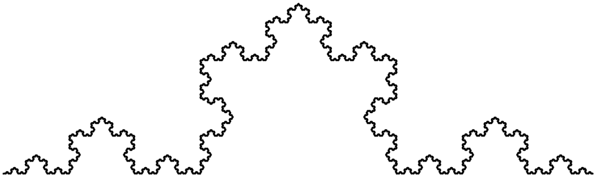


Some fractals have this same self-similar structure:



Koch Curve

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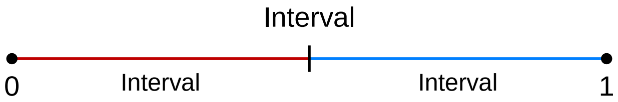

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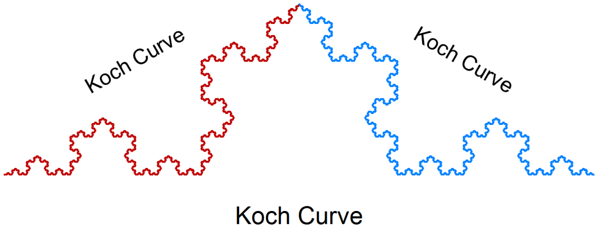
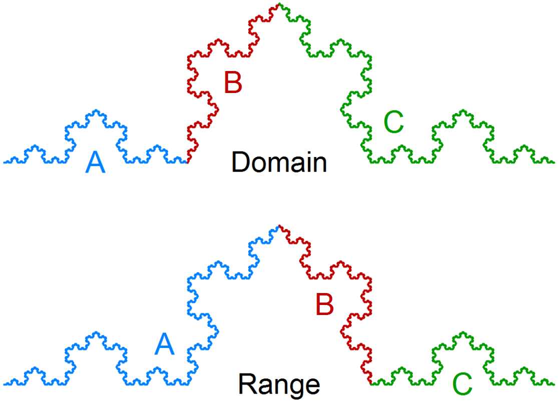


Self-Similarity

Thompson's group F acts on such a fractal by piecewise similarities.



Self-Similarity

Thompson's group F acts on such a fractal by piecewise similarities.

Idea: Find Thompson-like groups associated to other self-similar structures.

Self-Similarity

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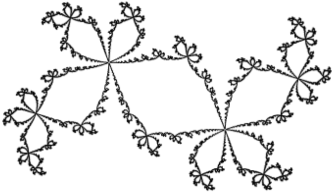
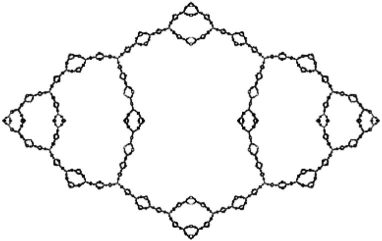
Idea: Find Thompson-like groups associated to other self-similar structures.

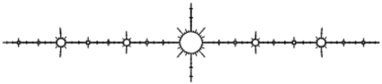
But where can we find other self-similar structures?

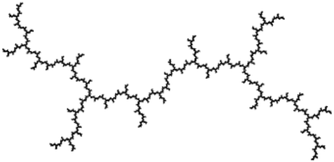
Julia Sets

Every rational function on the Riemann sphere has an associated ***Julia set***.



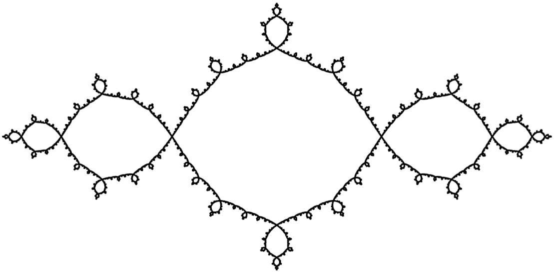






Julia Sets: The Basilica

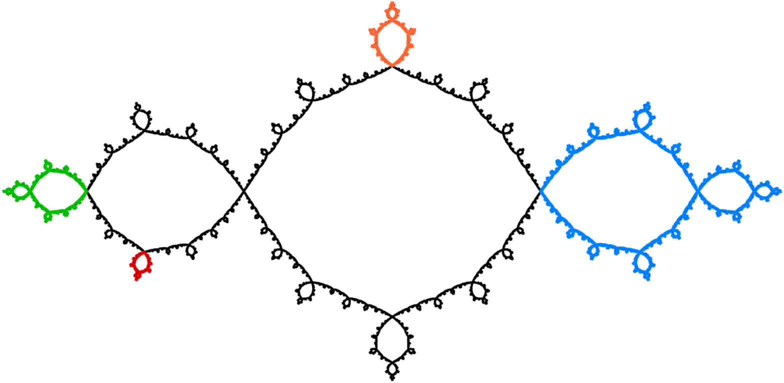
Example: The Julia set for $f(z) = z^2 - 1$ is called the ***Basilica***.



It is the simplest example of a fractal Julia set.

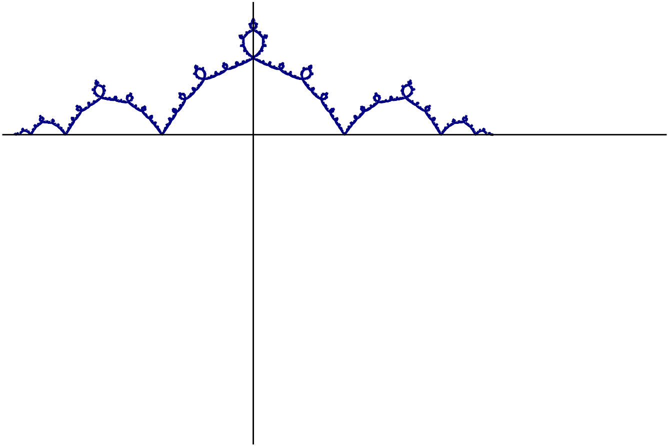
Julia Sets: The Basilica

The Basilica has a “self-similar” structure.



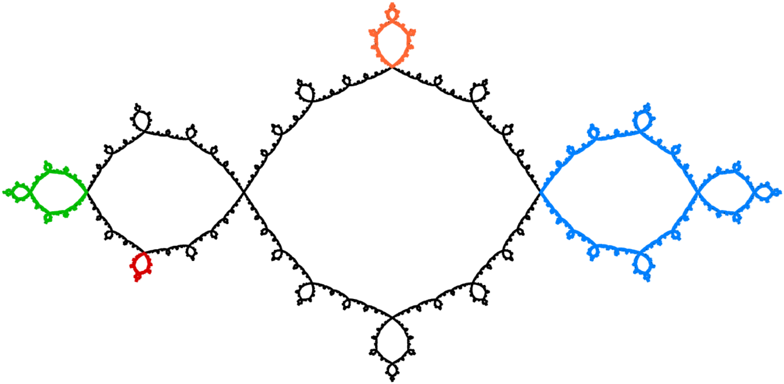
Invariance of the Basilica under $z^2 - 1$

The Basilica maps to itself under $f(z) = z^2 - 1$.



Julia Sets: The Basilica

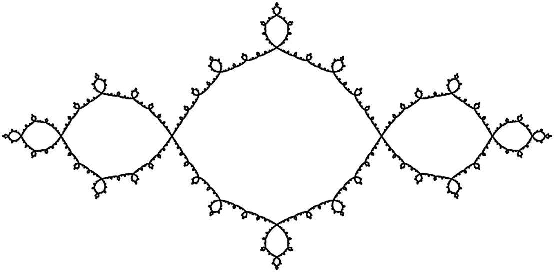
The Basilica has a ***conformally self-similar*** structure.



The Basilica Group

The Plan

Let's try to construct a Thompson-like group that acts on the Basilica.



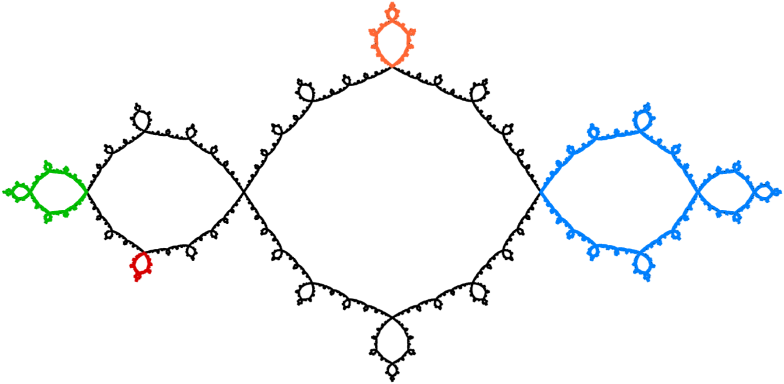
The Plan

Let's try to construct a Thompson-like group that acts on the Basilica.

Interval	**Basilica**
linear map	conformal map
dyadic subdivision	???

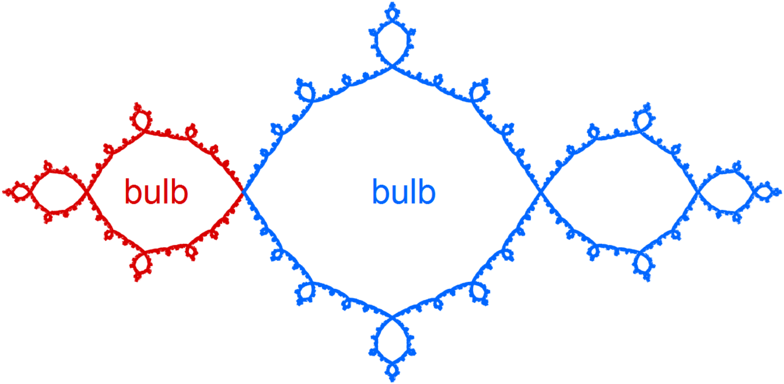
Structure of the Basilica

Terminology: Each of the highlighted sets below is a ***bulb***.



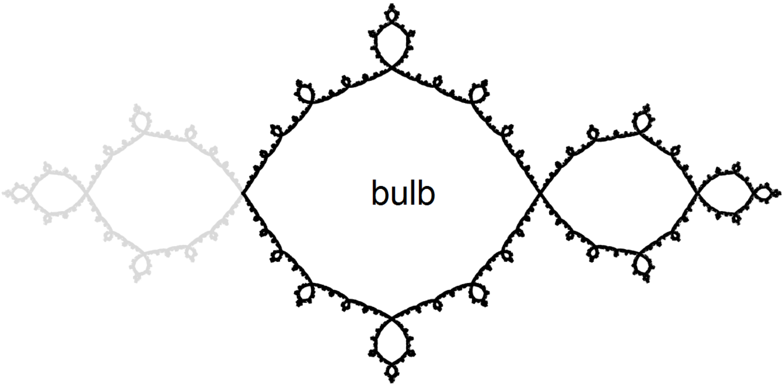
Structure of the Basilica

The Basilica is the union of two bulbs.



Structure of the Basilica

Each bulb has three parts.



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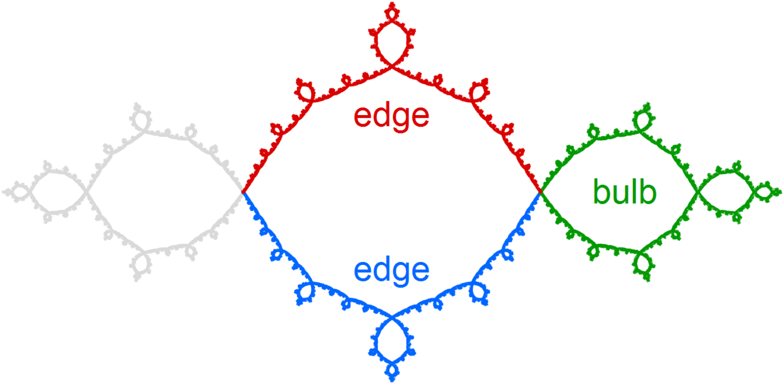
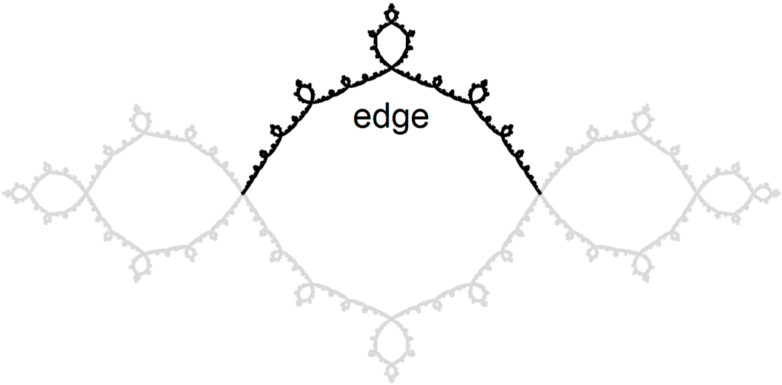


Structure of the Basilica

Each edge also has three parts.



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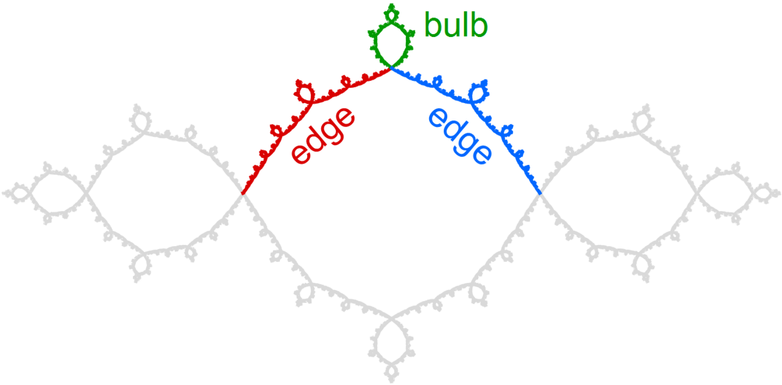
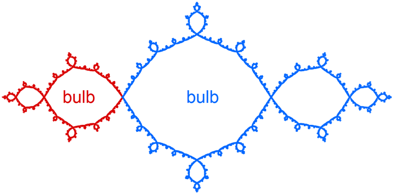
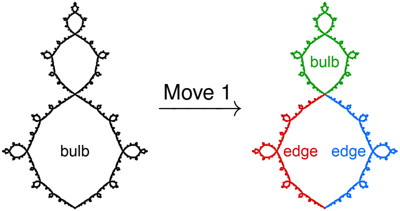


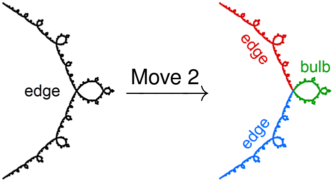
Allowed Subdivisions of the Basilica



Base

Allowed subdivision:
Start with the base and repeatedly apply the two subdivision moves.



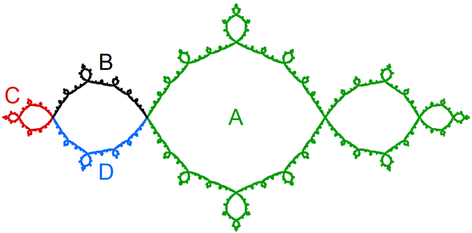


Rearrangements of the Basilica

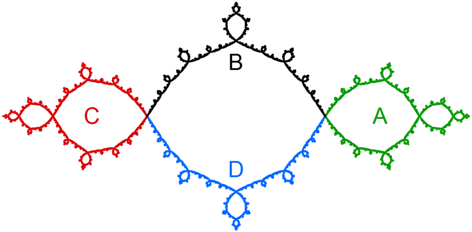
A ***rearrangement*** is a homeomorphism that maps conformally between the pieces of two allowed subdivisions.

Example 1

Domain:



Range:



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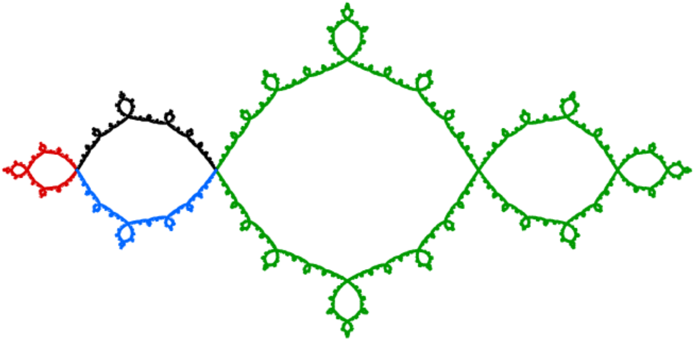
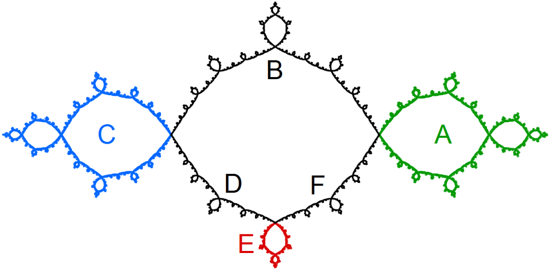


Example 2

Domain:



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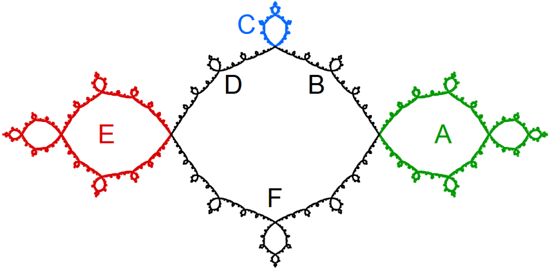

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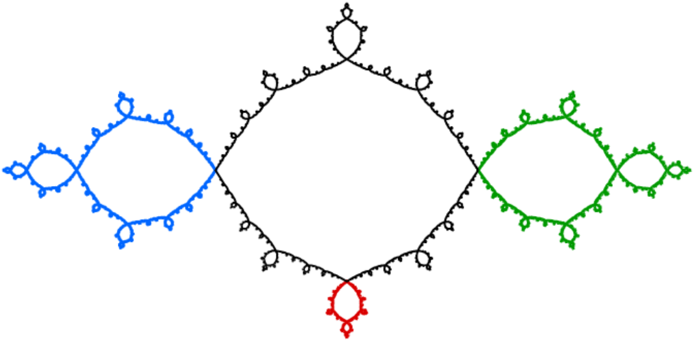
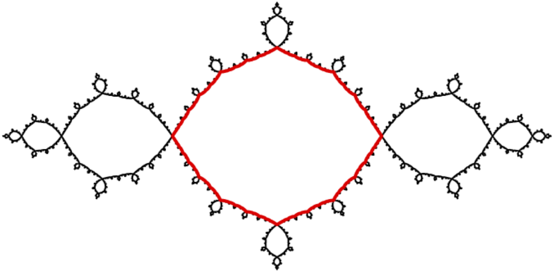


The Group T_B

Let T_B be the group of all rearrangements of the Basilica.

Theorem

1. *T_B contains isomorphic copies of Thompson's group T.*



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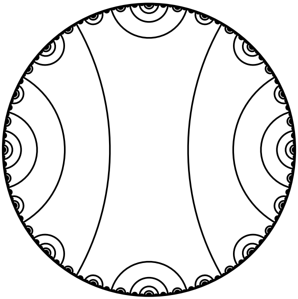

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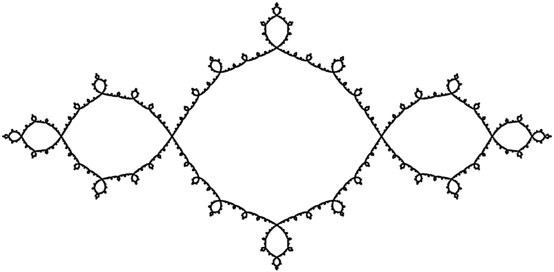
1. *T_B contains isomorphic copies of Thompson's group T.*
2. *T contains an isomorphic copy of T_B.*
3. *T_B is generated by four elements.*



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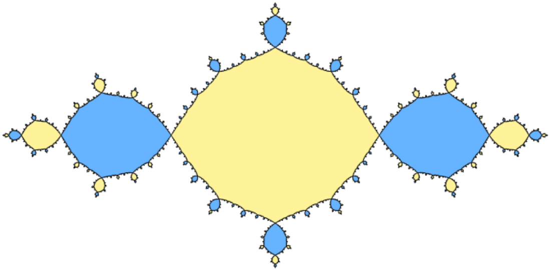
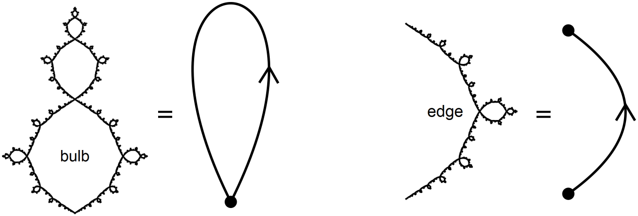
1. *T_B contains isomorphic copies of Thompson's group T.*
2. *T contains an isomorphic copy of T_B.*
3. *T_B is generated by four elements.*
4. *T_B has a simple subgroup of index two.*



Aside: Graphs and Diagram Groups

Everything here is combinatorial.

An allowed subdivision can be represented by a directed graph:



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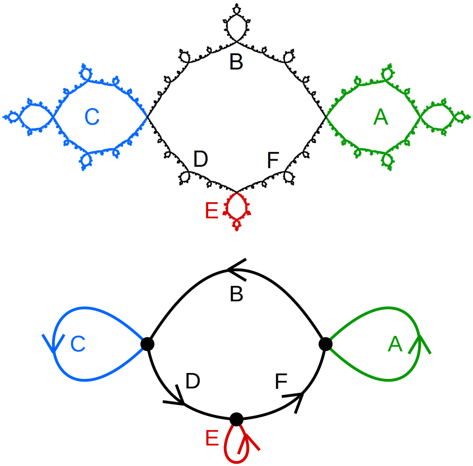
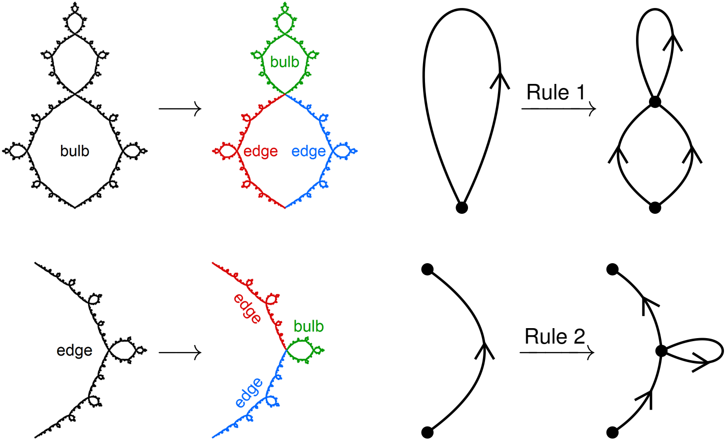


Aside: Graphs and Diagram Groups

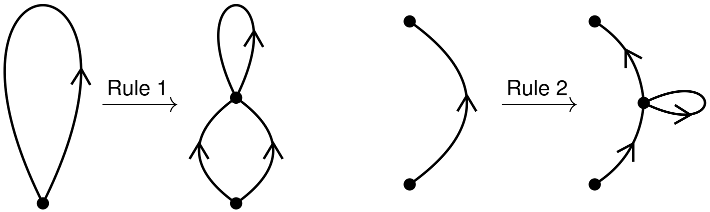
There are two ***replacement rules*** for these graphs:



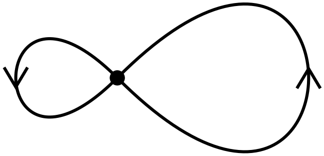
These constitute a ***graph rewriting system***.

Aside: Graphs and Diagram Groups

All we really need to define T_B are the graph rewriting system:



and the base graph:



Base Graph

Aside: Graphs and Diagram Groups

Victor Guba and Mark Sapir defined ***diagram groups***:

- Generalization of Thompson's groups
- Uses string rewriting systems.

T_B is similar to a diagram group, except that it uses graph rewriting.

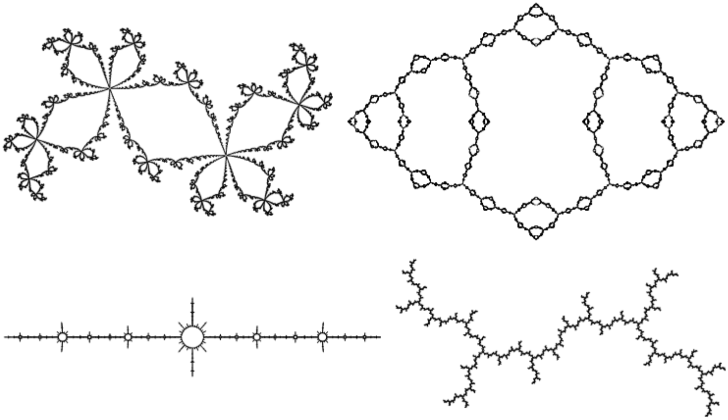
Theorem (Farley). *Every diagram group over a finite string rewriting system acts properly by isometries on a* CAT(0) *cubical complex.*

A similar construction gives a natural CAT(0) cubical complex on which T_B acts properly by isometries.

Other Julia Sets

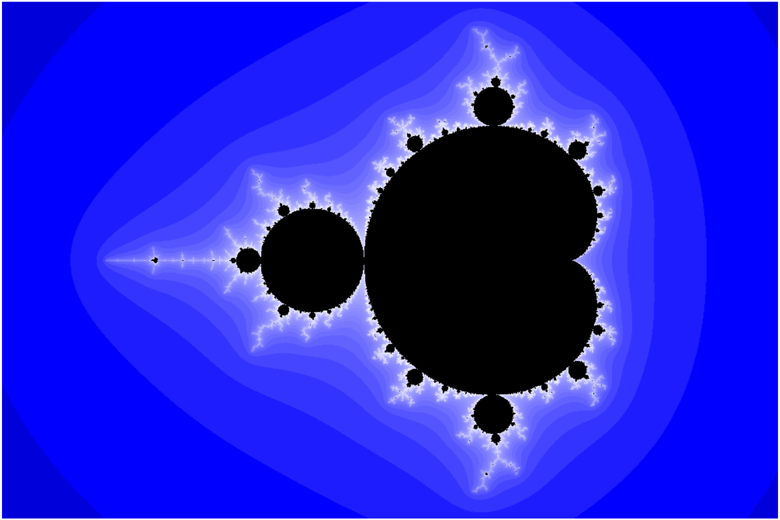
Julia Sets

Every rational function on the Riemann sphere has an associated Julia set.



The Mandelbrot Set

Julia sets for quadratic polynomials $f(z) = z^2 + c$ are parameterized by the ***Mandelbrot set***:



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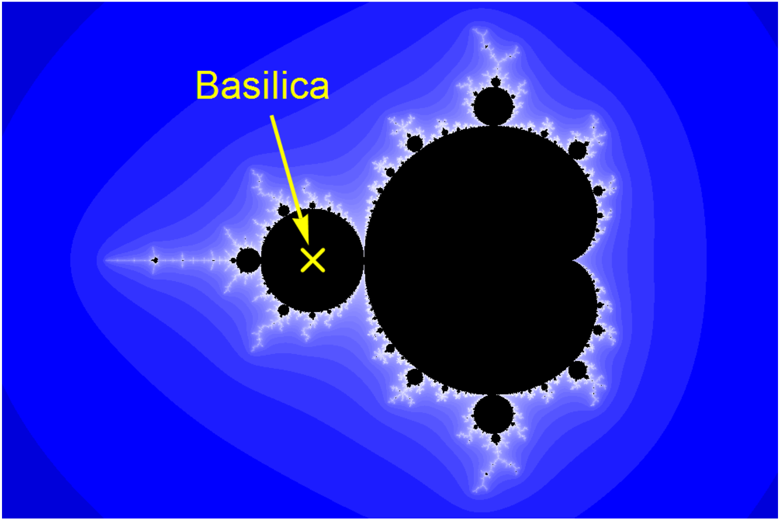
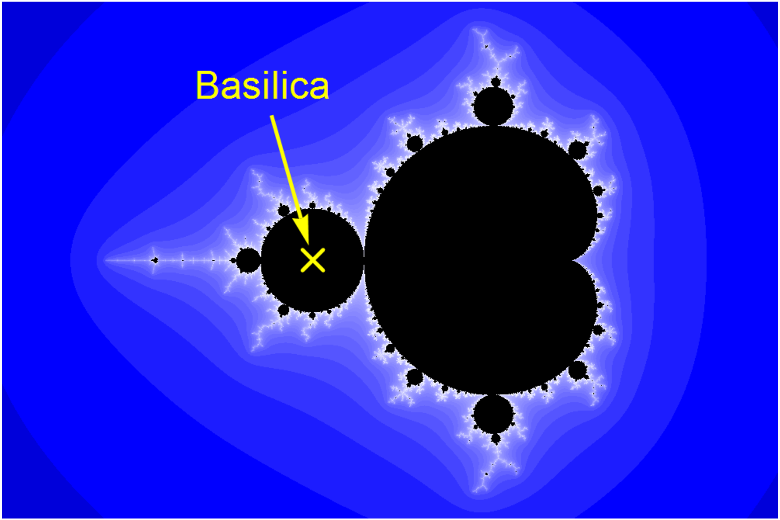


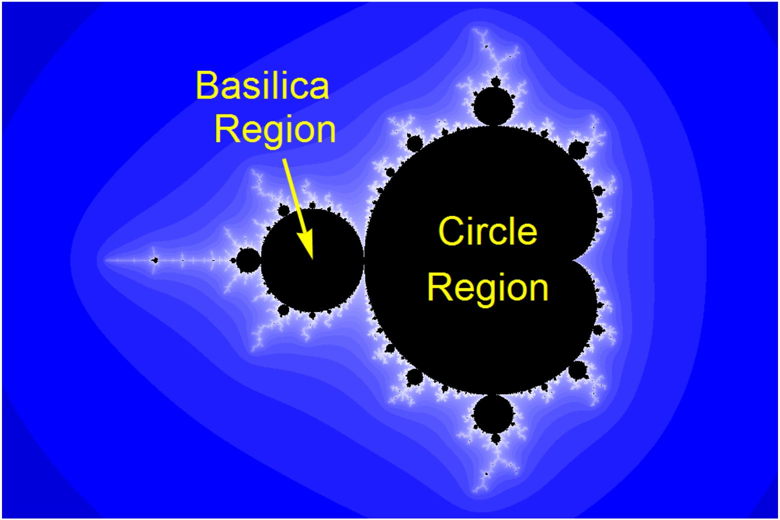
The Mandelbrot Set

Points in the interior of the Mandelbrot set are called ***hyperbolic***.



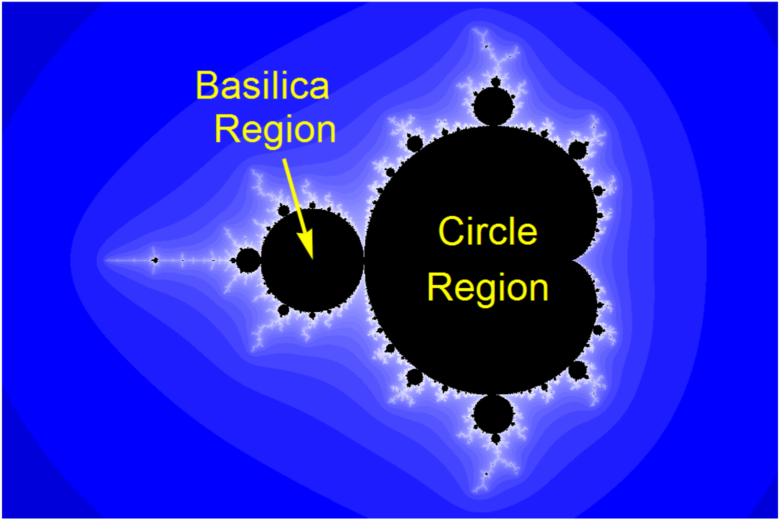
The Mandelbrot Set

Hyperbolic points from the same interior region give Julia sets with the same structure.



The Mandelbrot Set

We can construct a Thompson-like group T_J for each of these regions. (Hubbard tree $\to$ Graph rewriting system)



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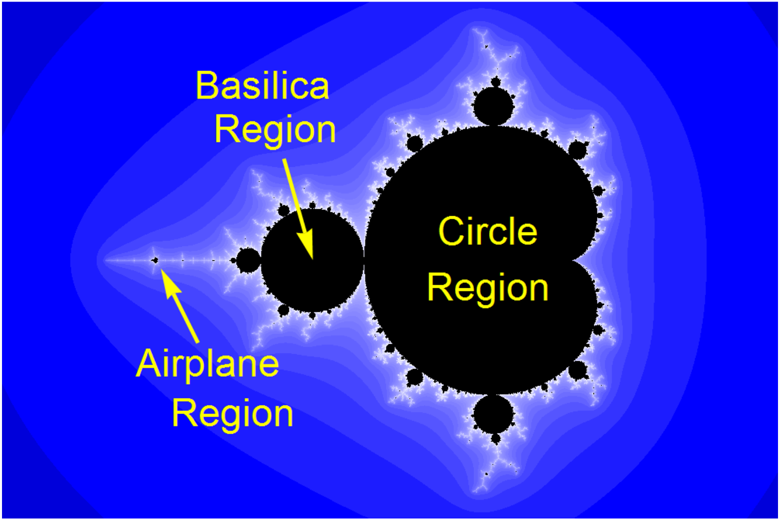
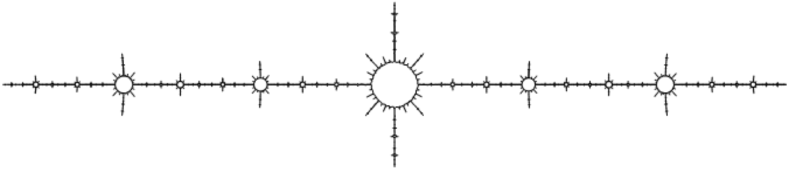


The Airplane Group



Let T_A be the group of rearrangements of the airplane Julia set.

Theorem.

1. T_A has a simple subgroup of index 3.
2. T_A has type F_∞.

The proof of (2) involves discrete Morse Theory on the CAT(0) cubical complex for T_A.

Questions

- Are all the T_J finitely generated? Is there a uniform method to find a generating set?
- Which T_J are finitely presented? Which have type F_∞?
- Which of these groups are virtually simple?
- What is the relation between these groups? For which Julia sets J and J' does T_J contain an isomorphic copy of $T_{J'}$?
- For which rational Julia sets can we construct a Thompson-like group? Are there Thompson-like groups associated to other families of fractals?

The End