Thompson-Like Groups Acting on Julia Sets

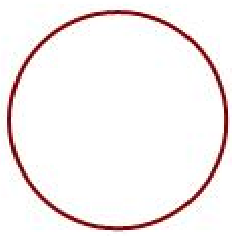
Jim Belk* and Bradley Forrest

October 2013

Thompson's Groups

In the 1960's, Richard J. Thompson defined three infinite groups:

F acts on the unit interval.



T acts on the unit circle.

V acts on the Cantor set.

^

Definition of *F*

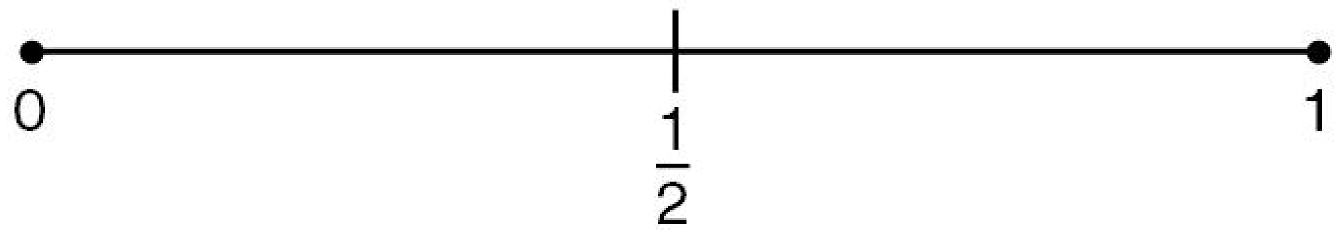
^

Definition of F

A ***dyadic subdivision*** of [0,1] is any subdivision obtained by repeatedly cutting intervals in half:

Definition of *F*

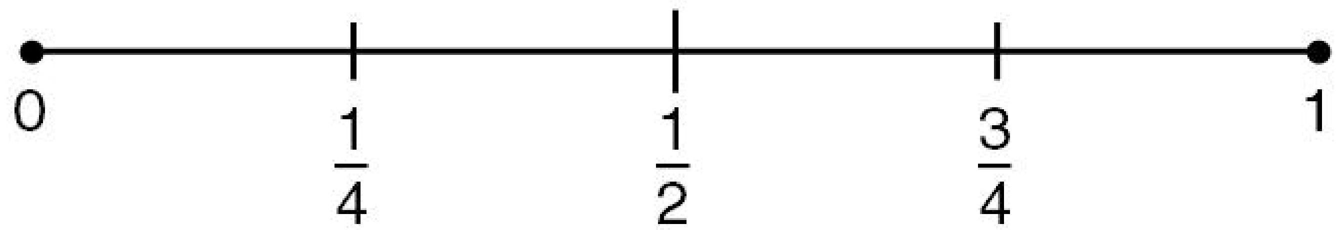
A ***dyadic subdivision*** of [0,1] is any subdivision obtained by repeatedly cutting intervals in half:



^

Definition of F

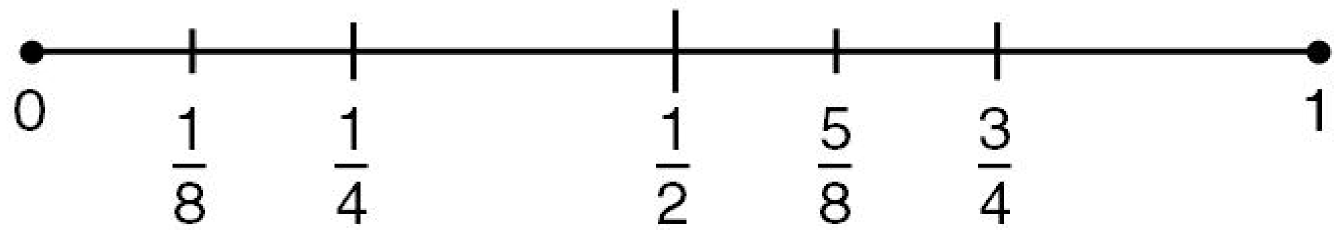
A ***dyadic subdivision*** of [0,1] is any subdivision obtained by repeatedly cutting intervals in half:



^

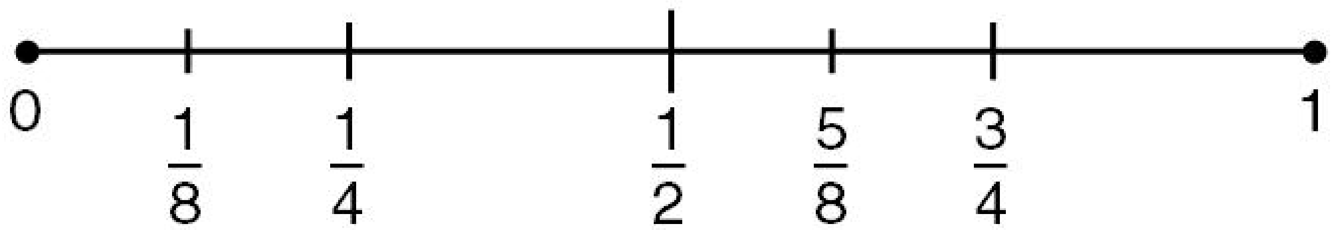
Definition of *F*

A ***dyadic subdivision*** of [0,1] is any subdivision obtained by repeatedly cutting intervals in half:



Definition of *F*

A ***dyadic subdivision*** of [0,1] is any subdivision obtained by repeatedly cutting intervals in half:

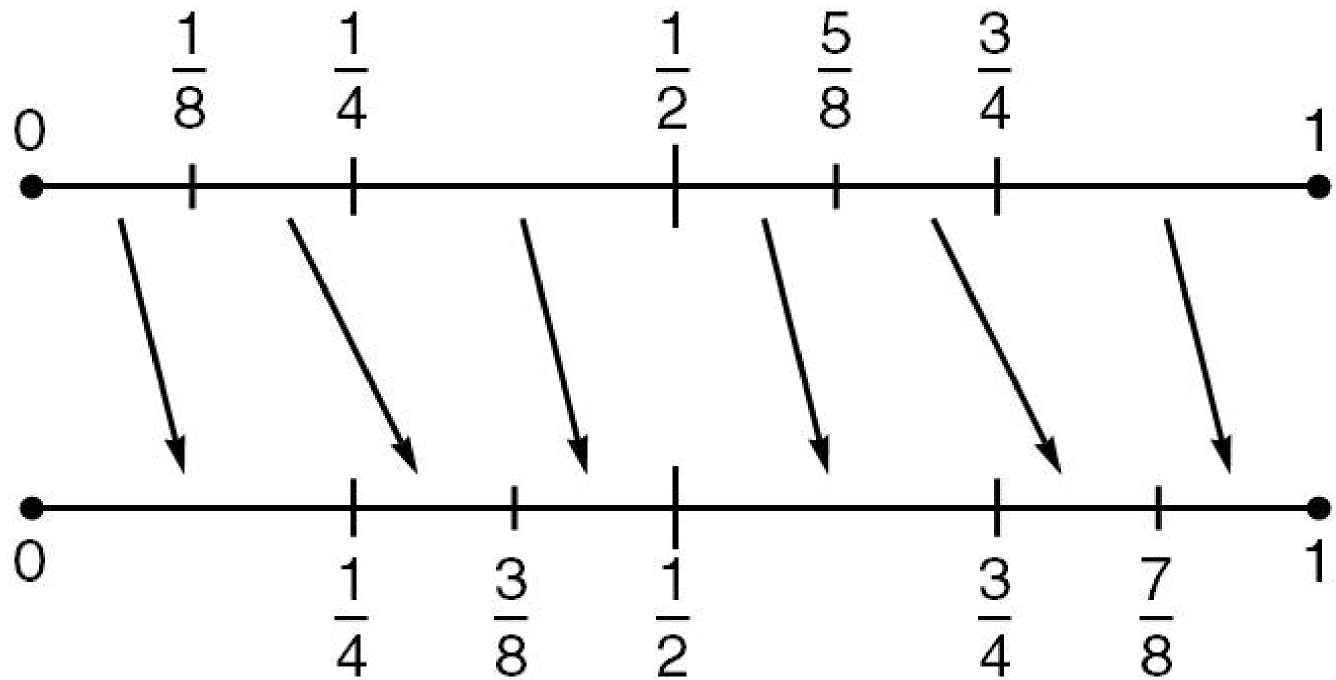


The intervals of such a subdivision are ***standard dyadic intervals***:

$$\left[\frac{k}{2^m}, \frac{k+1}{2^m}\right]$$

Definition of F

A ***dyadic rearrangement*** of [0,1] is a PL homeomorphism that maps linearly between the intervals of two dyadic subdivisions:

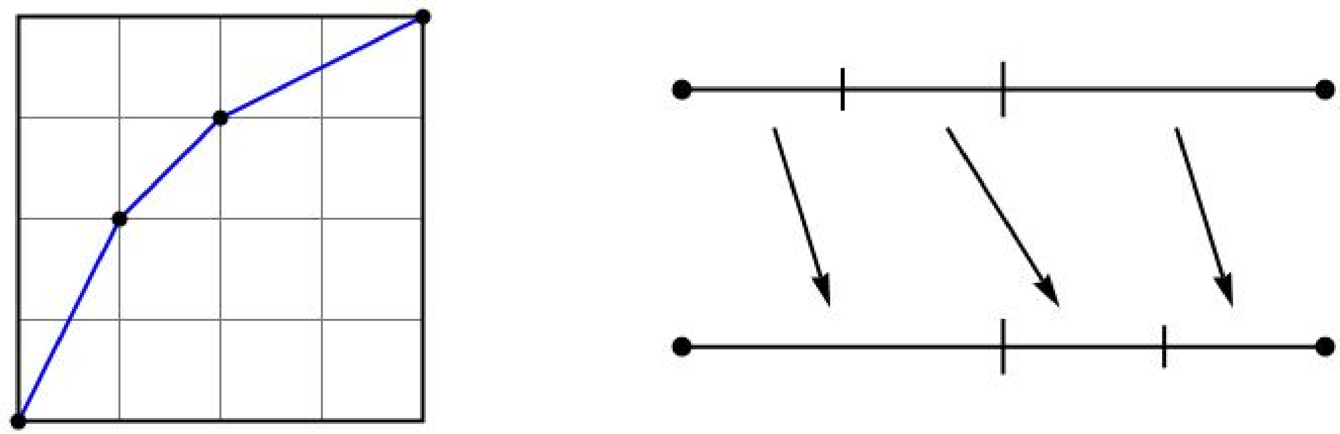


The group of all dyadic rearrangements is ***Thompson's group F***.

^

Definition of *F*

The group of all dyadic rearrangements is ***Thompson's group F***.



Given an element of *F*:

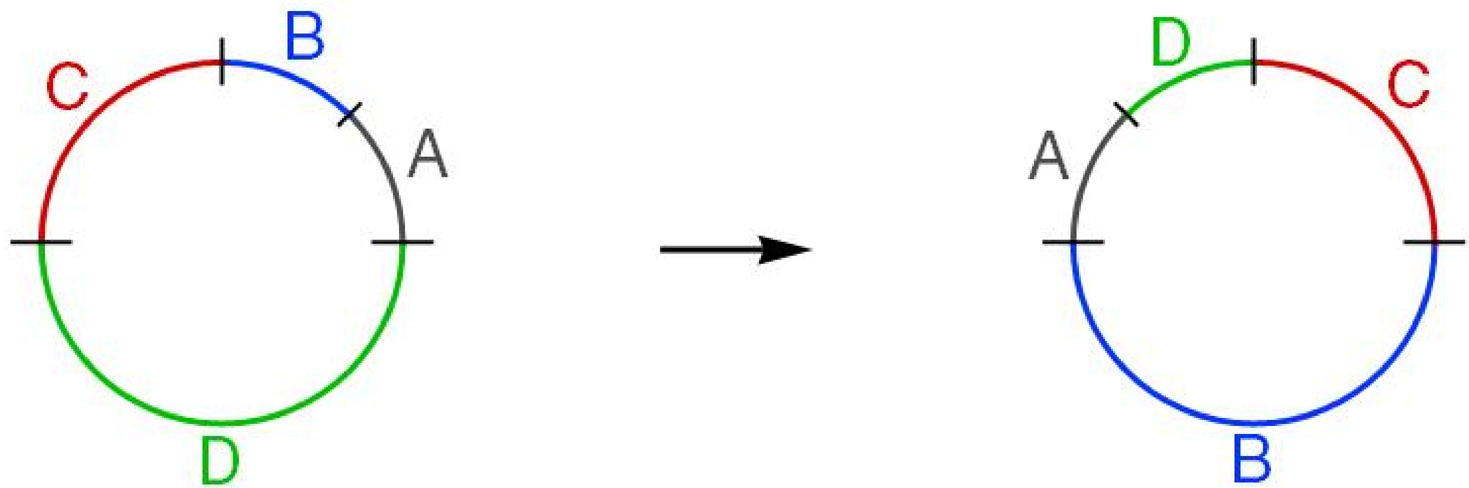
- Each segment has slope 2^m.
- Each breakpoint has dyadic rational coordinates.

These conditions characterize the elements of *F*.

ˆ

Definitions of *T* and *V*

Thompson's group *T* acts on the circle.



Definitions of *T* and *V*

Thompson's group *T* acts on the circle.

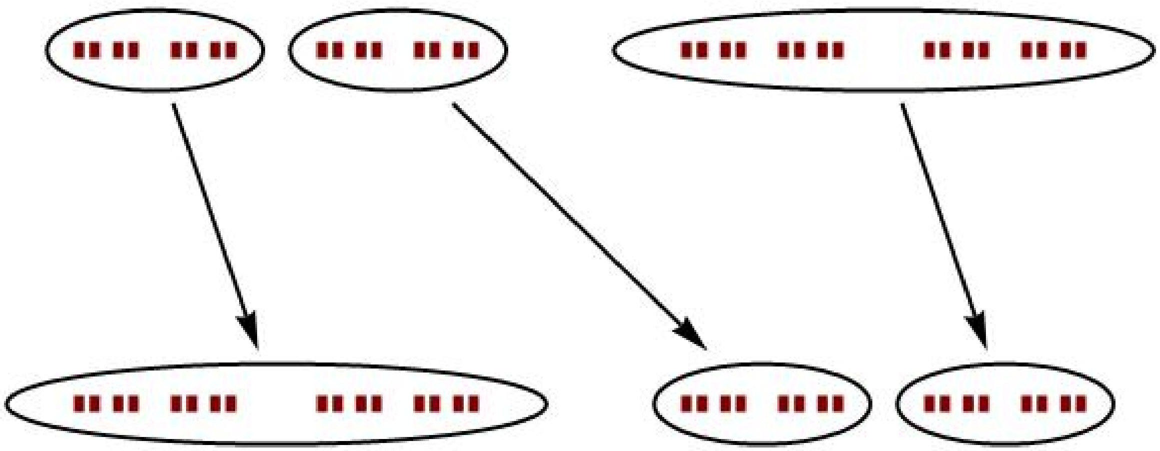
Thompson's group *V* acts on the Cantor set.

^

Definitions of T and V

Thompson's group T acts on the circle.

Thompson's group V acts on the Cantor set.

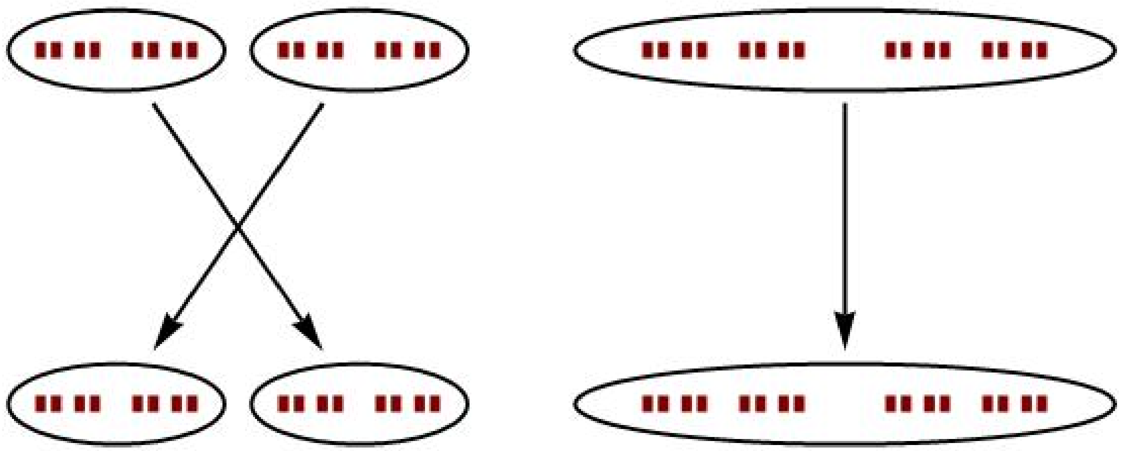


ˆ

Definitions of *T* and *V*

Thompson’s group *T* acts on the circle.

Thompson’s group *V* acts on the Cantor set.

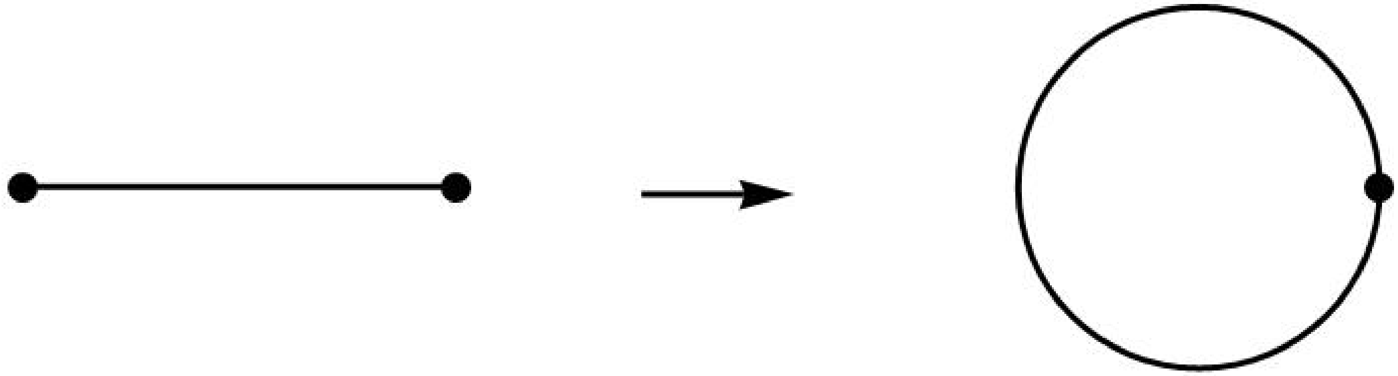


Properties of *F*, *T*, and *V*

^

Properties of *F*, *T*, and *V*

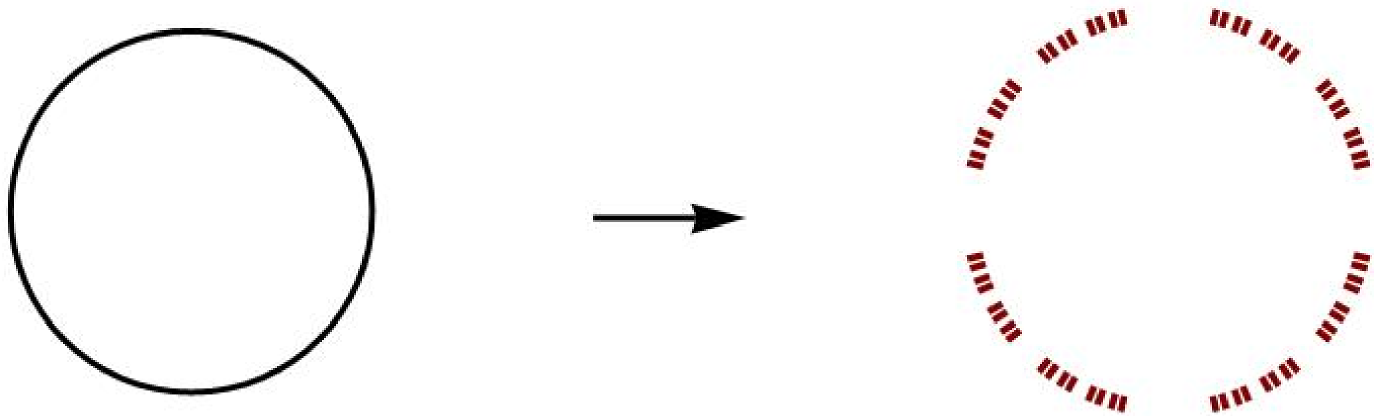
- *F* embeds into *T*.



^

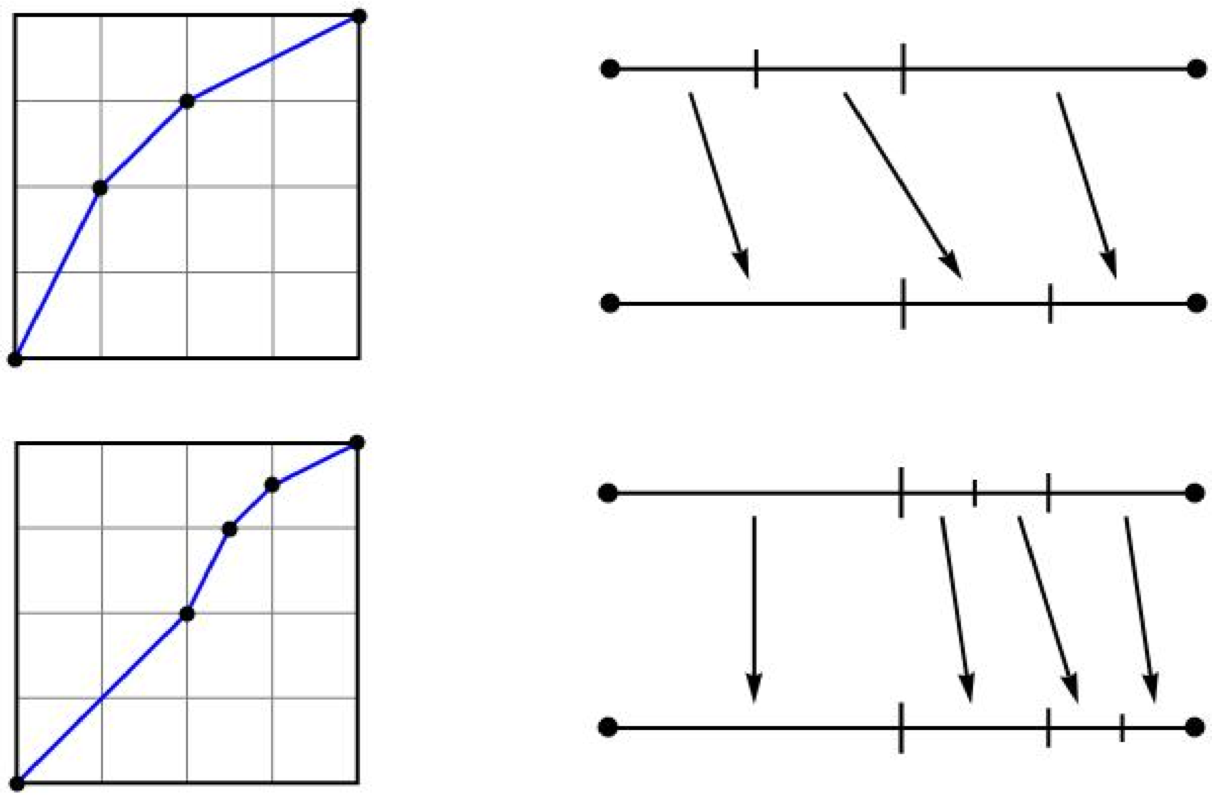
Properties of *F*, *T*, and *V*

- *F* embeds into *T*, and *T* embeds into *V*.



Properties of *F*, *T*, and *V*

- *F* embeds into *T*, and *T* embeds into *V*.
- *F*, *T*, and *V* are finitely generated.



Properties of *F*, *T*, and *V*

- *F* embeds into *T*, and *T* embeds into *V*.
- *F*, *T*, and *V* are finitely generated.
- *T* and *V* are ***infinite, finitely presented simple groups***.

^

Properties of F, T, and V

- F embeds into T, and T embeds into V.
- F, T, and V are finitely generated.
- T and V are ***infinite, finitely presented simple groups***.
- F is finitely presented, but not simple (though $[F, F]$ is simple).

^

Properties of *F*, *T*, and *V*

- *F* embeds into *T*, and *T* embeds into *V*.
- *F*, *T*, and *V* are finitely generated.
- *T* and *V* are ***infinite, finitely presented simple groups***.
- *F* is finitely presented, but not simple (though $[F, F]$ is simple).
- Finiteness property: *F*, *T*, and *V* have ***type*** $\boldsymbol{F_\infty}$.

^

Properties of *F*, *T*, and *V*

- *F* embeds into *T*, and *T* embeds into *V*.
- *F*, *T*, and *V* are finitely generated.
- *T* and *V* are ***infinite, finitely presented simple groups***.
- *F* is finitely presented, but not simple (though $[F, F]$ is simple).
- Finiteness property: *F*, *T*, and *V* have ***type*** $\boldsymbol{F_\infty}$.
- *F*, *T*, and *V* act properly and isometrically on CAT(0) cubical complexes.

^

Generalizations

Basic Questions:

Why are there *three* Thompson groups?

What do the interval, circle, and Cantor set have in common?

^

Generalizations

Basic Questions:

Why are there *three* Thompson groups?

What do the interval, circle, and Cantor set have in common?

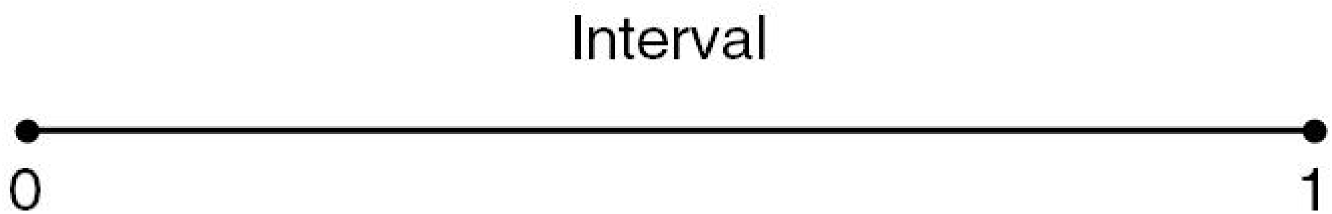
Generalizations:

- $F(n)$, $T(n)$, $V(n)$ (Higman 1974, Brown 1987)
- Other PL groups (Bieri & Strebel 1985, Stein 1992)
- Diagram Groups (Guba & Sapir 1997)
- "Braided" V (Brin 2004, Dehornoy 2006)
- $2V$ (Brin 2004)

Self-Similarity

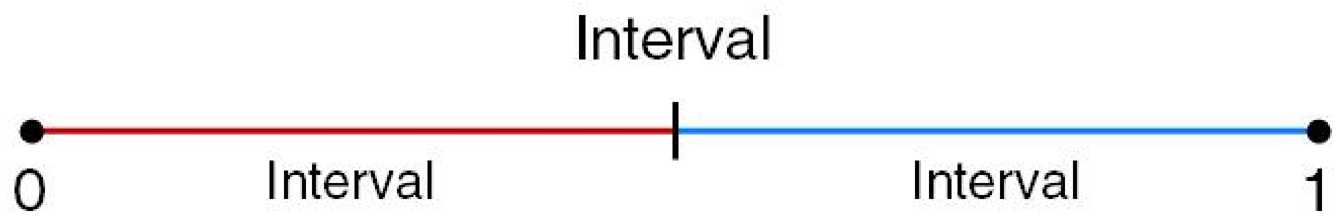
Self-Similarity

The definition of *F* depends on the ***self-similarity*** of the interval.



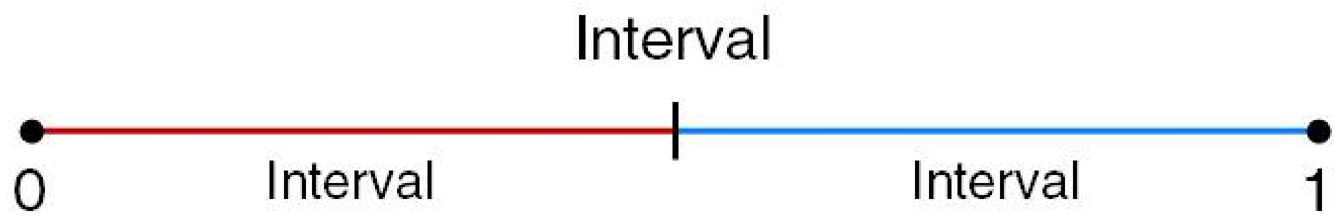
Self-Similarity

The definition of *F* depends on the ***self-similarity*** of the interval.

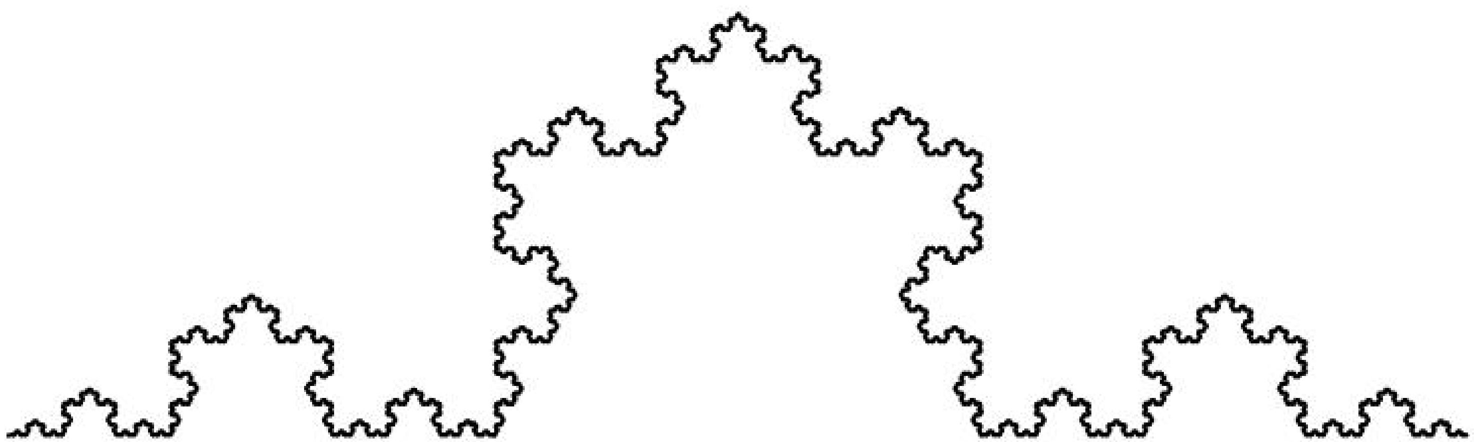


Self-Similarity

The definition of *F* depends on the ***self-similarity*** of the interval.



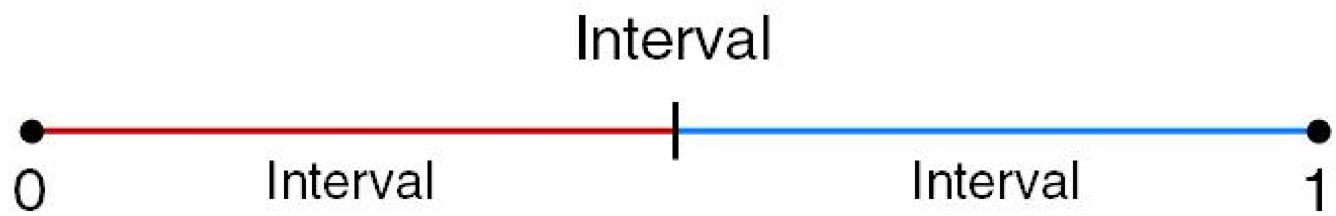
Some fractals have the same self-similar structure as an interval:



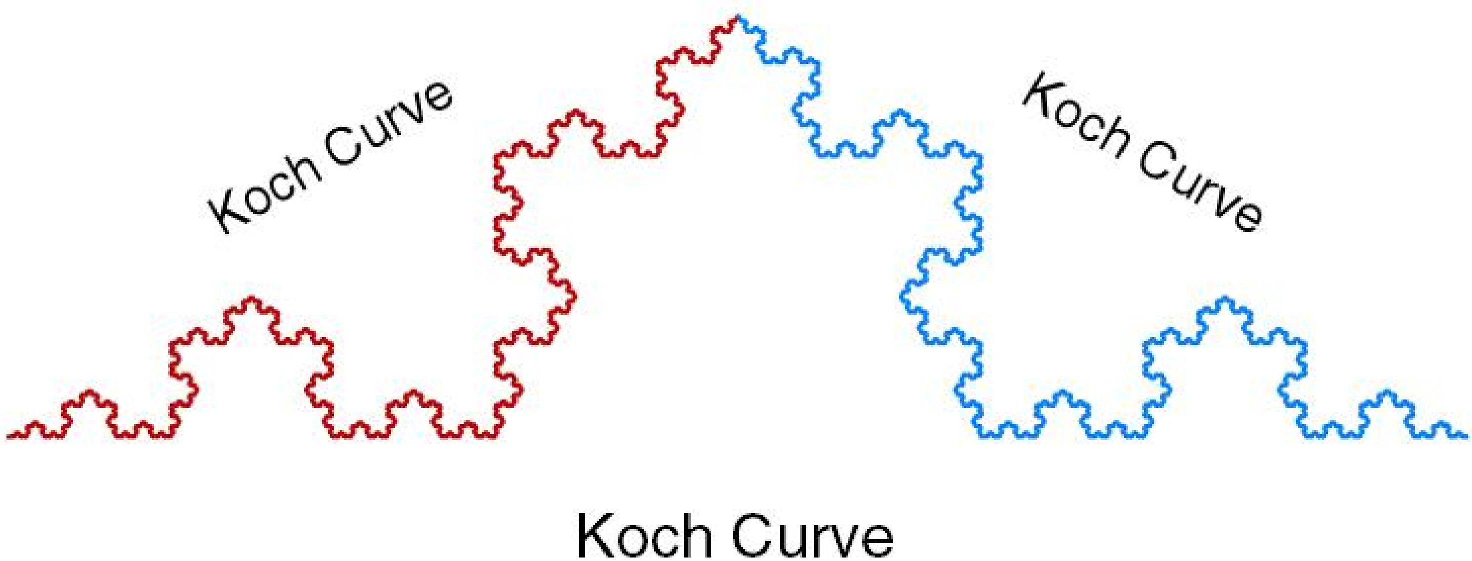
Koch Curve

Self-Similarity

The definition of *F* depends on the ***self-similarity*** of the interval.

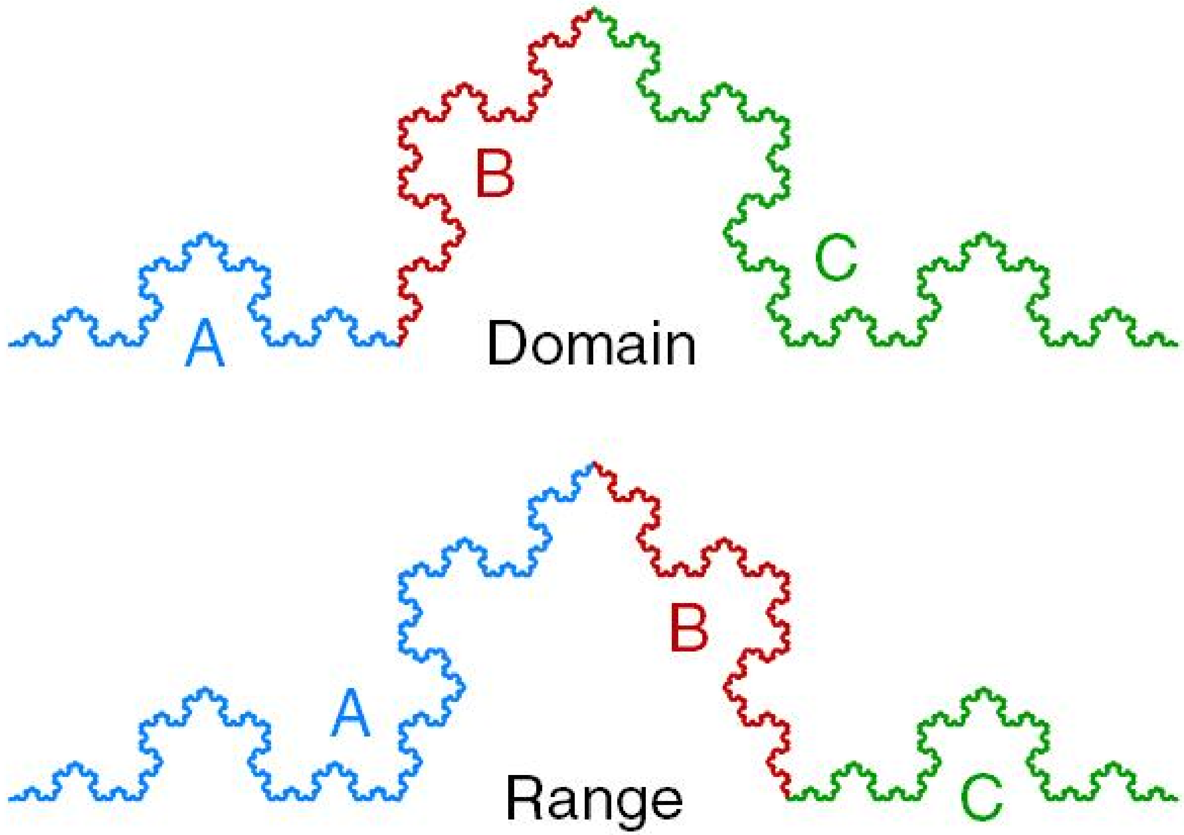


Some fractals have the same self-similar structure as an interval:



Self-Similarity

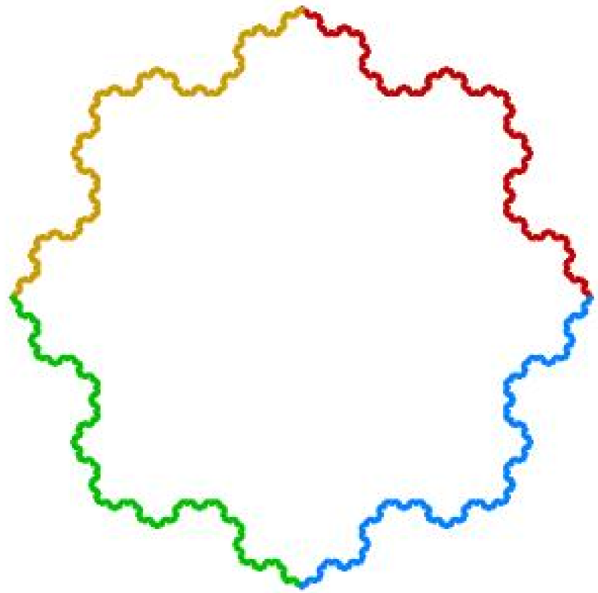
Thompson's group *F* acts on such a fractal by piecewise similarities.



Self-Similarity

Thompson's group F acts on such a fractal by piecewise similarities.

There are also fractals on which T or V acts by piecewise similarities.



Self-Similarity

Thompson's group *F* acts on such a fractal by piecewise similarities.

There are also fractals on which *T* or *V* acts by piecewise similarities.

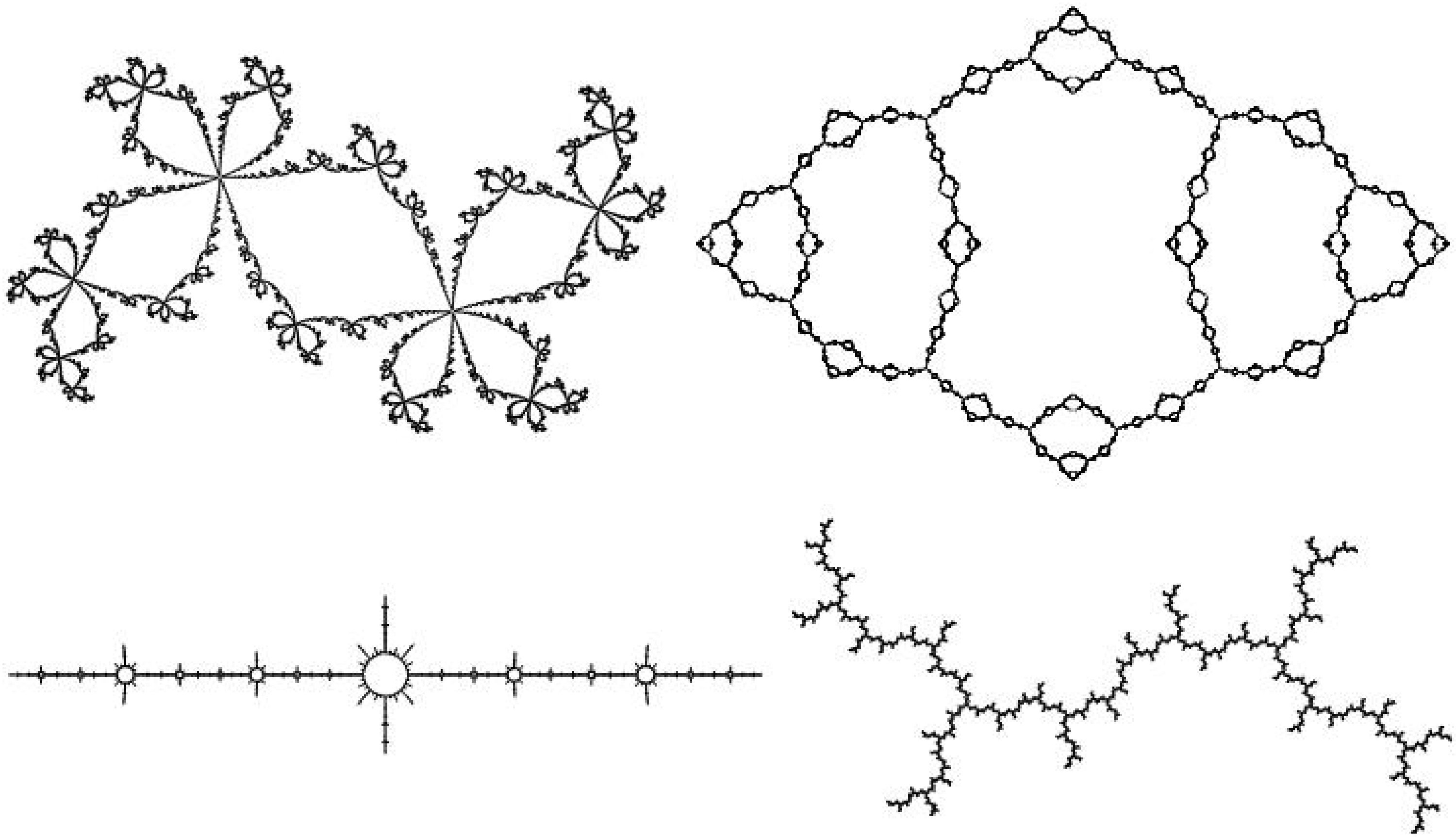
Main Idea:

Perhaps there are Thompson-like groups acting on other fractal shapes.

Julia Sets

Julia Sets

Every rational map on the Riemann sphere has an associated ***Julia set***.



Julia Sets

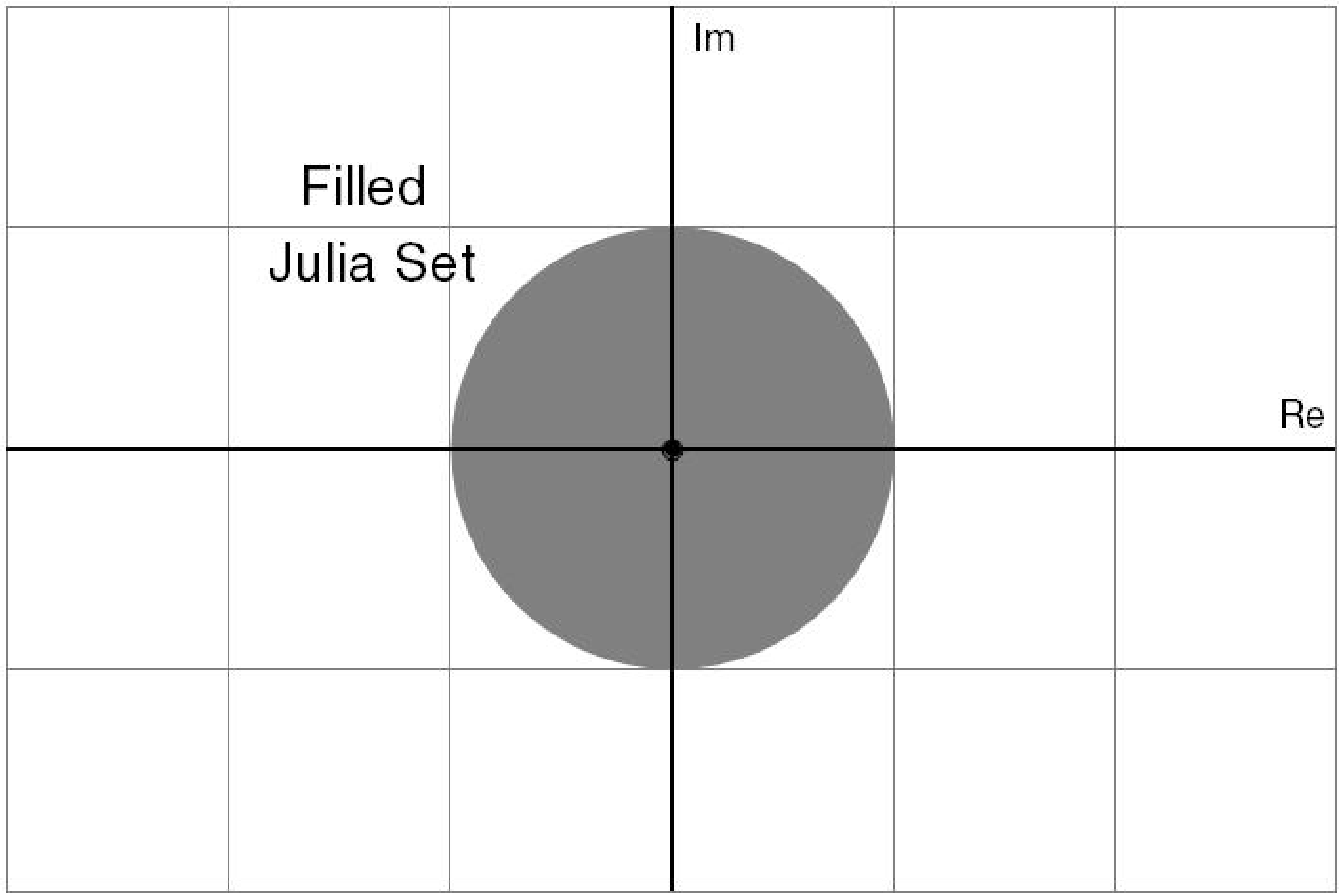
Let $f(z) = z^2 + c$, where z and c are complex.

The ***filled Julia set*** for f is the set

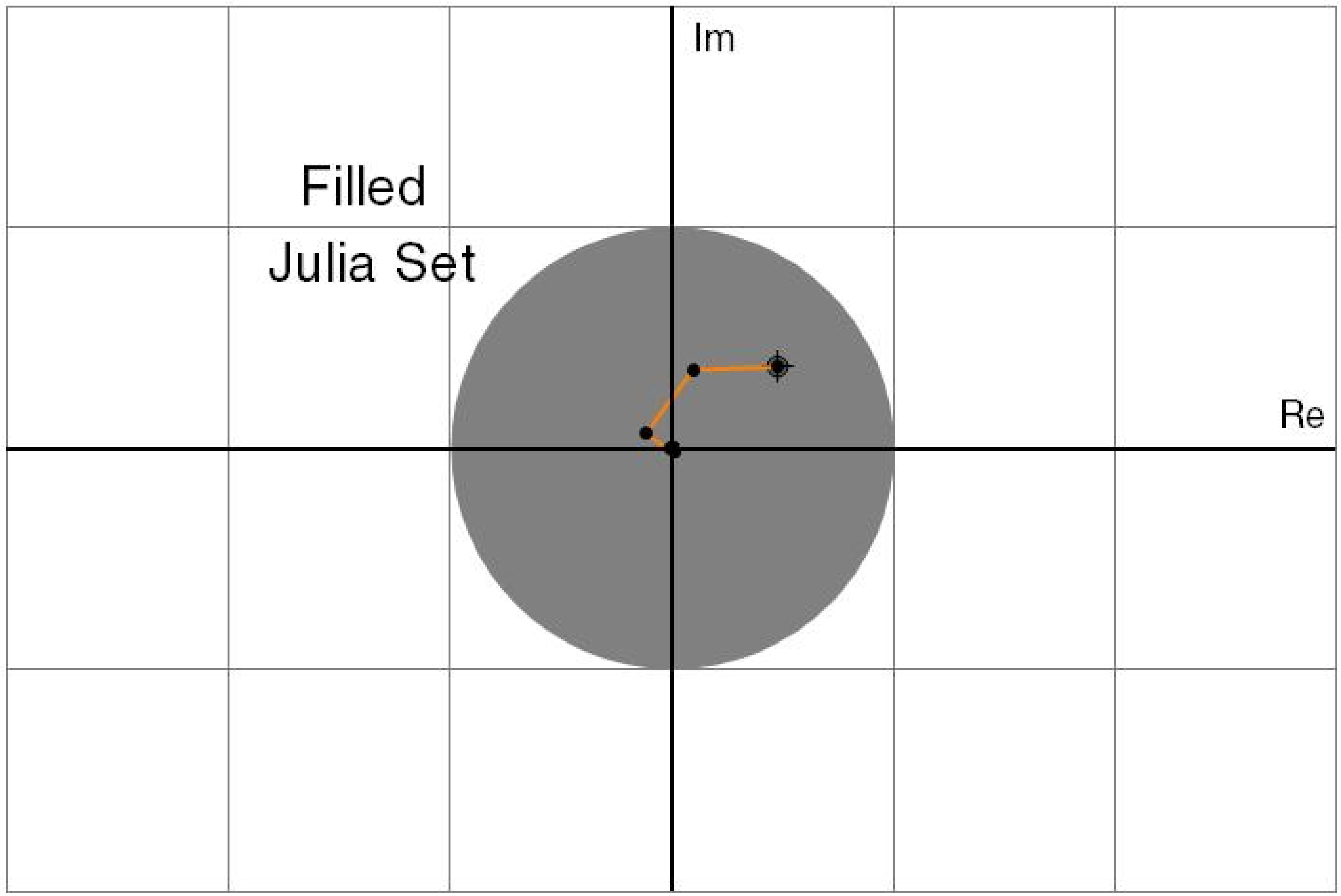
$$\{p \in \mathbb{C} \mid \text{the orbit of } p \text{ under } f \text{ is bounded}\}$$

Example: The filled Julia set for $f(z) = z^2$ is the closed unit disk.

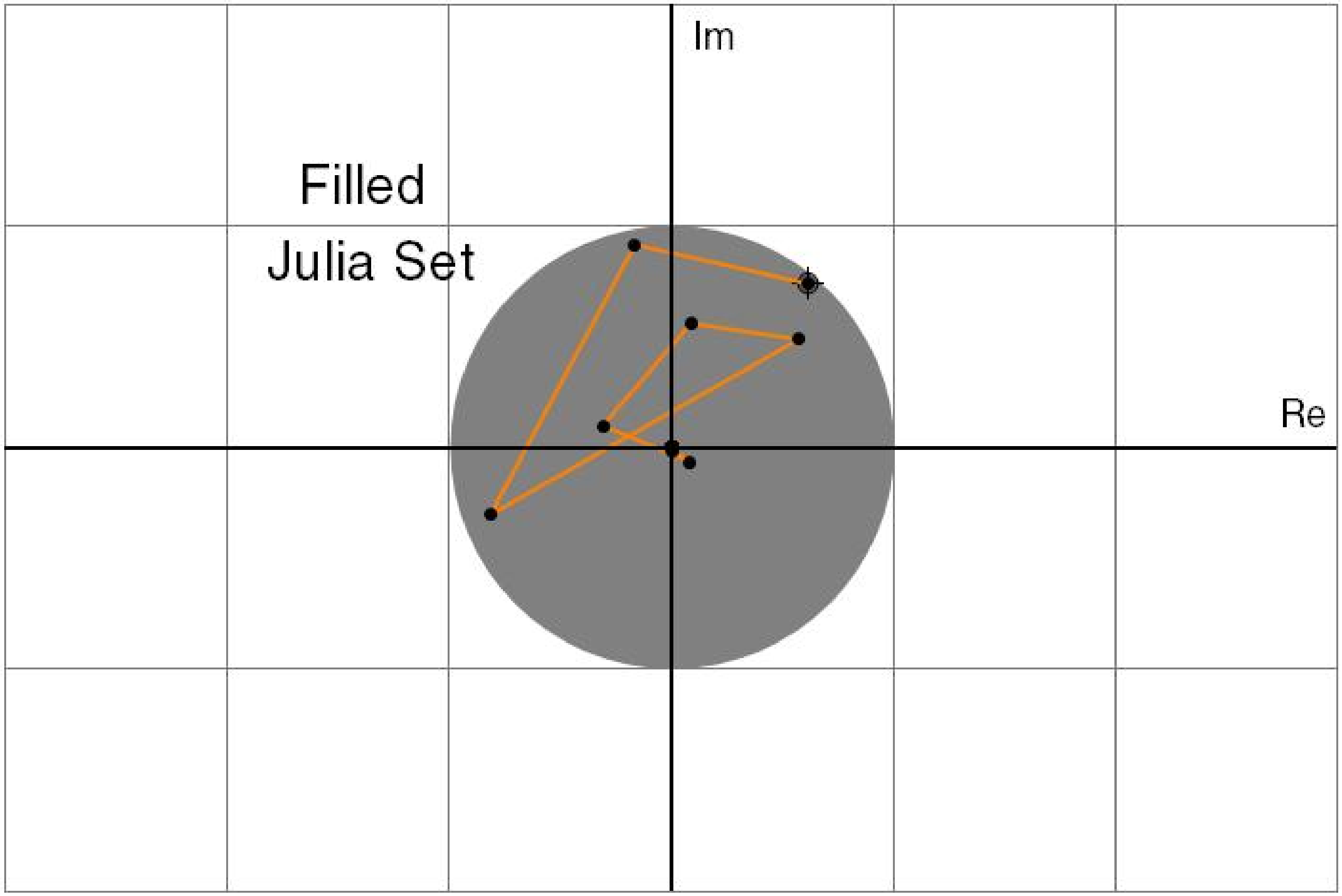
Orbits Under z^2



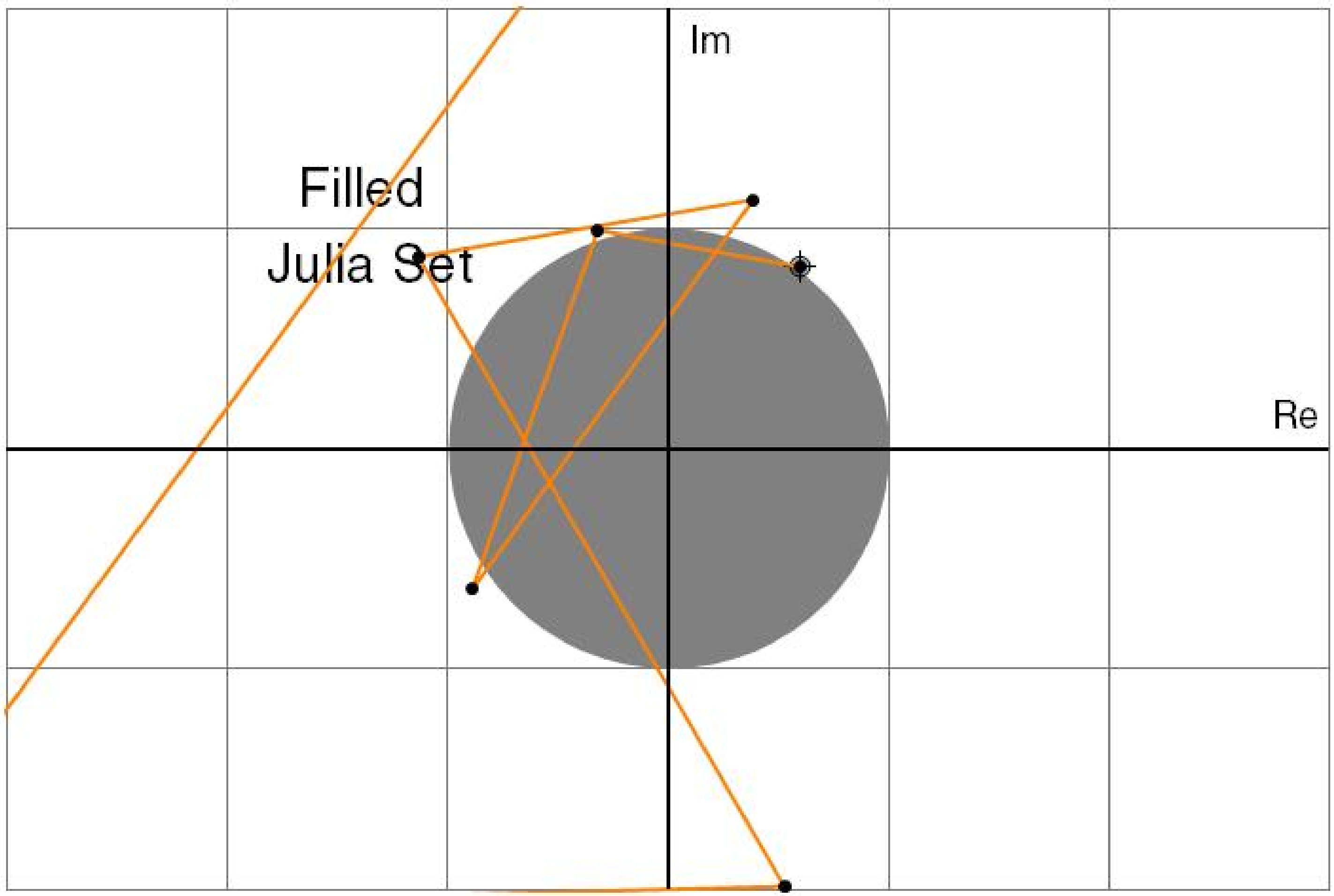
Orbits Under z^2



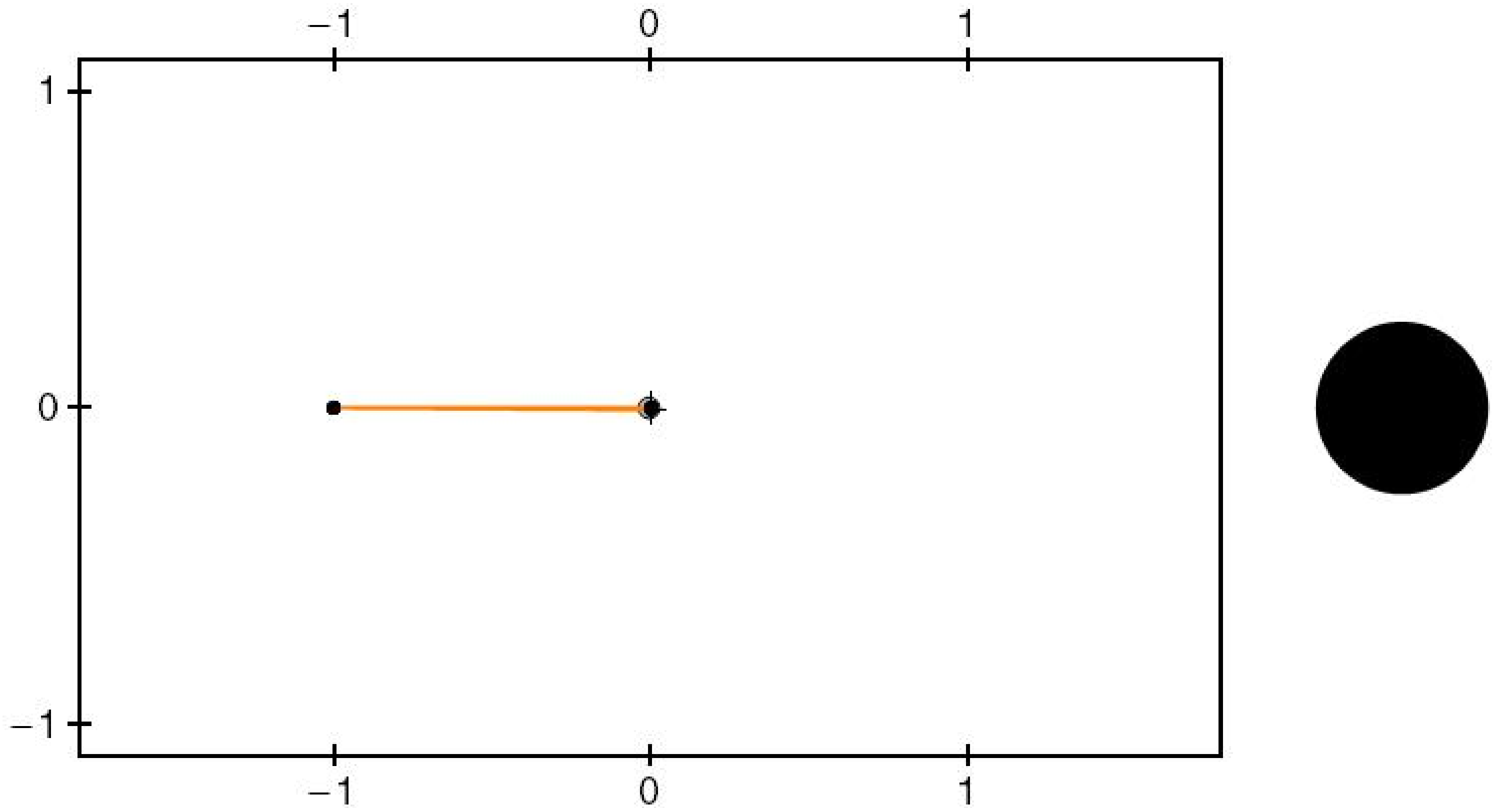
Orbits Under z^2



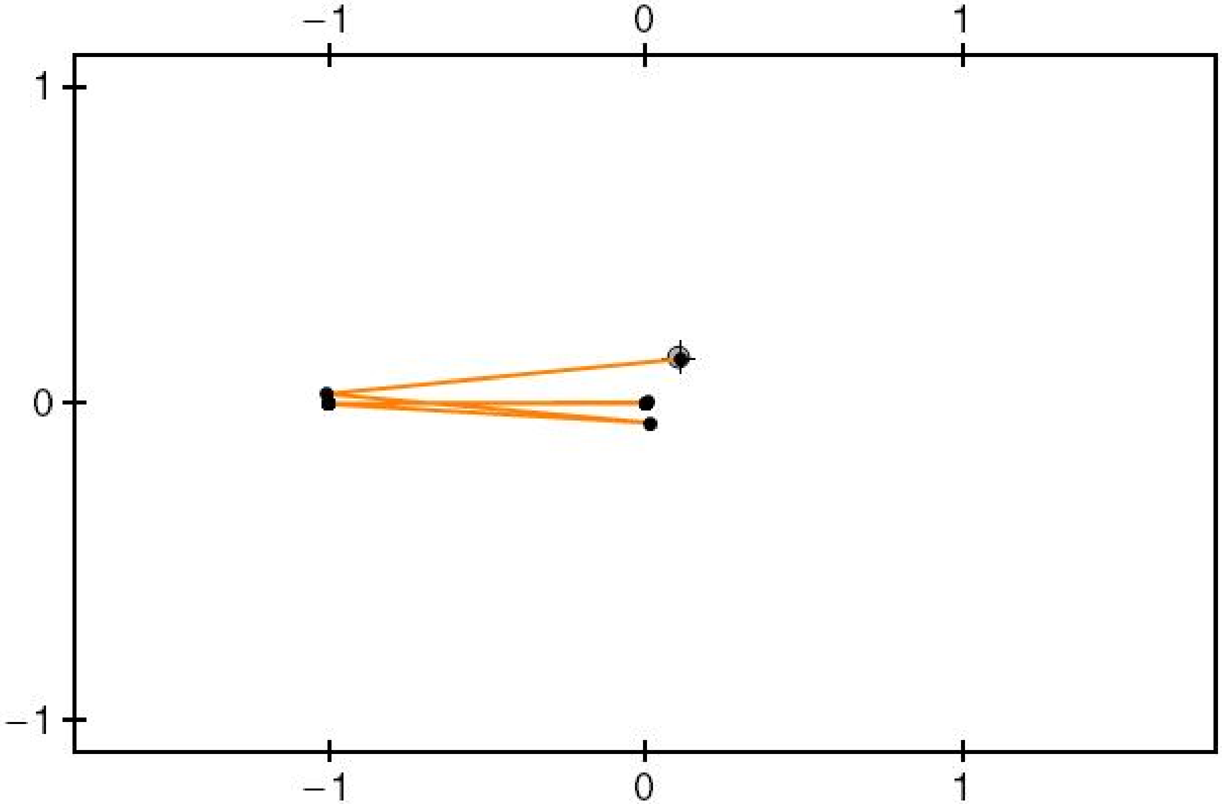
Orbits Under z^2



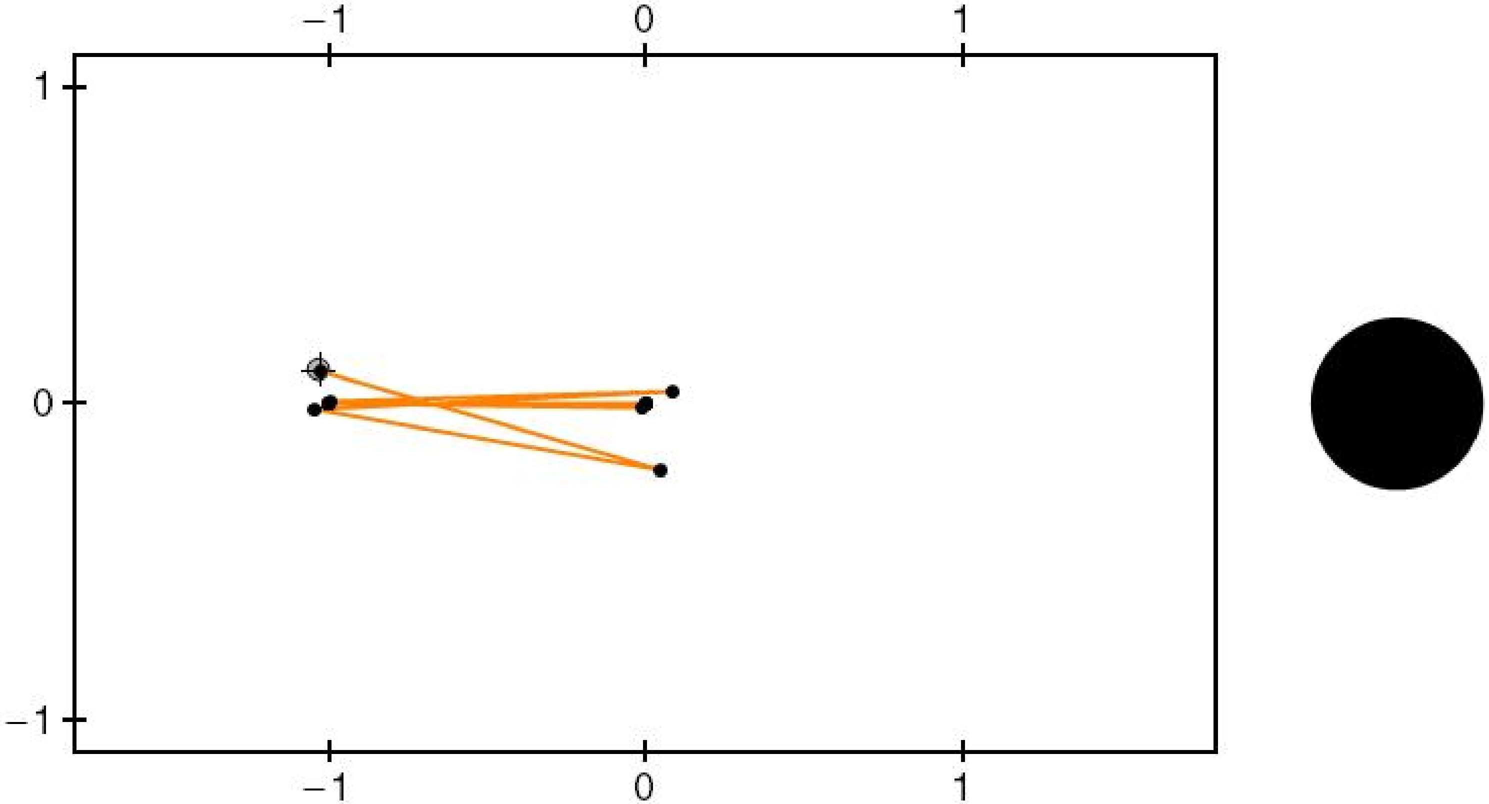
Orbits Under $z^2 - 1$



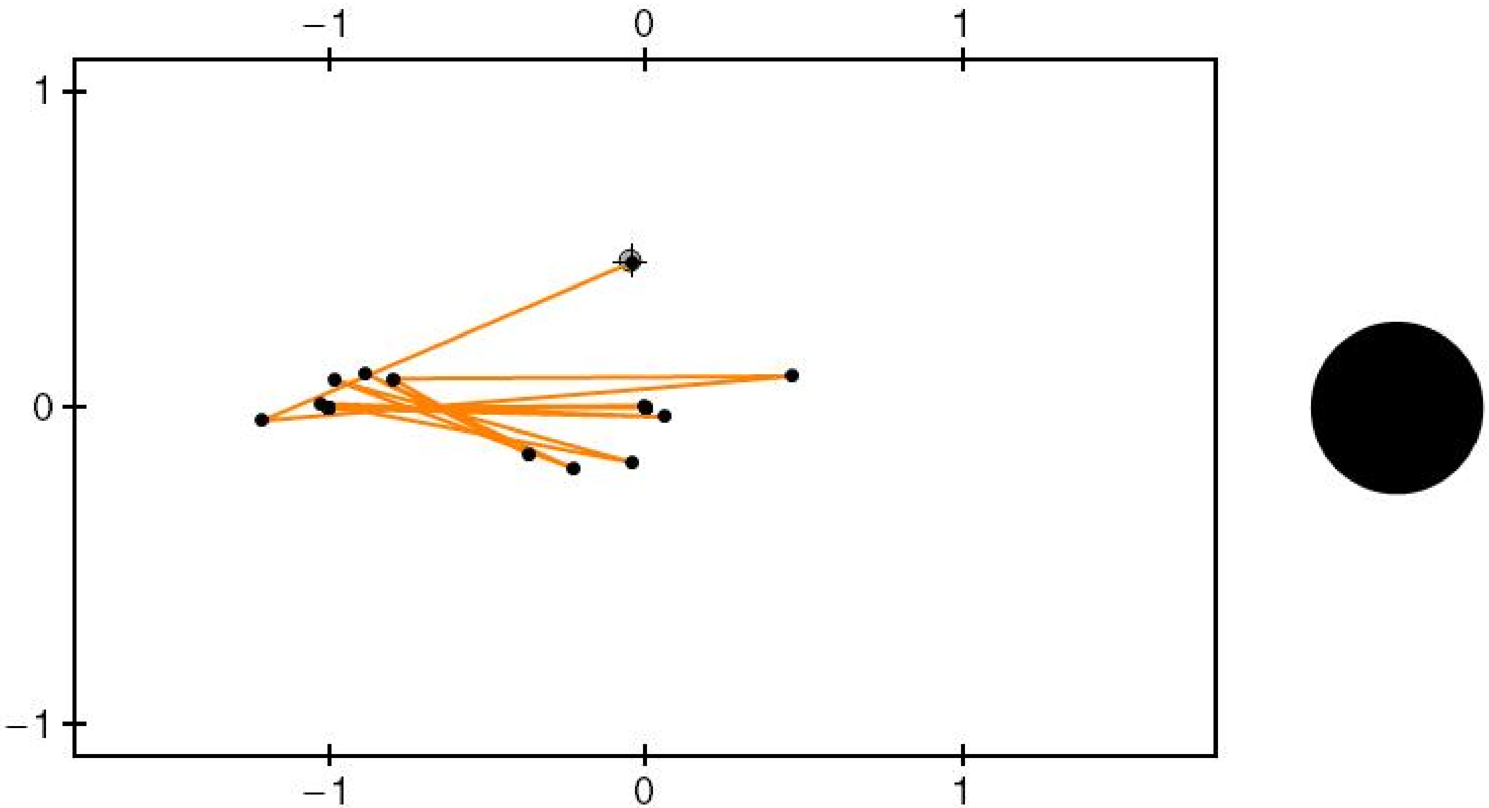
Orbits Under $z^2 - 1$



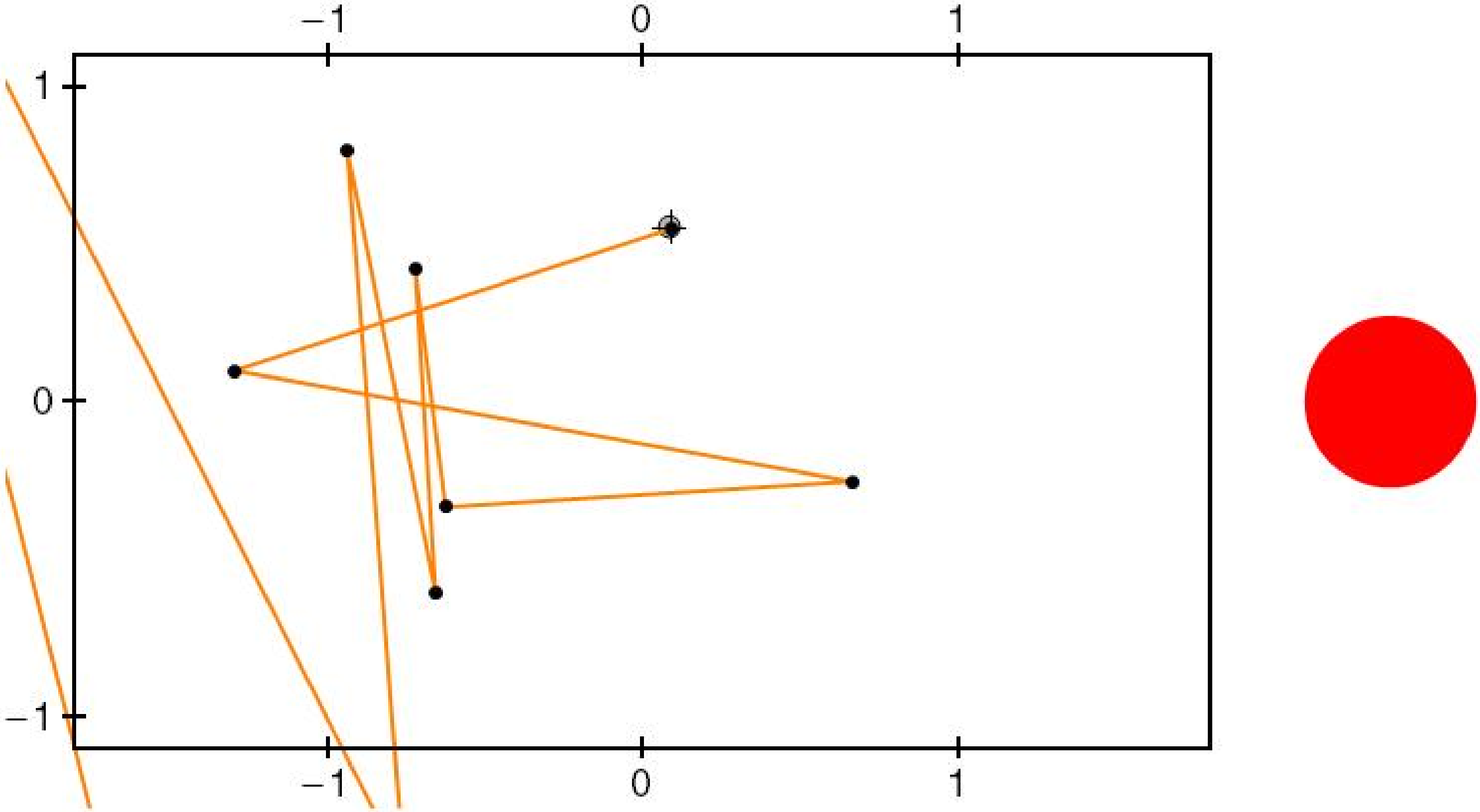
Orbits Under $z^2 - 1$



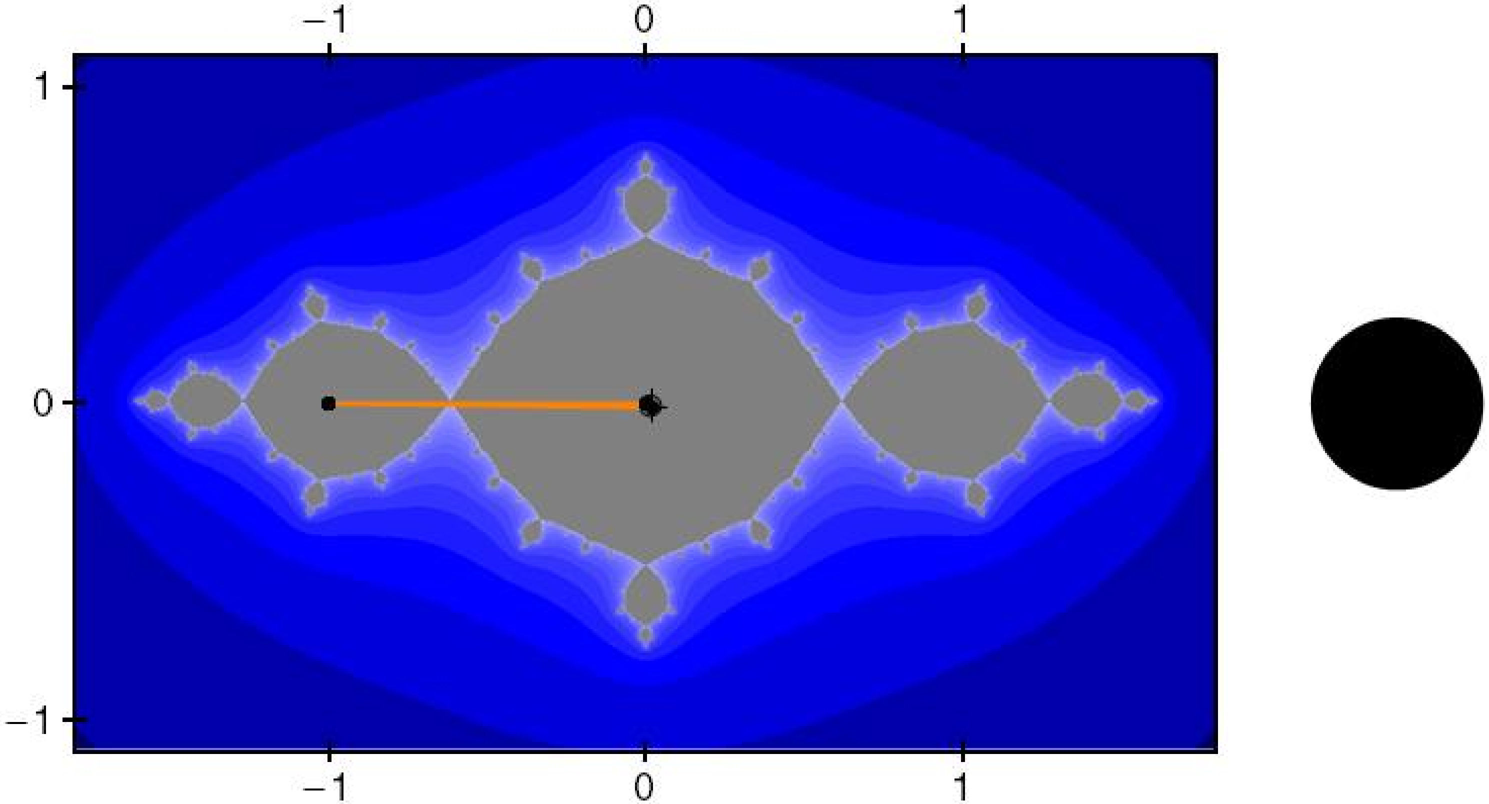
Orbits Under $z^2 - 1$



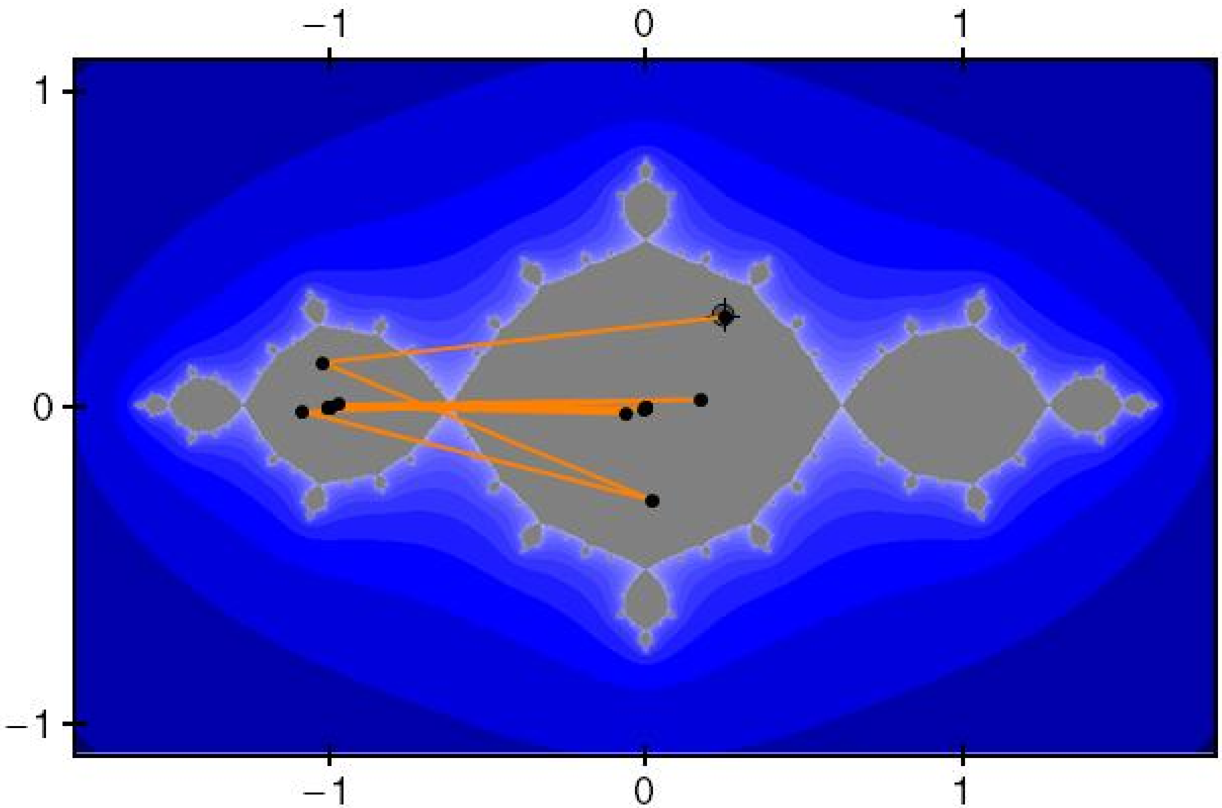
Orbits Under $z^2 - 1$

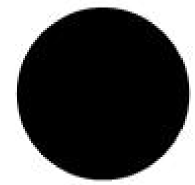


Orbits Under $z^2 - 1$

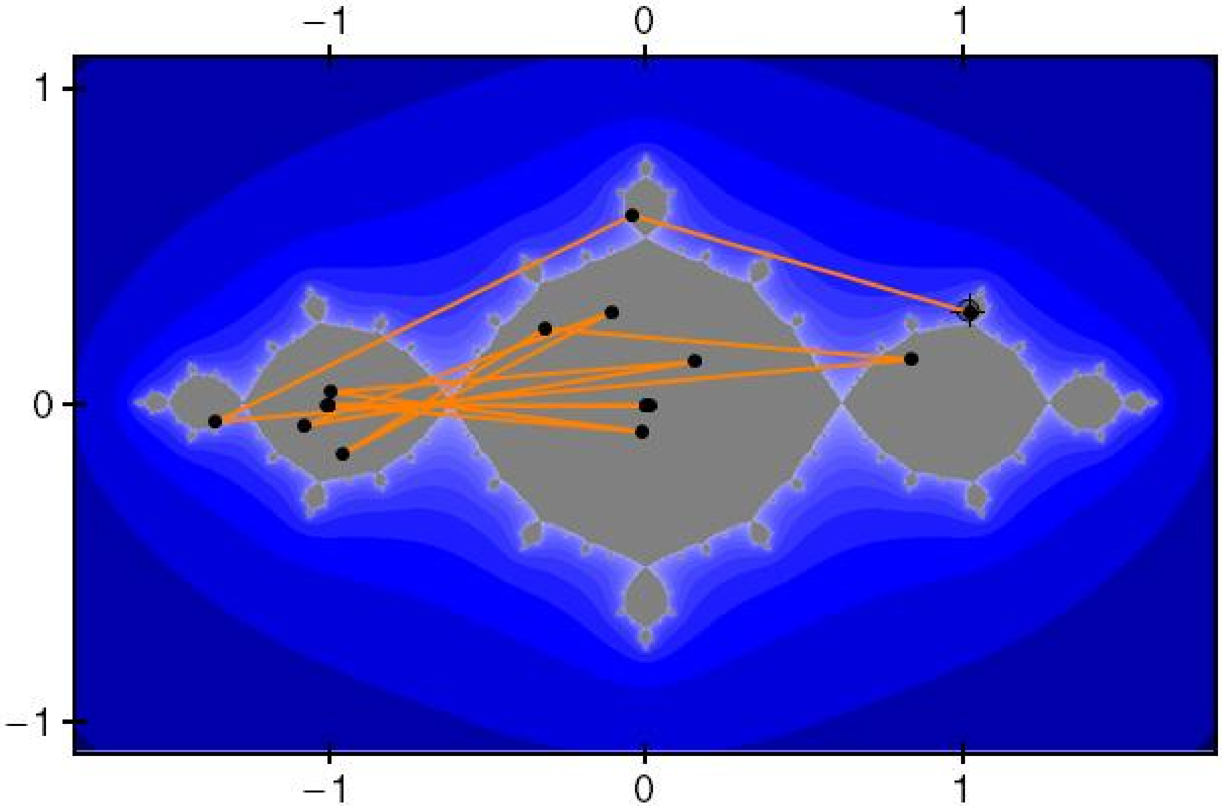


Orbits Under $z^2 - 1$

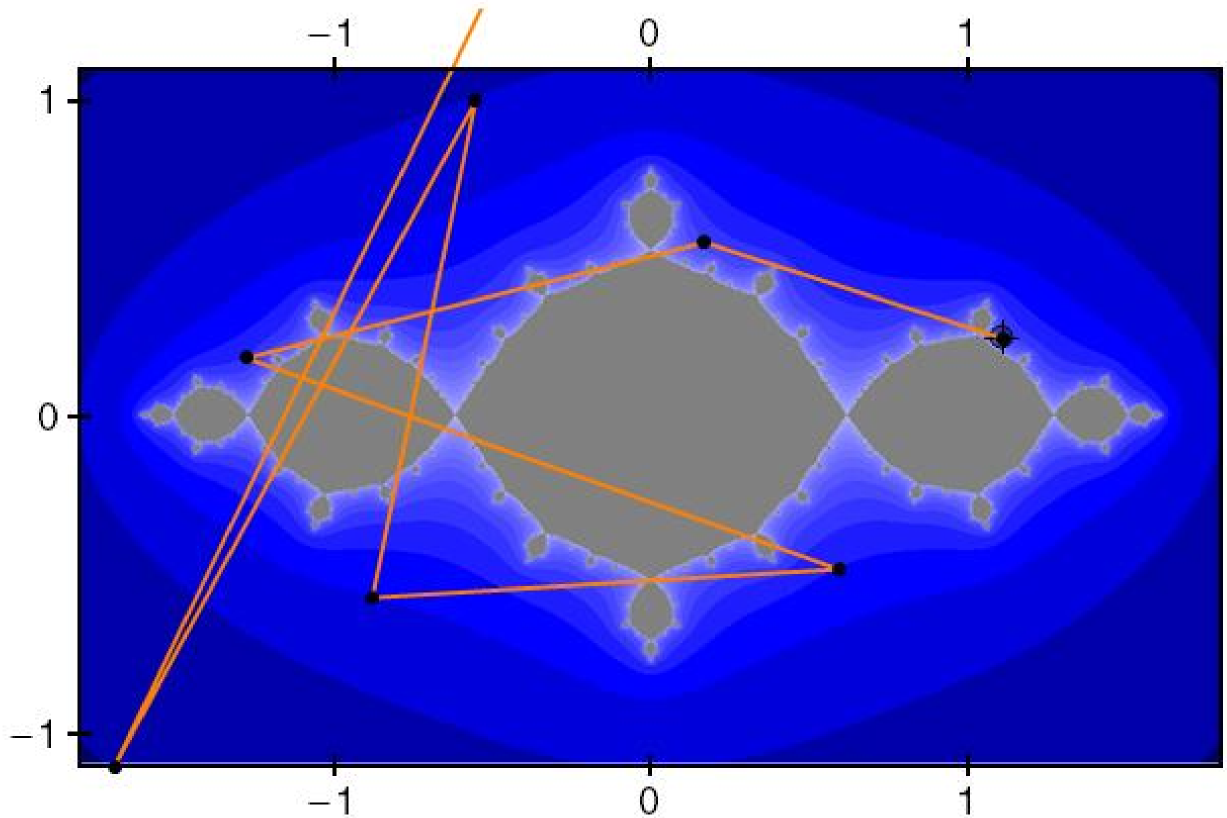


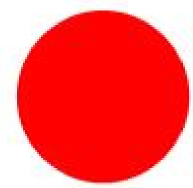


Orbits Under $z^2 - 1$

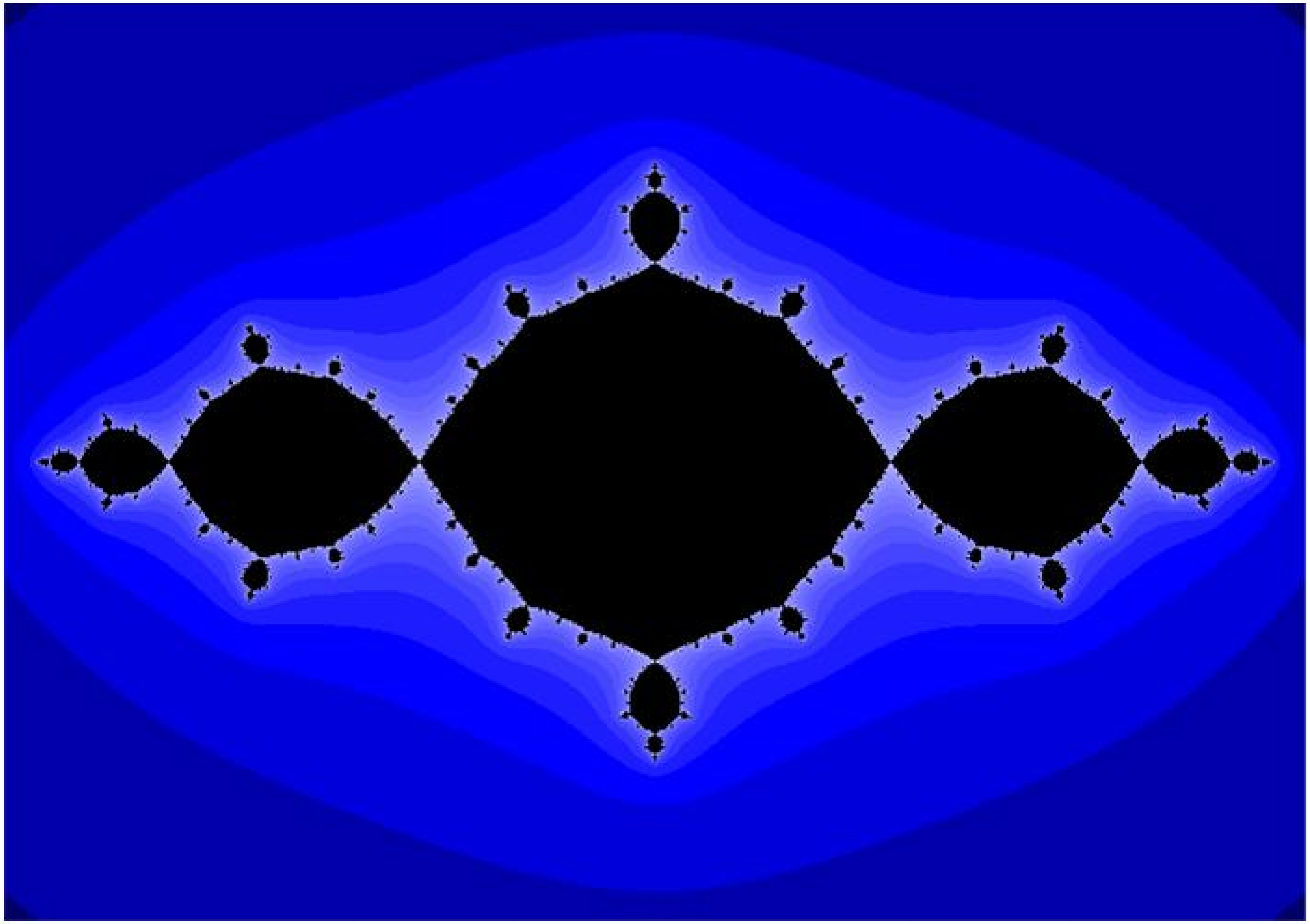


Orbits Under $z^2 - 1$



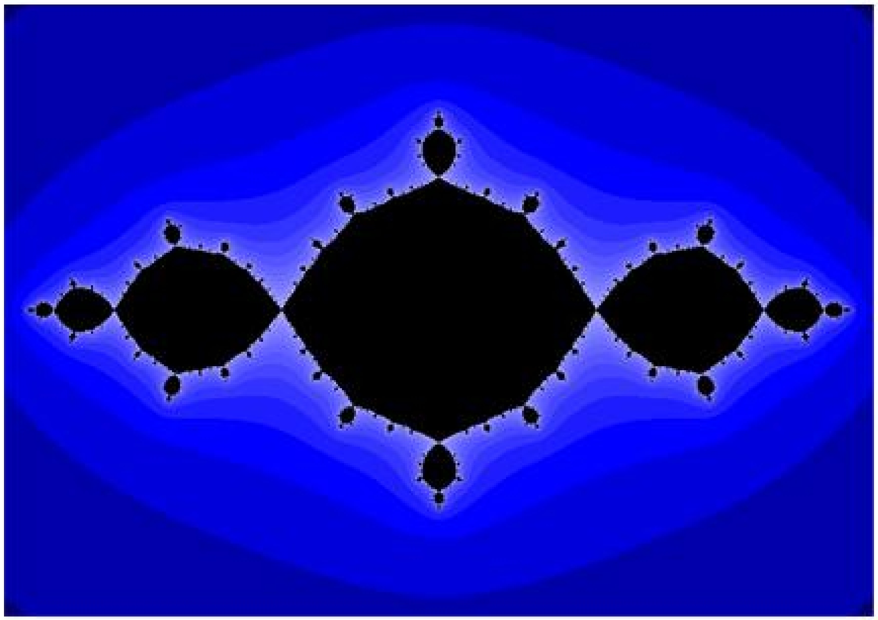


Filled Julia Set for $z^2 - 1$

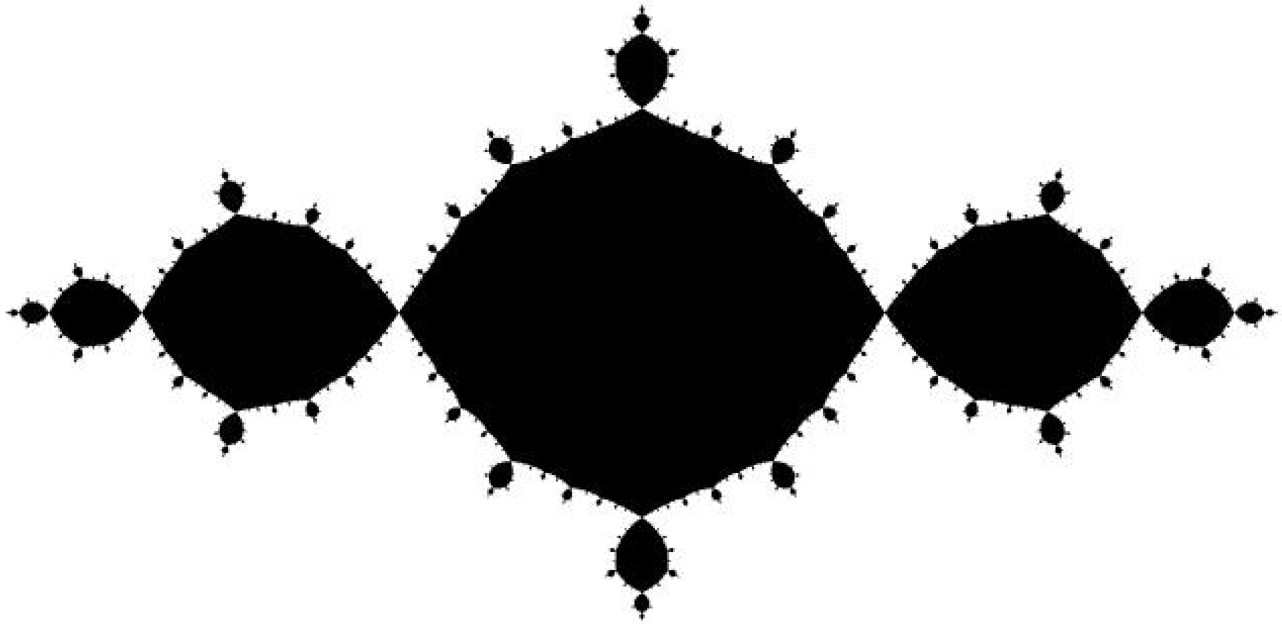


Filled Julia Set for $z^2 - 1$

This Julia set is known as the ***Basilica***.

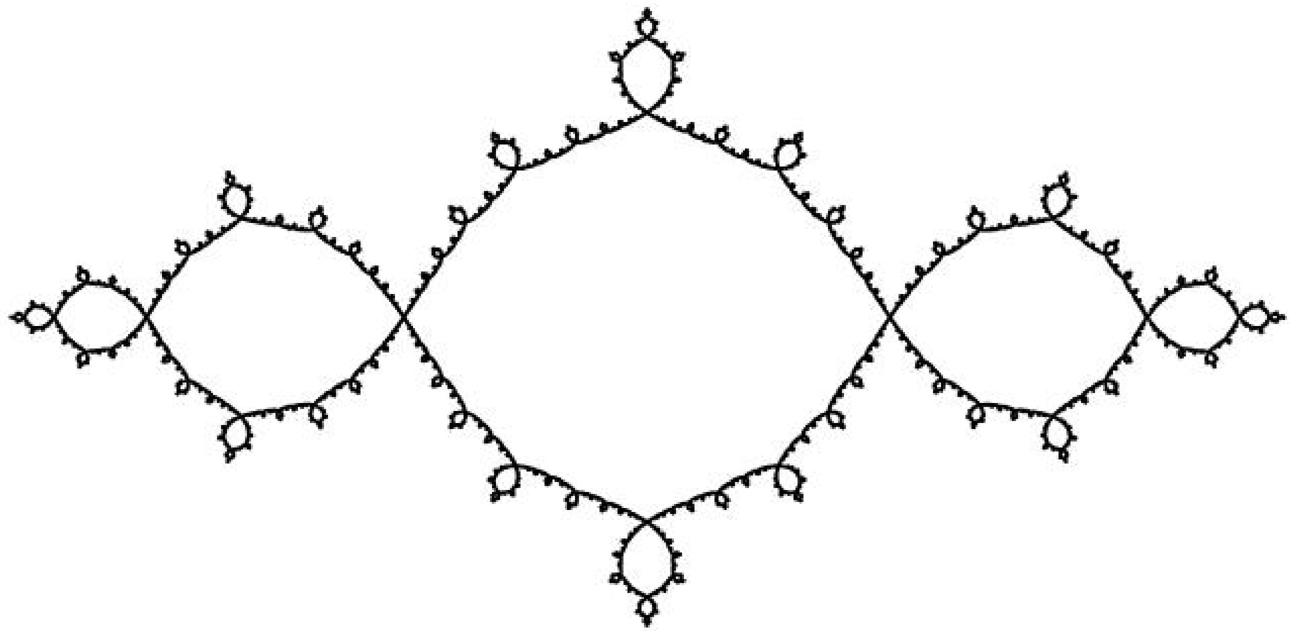


Filled Julia Set



^

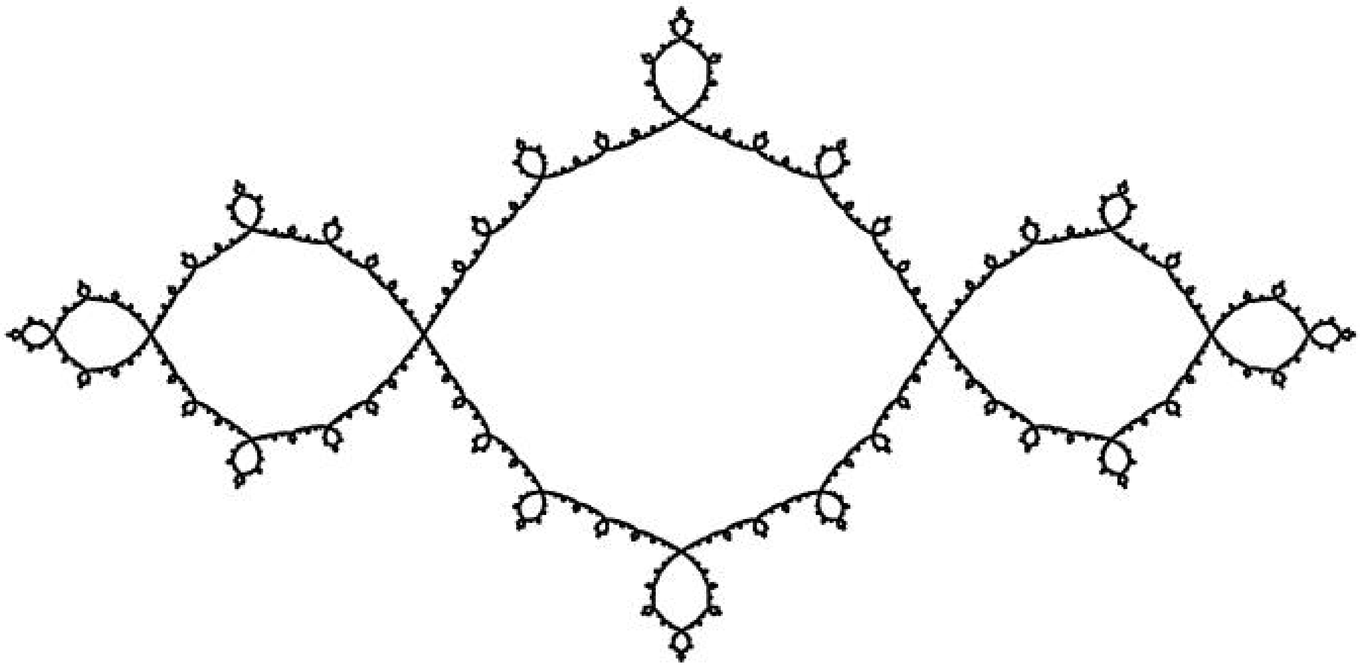
Julia Set



^

Julia Sets: The Basilica

Example: The Julia set for $f(z) = z^2 - 1$ is called the ***Basilica***.

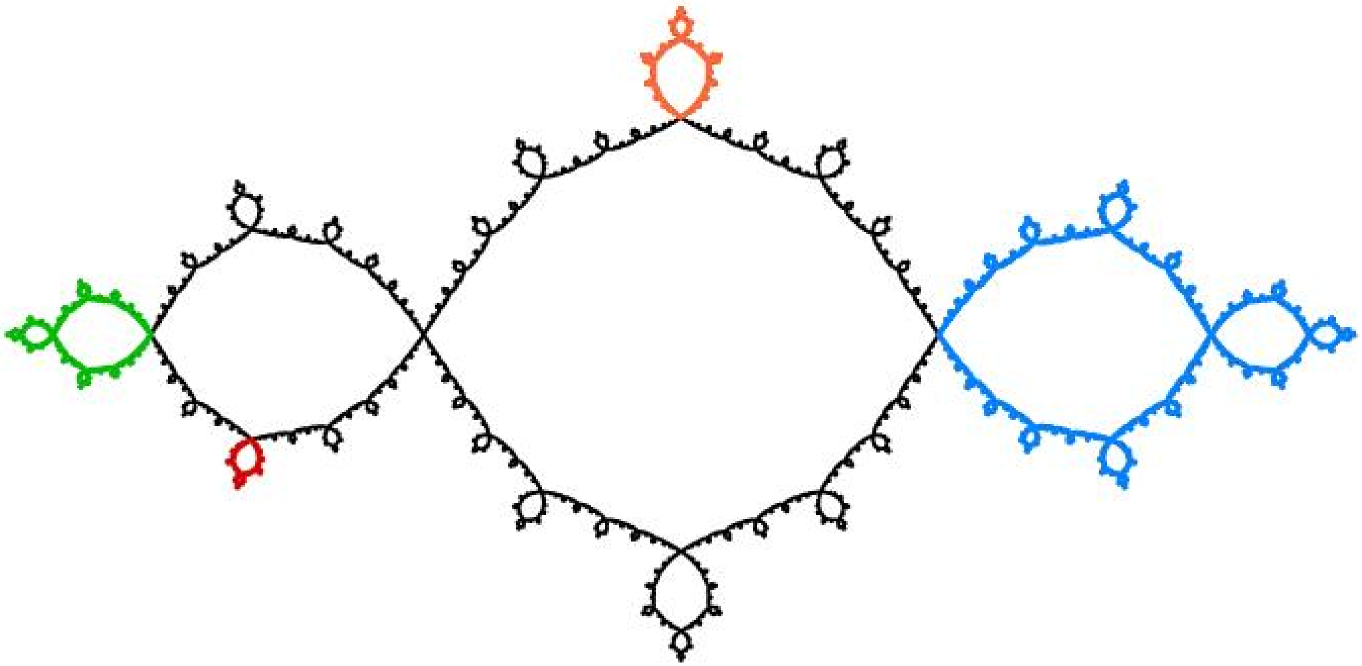


It is the simplest example of a fractal Julia set.

^

Julia Sets: The Basilica

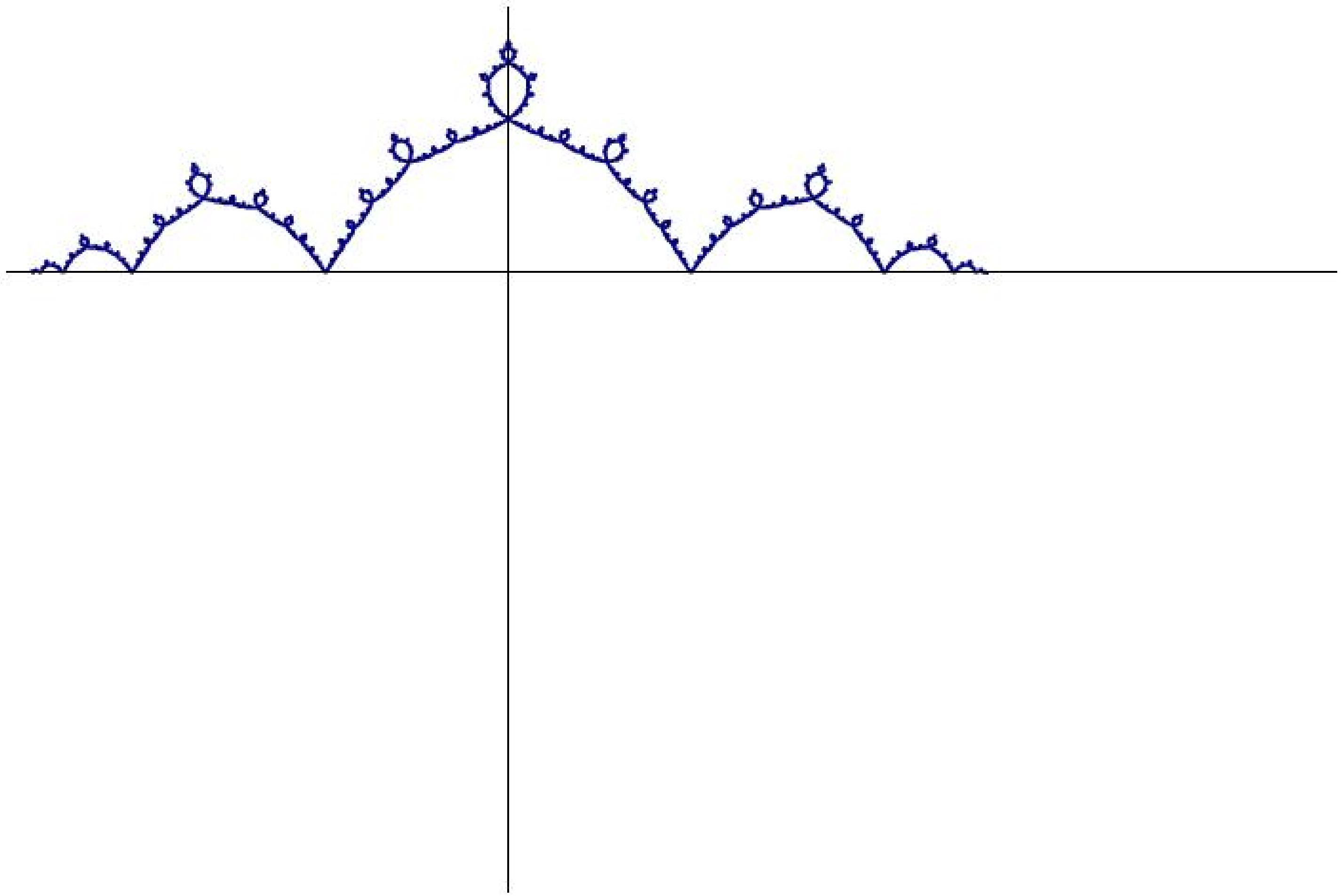
The Julia set has a self-similar structure:



Note: The “similarity” maps are not actually linear.

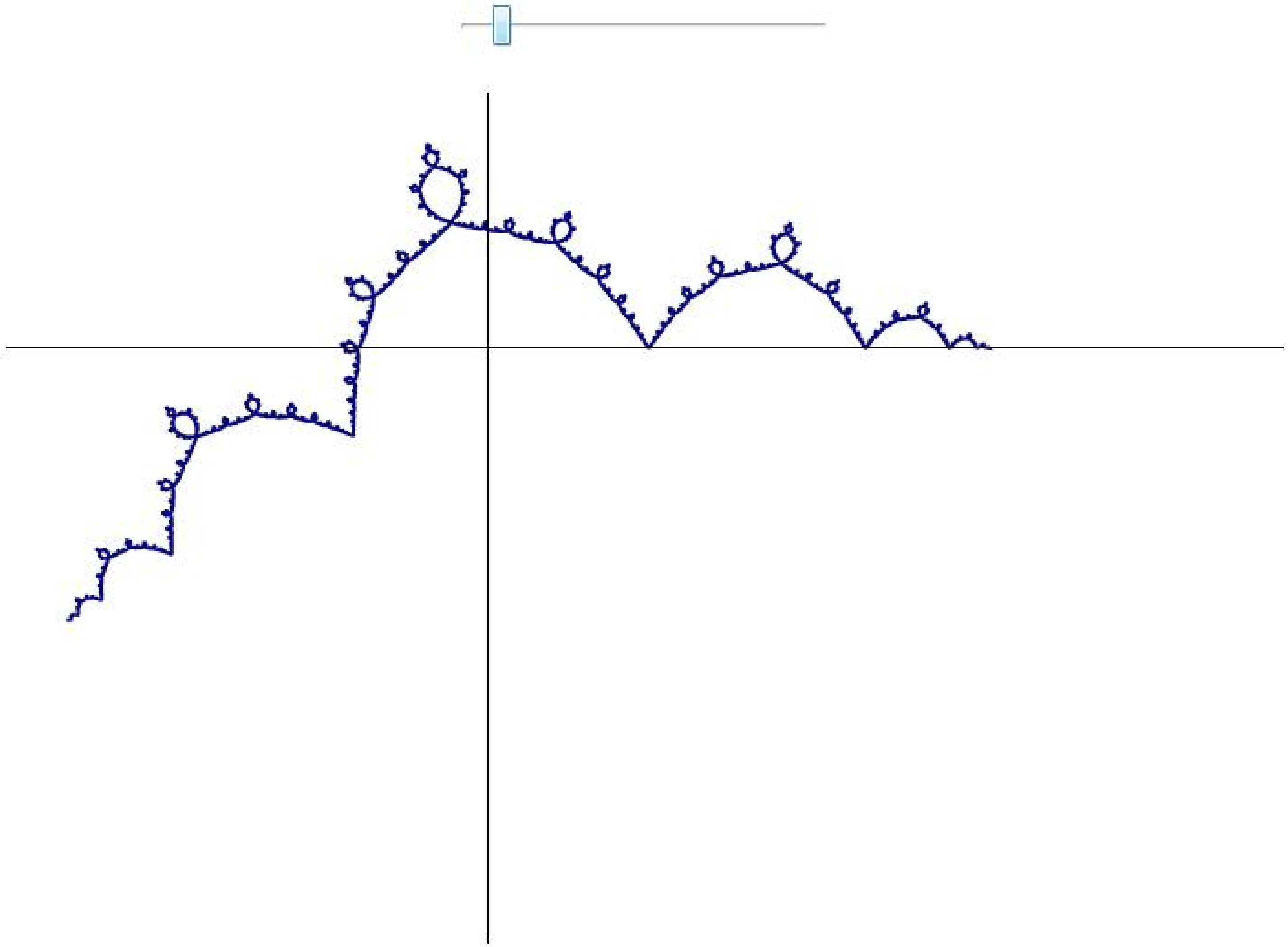
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



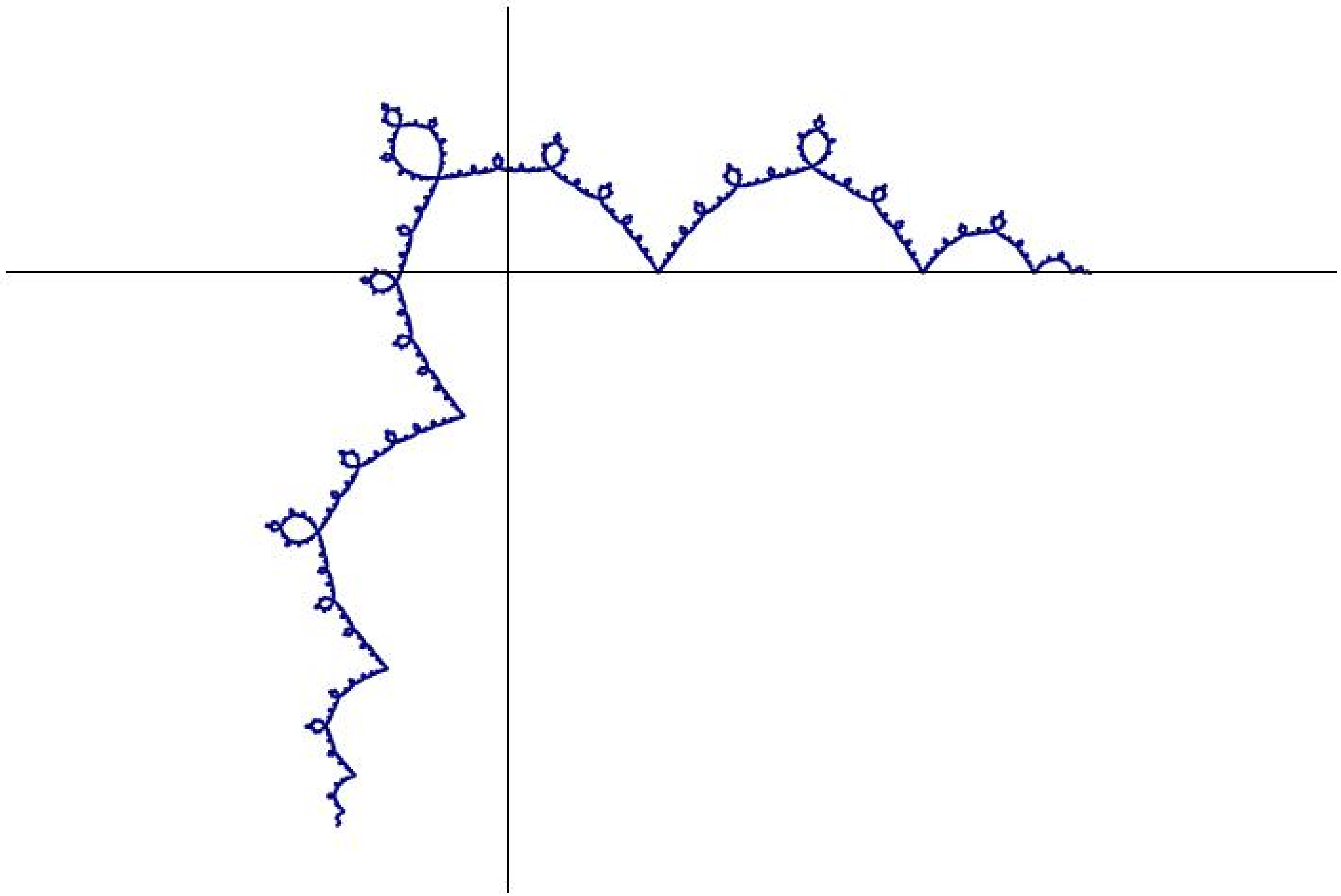
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



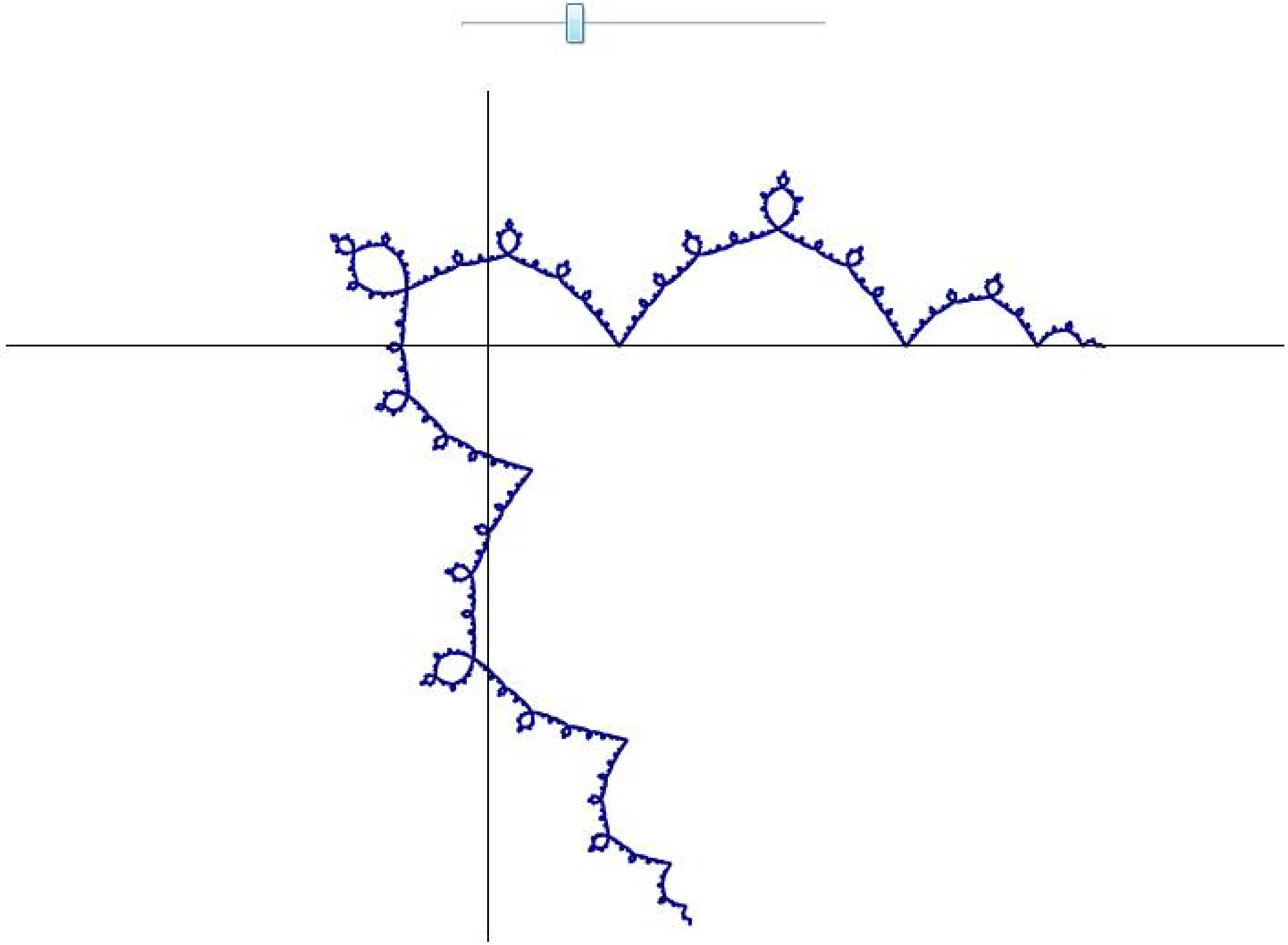
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



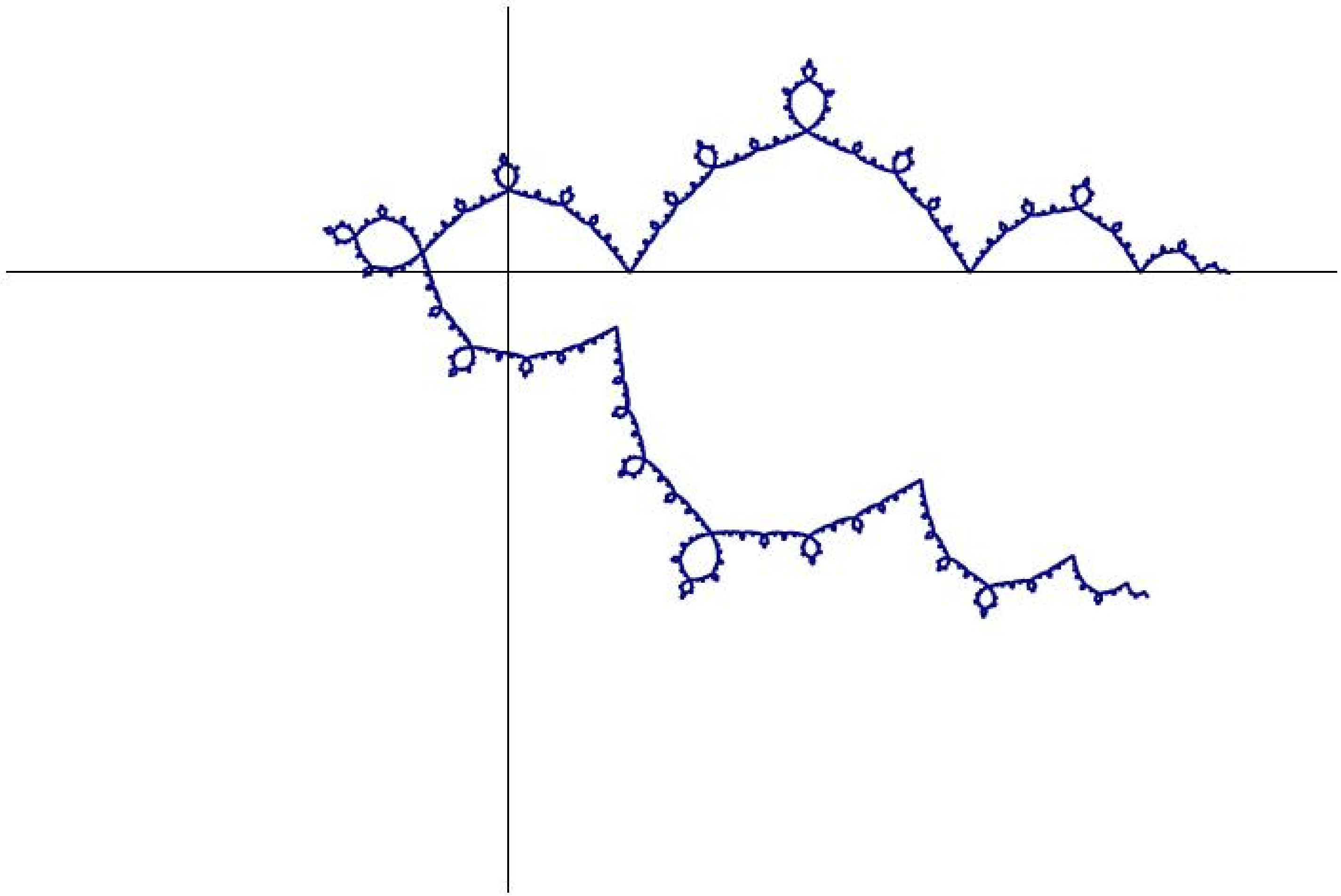
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



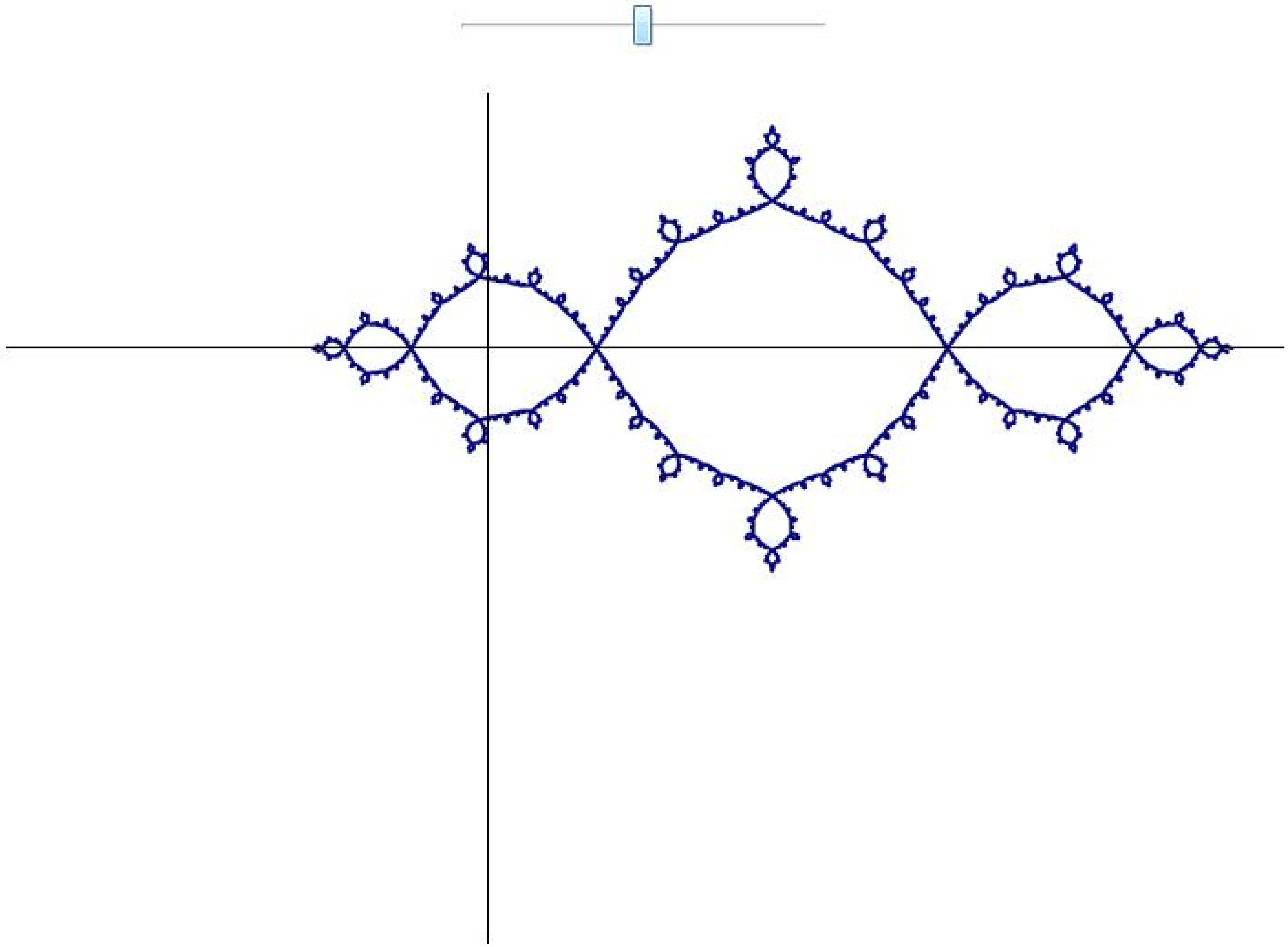
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



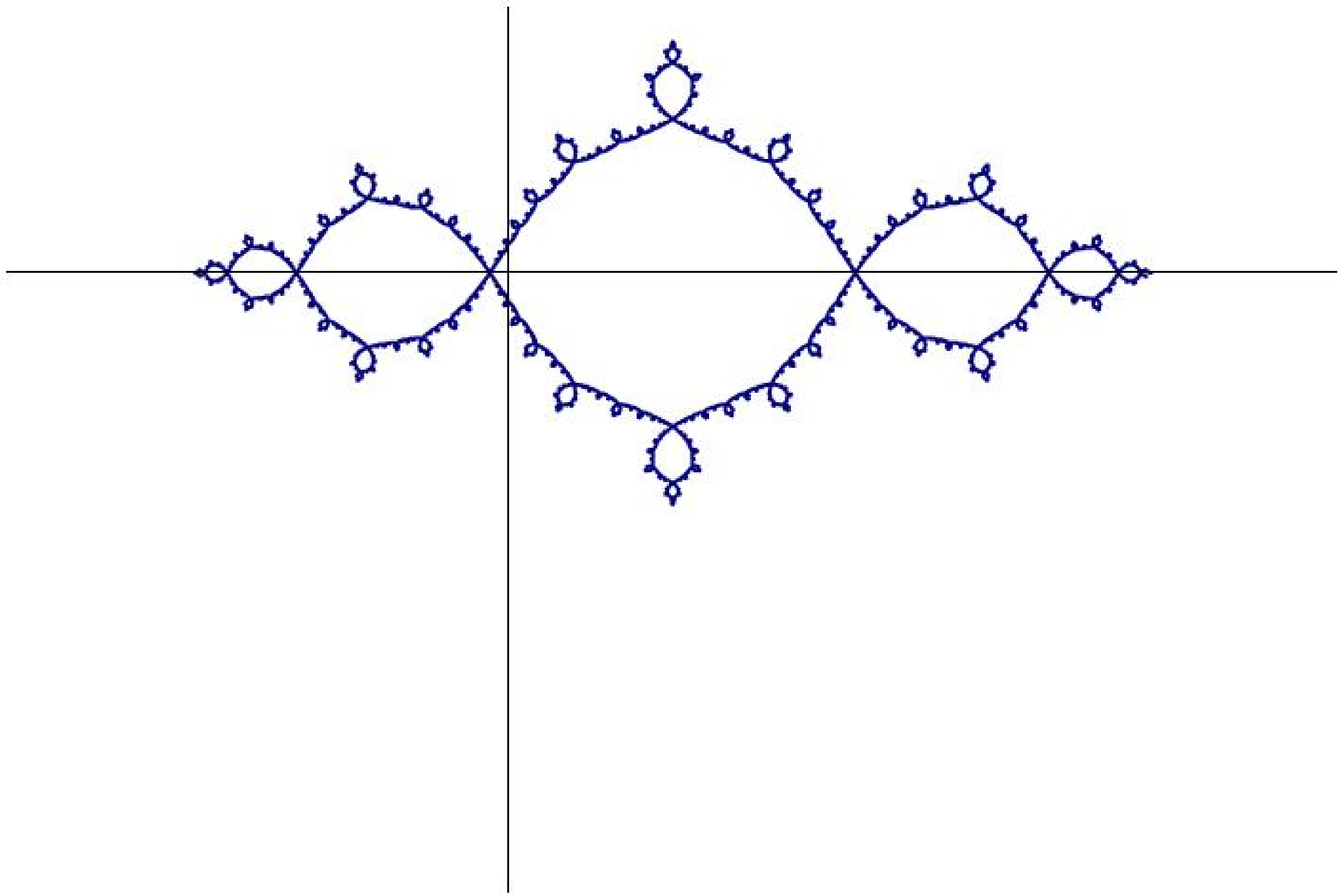
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



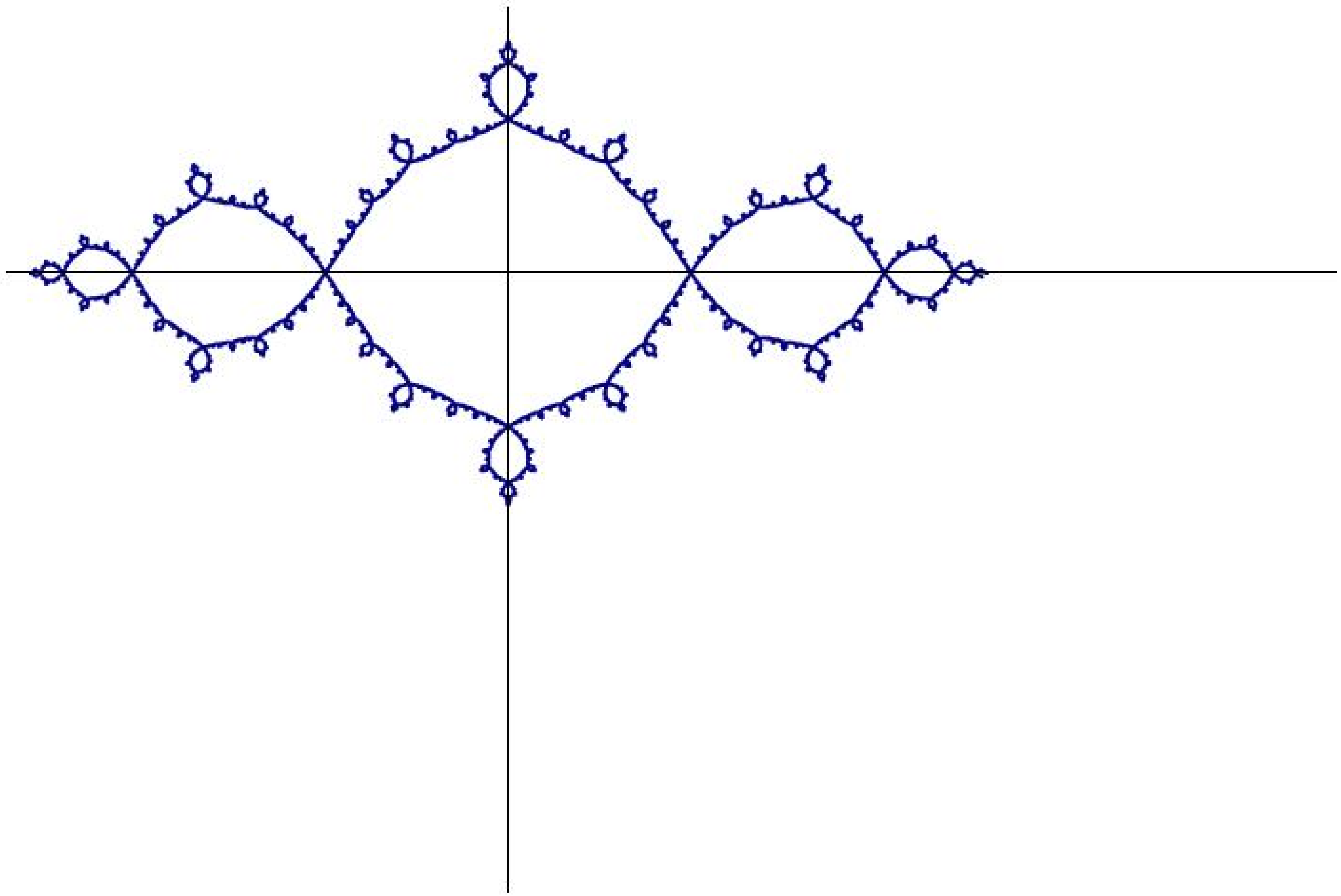
Invariance of the Basilica

The Basilica maps to itself under $f(z) = z^2 - 1$.



Invariance of the Basilica

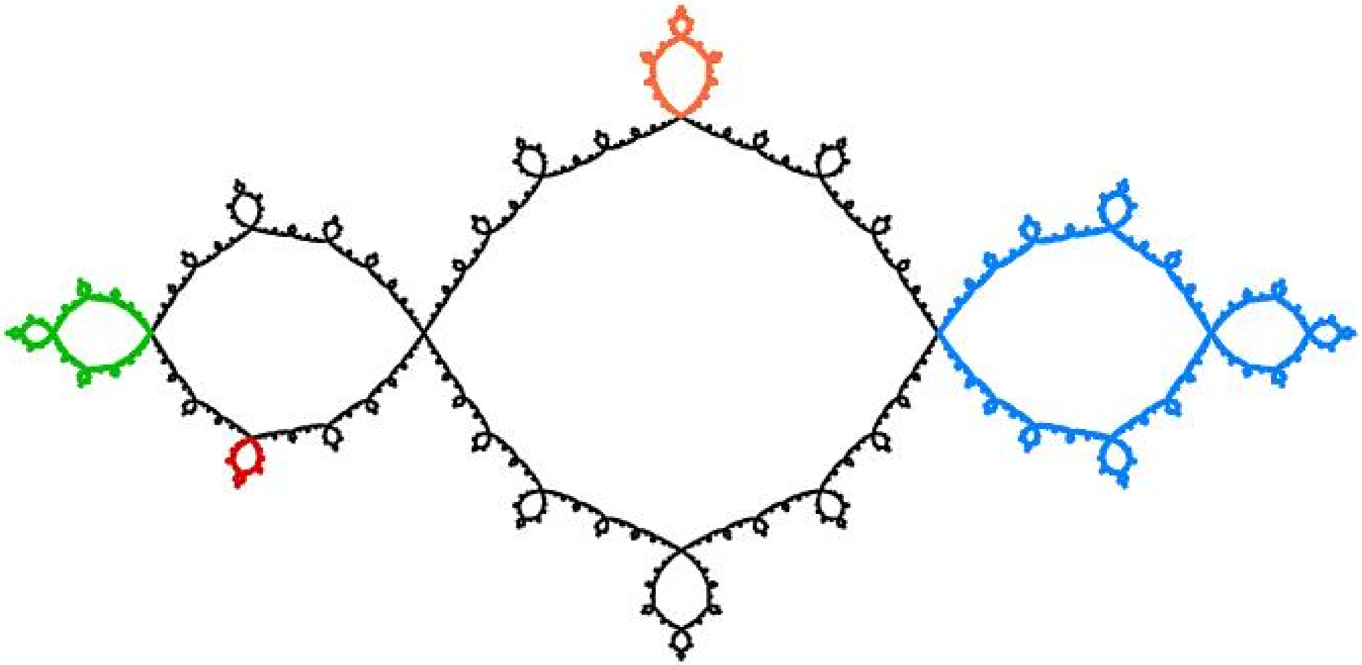
The Basilica maps to itself under $f(z) = z^2 - 1$.



^

Julia Sets: The Basilica

The “self-similarities” of the Basilica are ***conformal maps***.



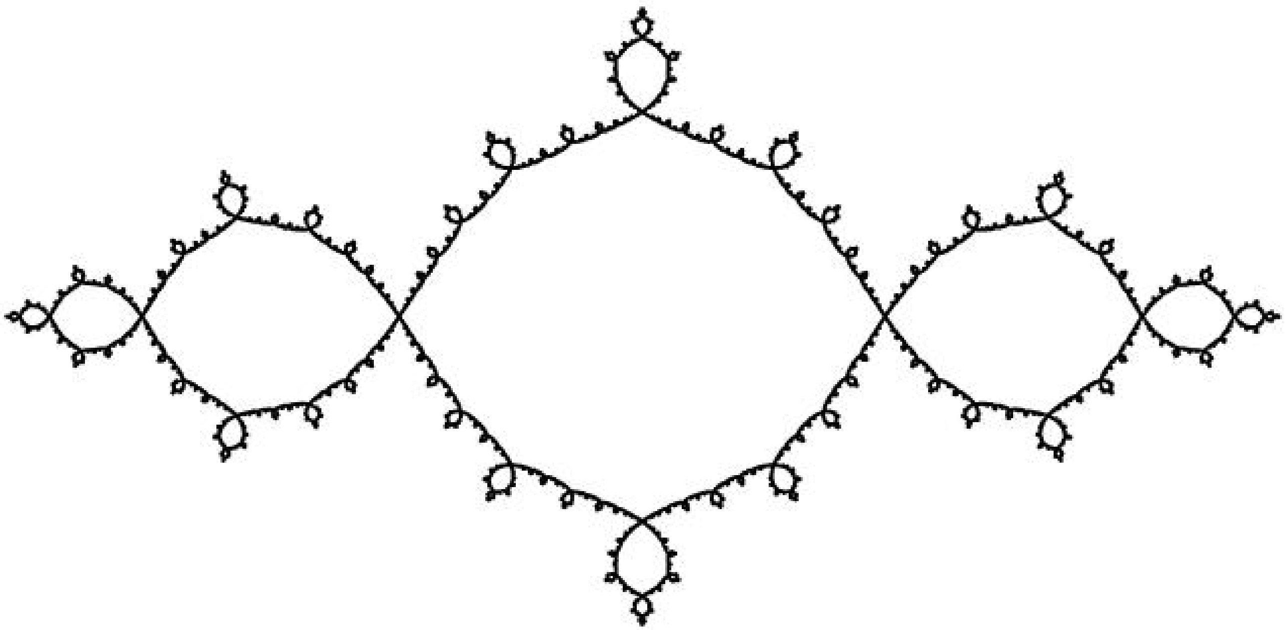
We want a Thompson-like group of ***piecewise-conformal*** homeomorphisms.

The Basilica Group

^

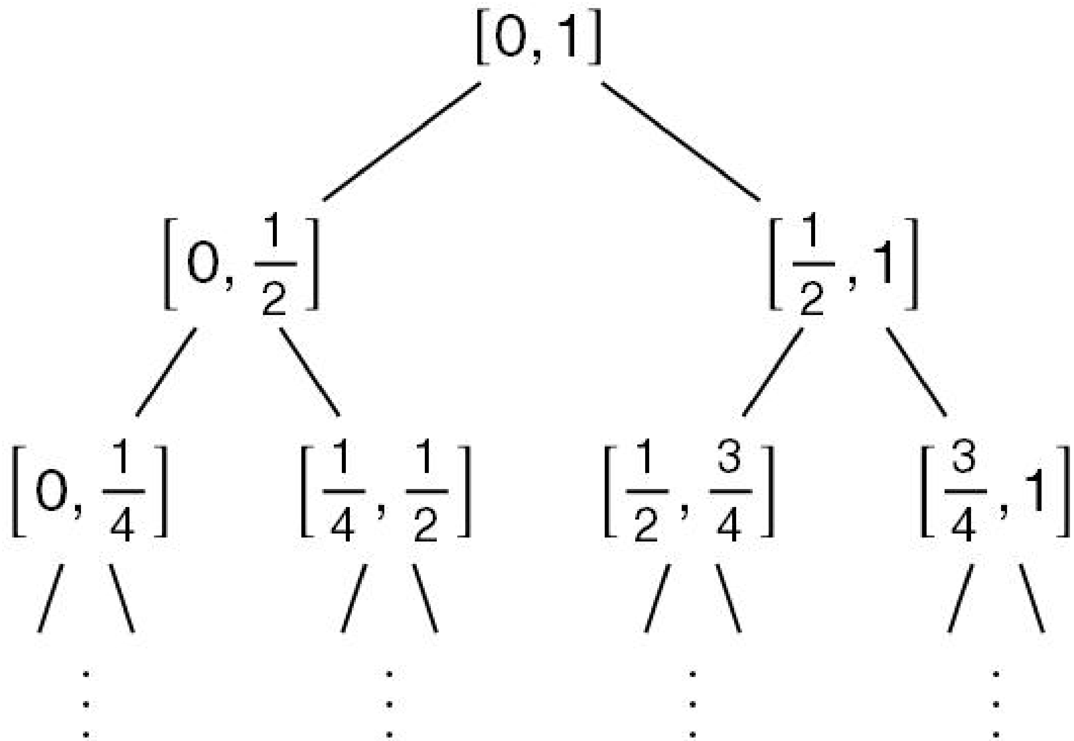
The Basilica Group

We need an analogue of “dyadic subdivision” for the Basilica.



The Basilica Group

Dyadic subdivisions of $[0,1]$ are made of ***standard dyadic intervals***.

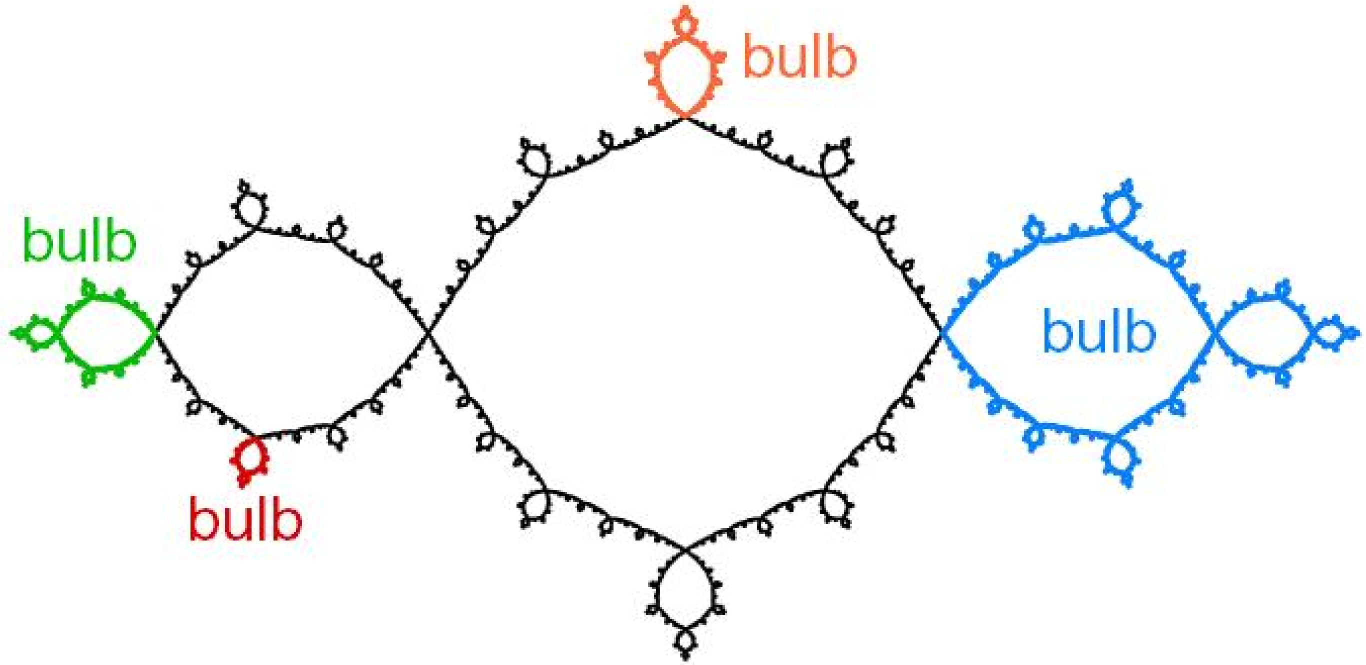


We need similar “building blocks” for the Basilica.

^

Bulbs and Edges

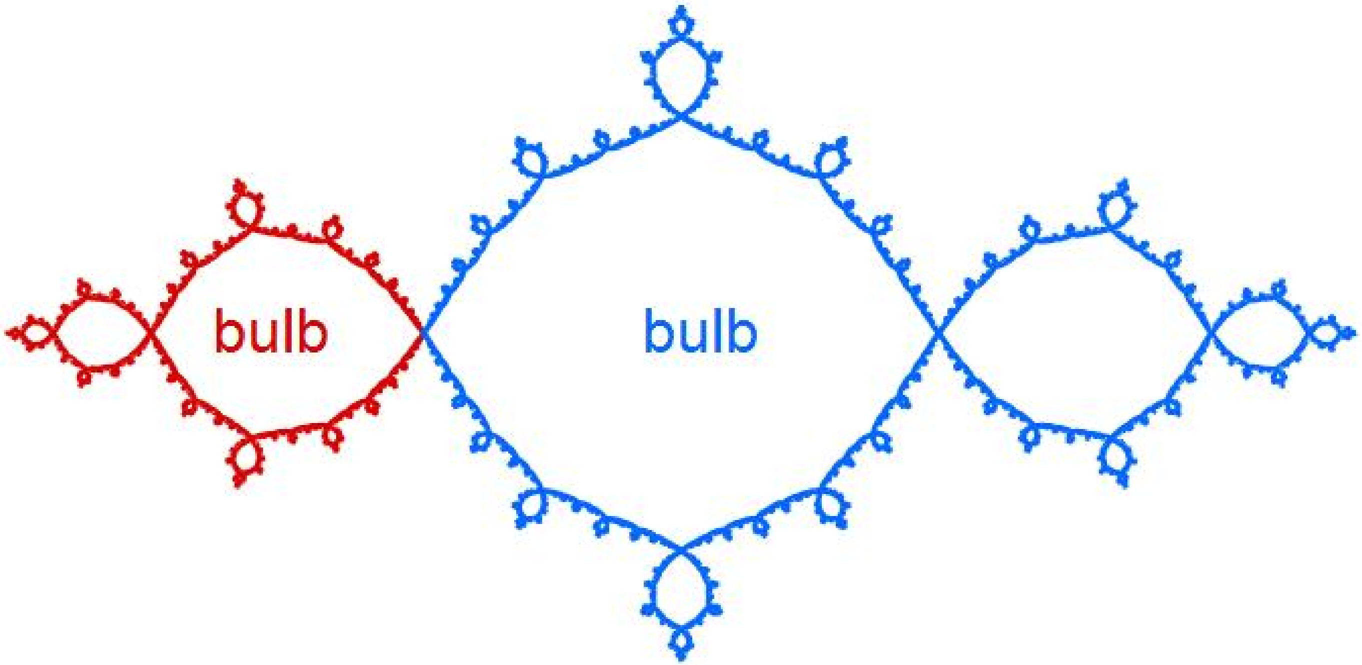
These structures are all conformally isomorphic. Let's call them ***bulbs***.



^

Bulbs and Edges

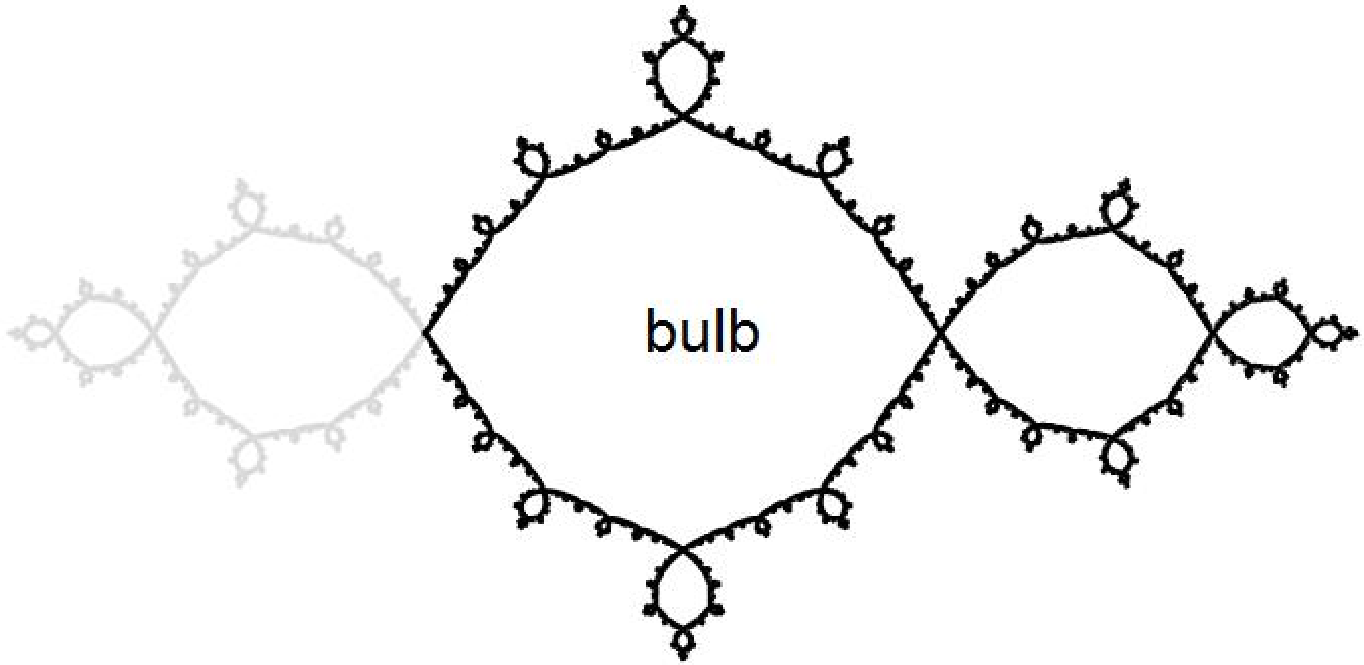
The Basilica is actually the union of two bulbs.



^

Bulbs and Edges

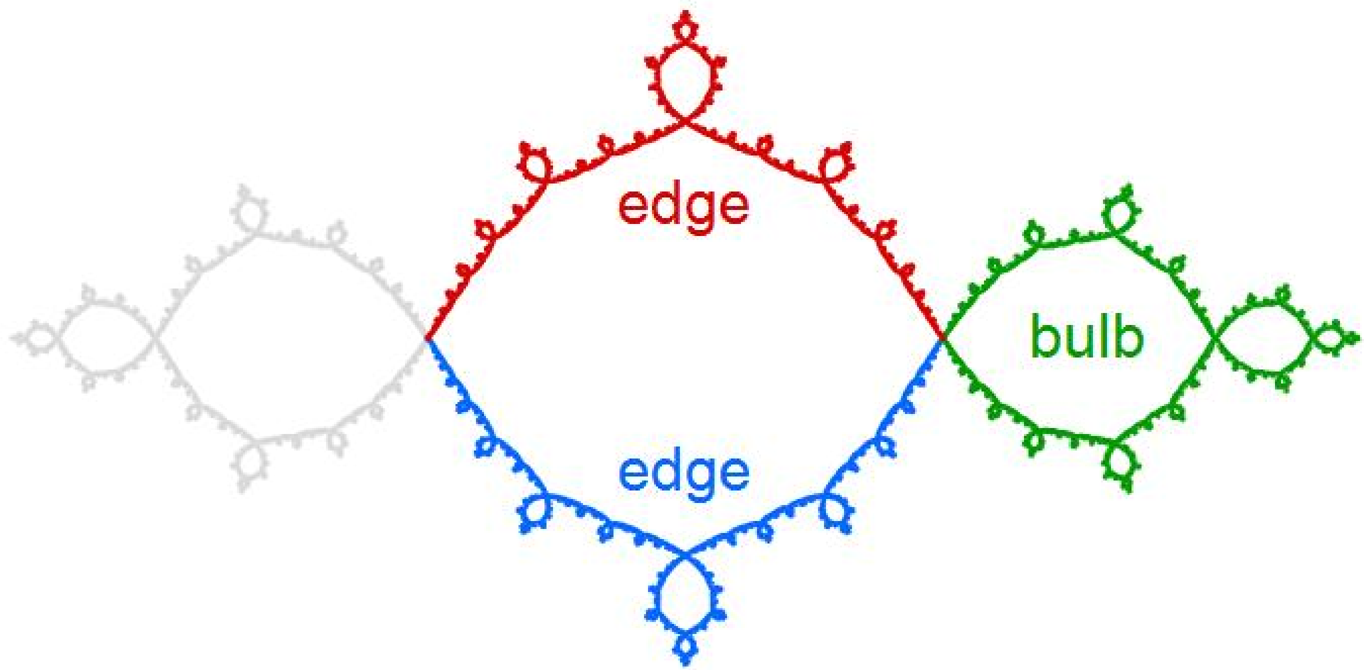
Each bulb can be partitioned into one smaller bulb and two ***edges***.



^

Bulbs and Edges

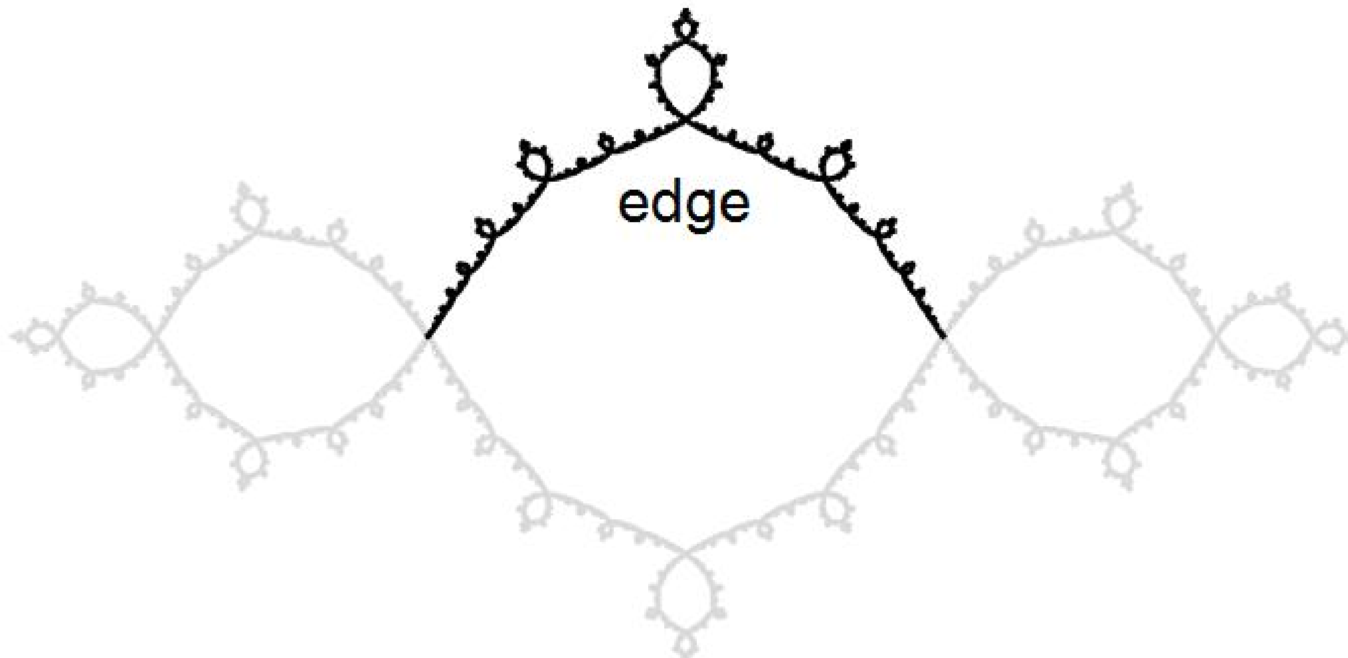
Each bulb can be partitioned into one smaller bulb and two ***edges***.



^

Bulbs and Edges

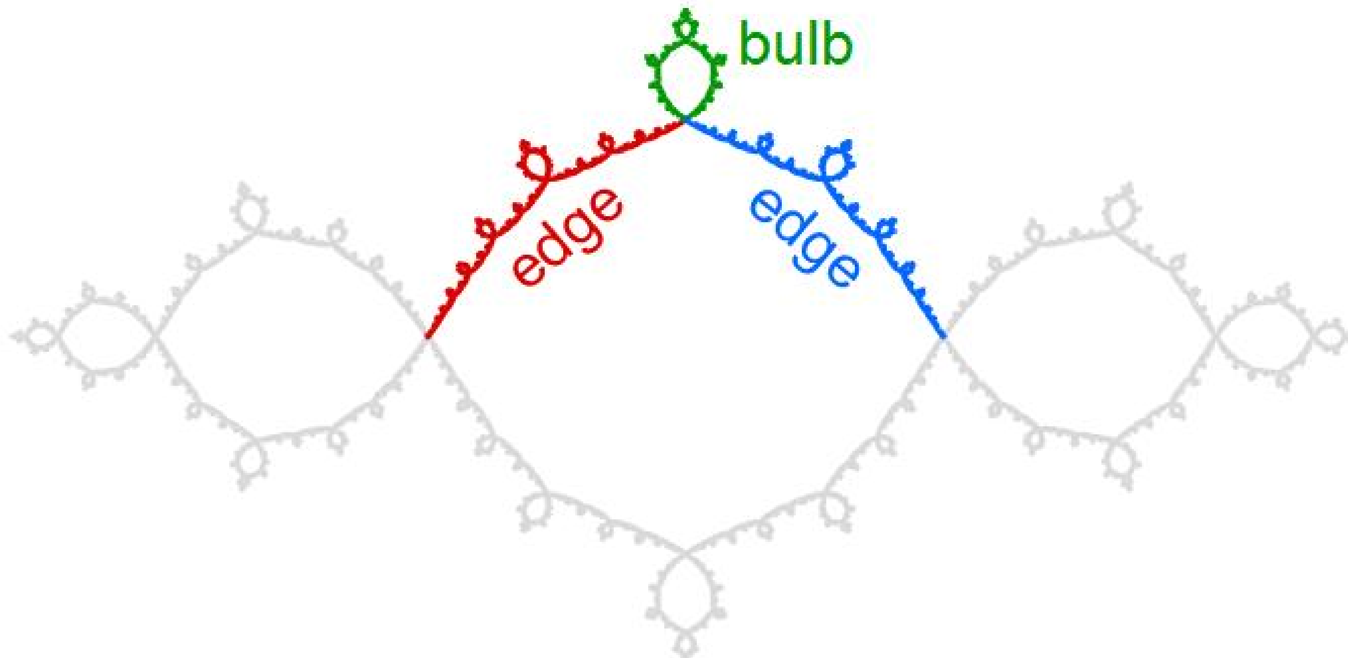
Similarly, each edge can be partitioned into one bulb and two edges.



^

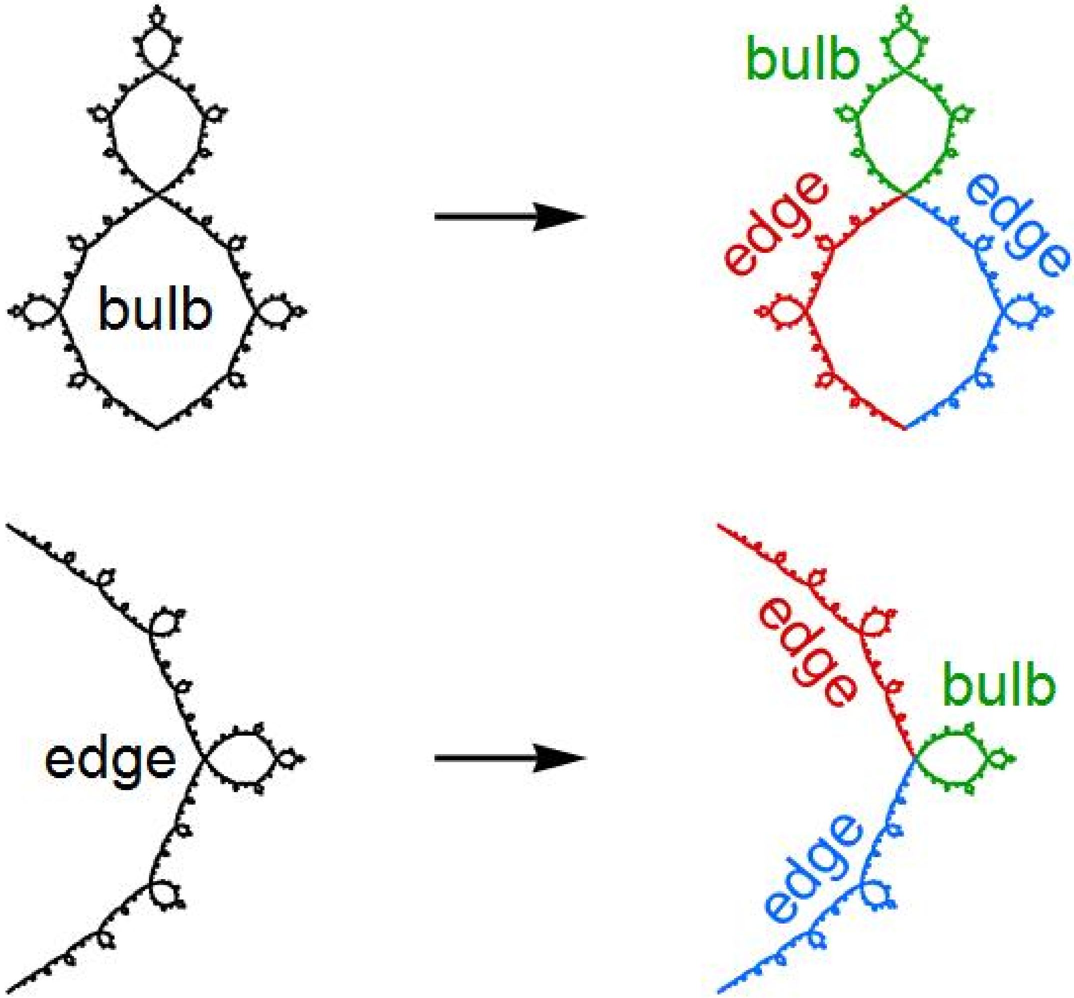
Bulbs and Edges

Similarly, each edge can be partitioned into one bulb and two edges.



^

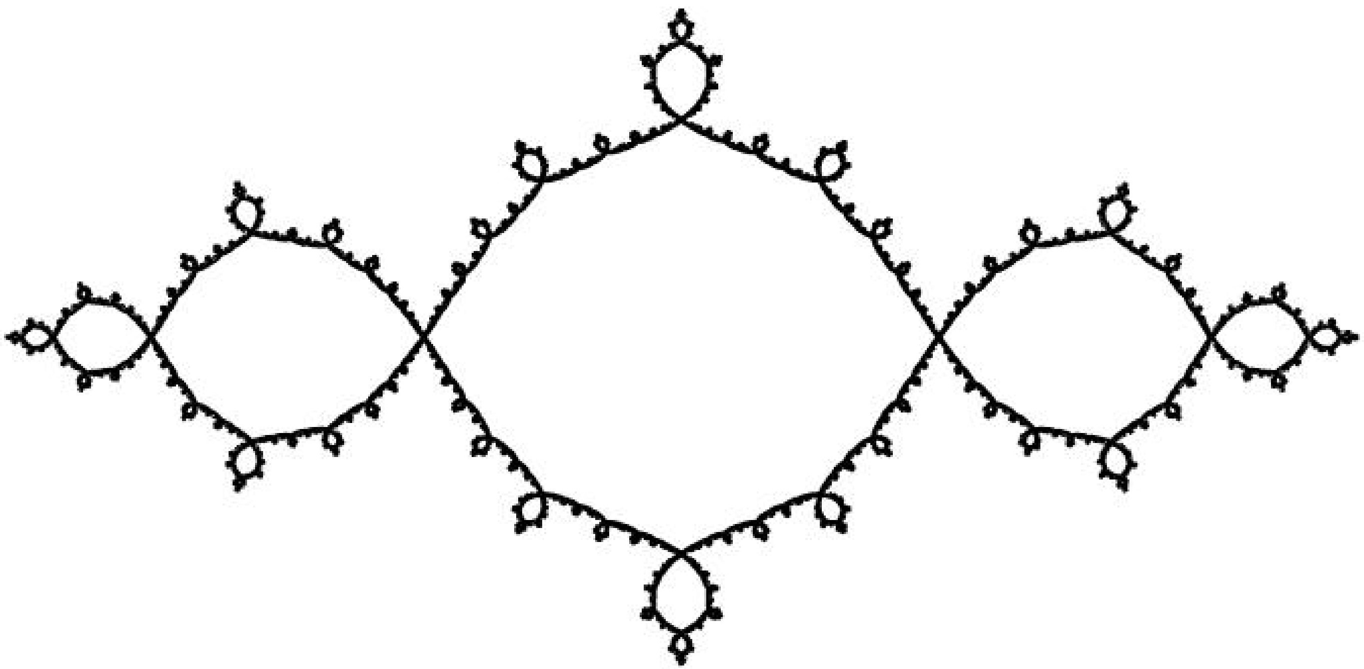
Two Subdivision Moves



^

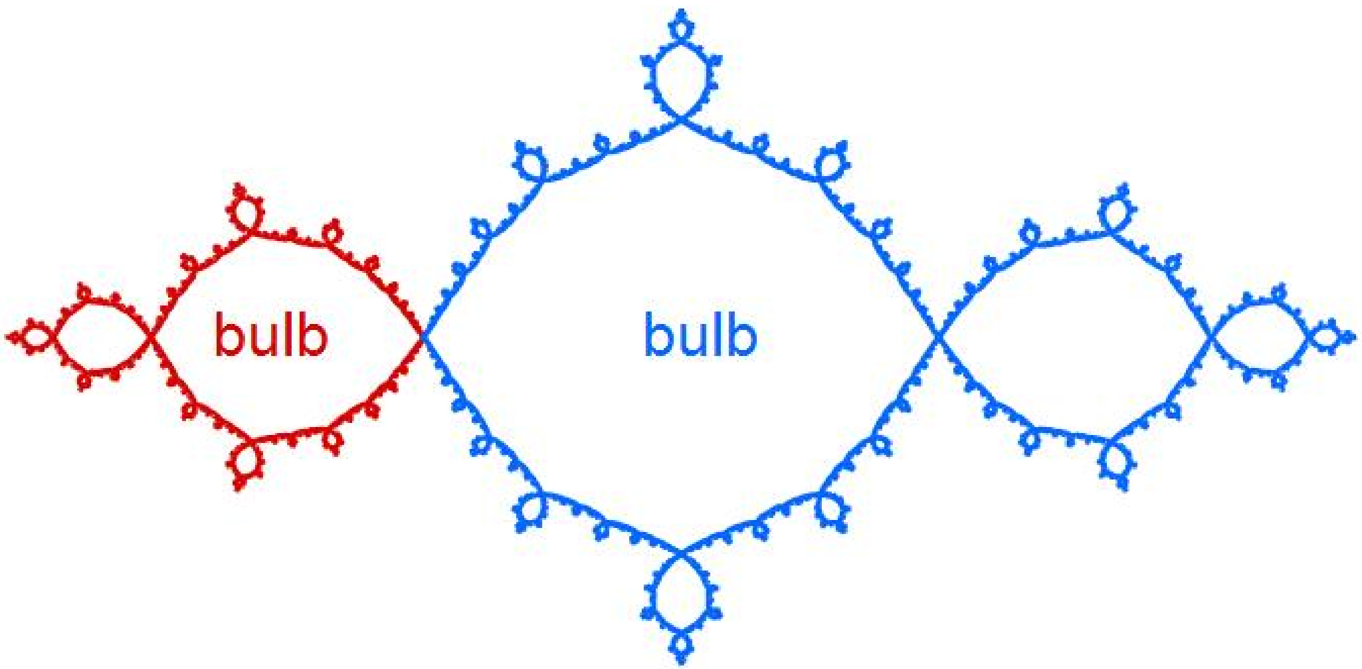
Allowed Subdivisions

An ***allowed subdivision*** of the Basilica is obtained by repeatedly applying these two moves, starting with the two main bulbs.



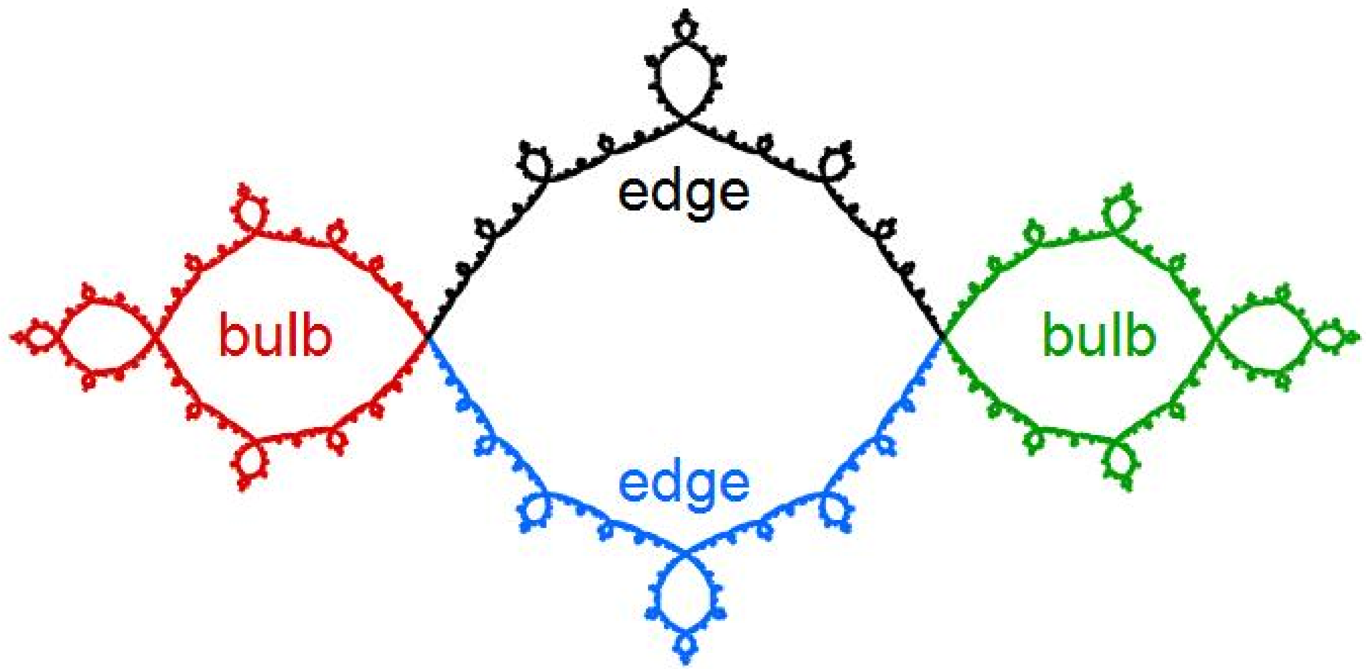
Allowed Subdivisions

An ***allowed subdivision*** of the Basilica is obtained by repeatedly applying these two moves, starting with the two main bulbs.



Allowed Subdivisions

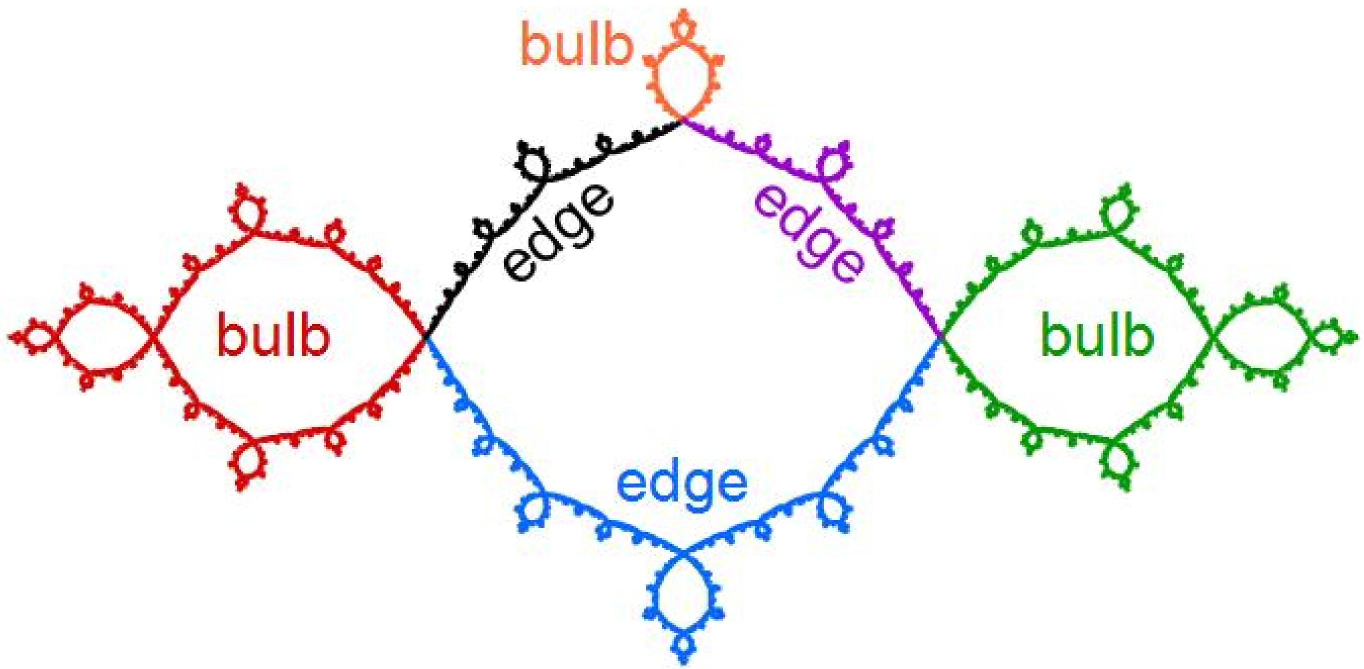
An ***allowed subdivision*** of the Basilica is obtained by repeatedly applying these two moves, starting with the two main bulbs.



^

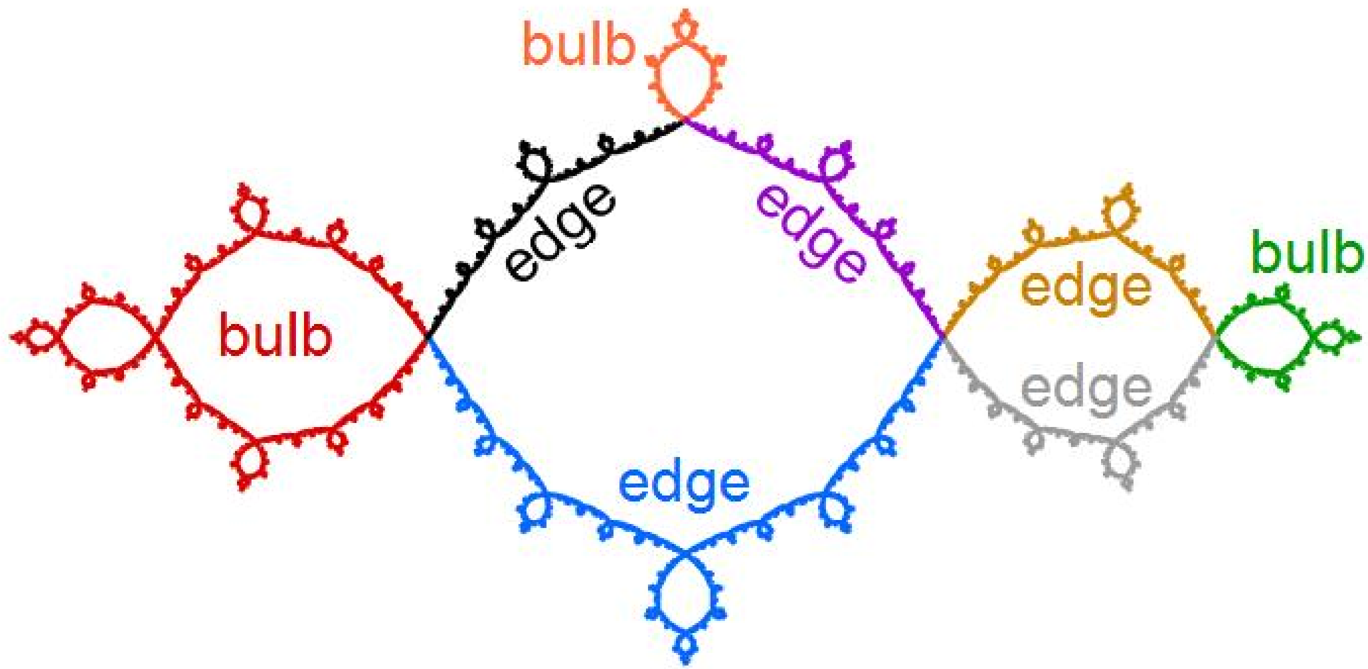
Allowed Subdivisions

An ***allowed subdivision*** of the Basilica is obtained by repeatedly applying these two moves, starting with the two main bulbs.



Allowed Subdivisions

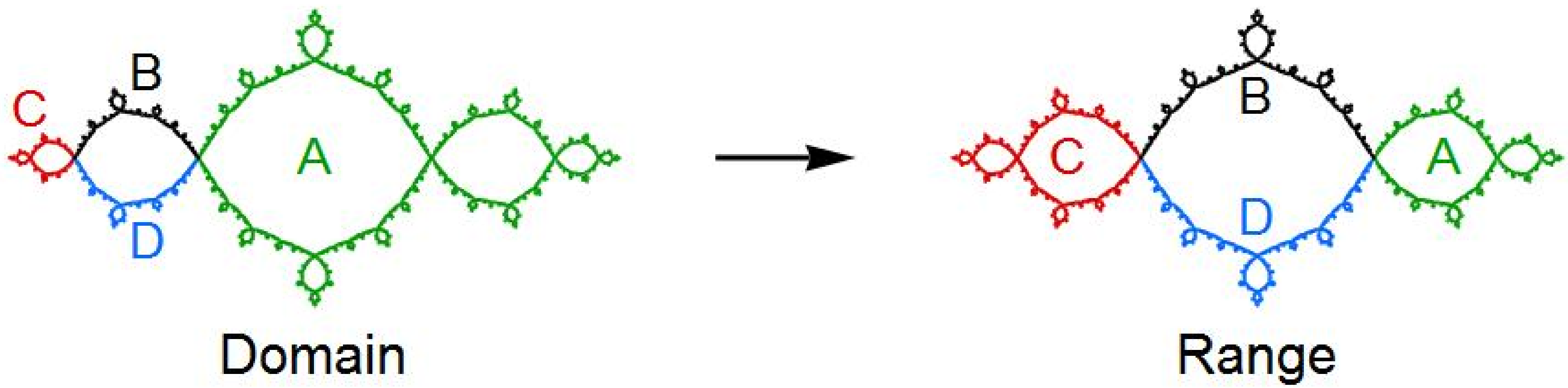
An ***allowed subdivision*** of the Basilica is obtained by repeatedly applying these two moves, starting with the two main bulbs.



^

The Basilica Group

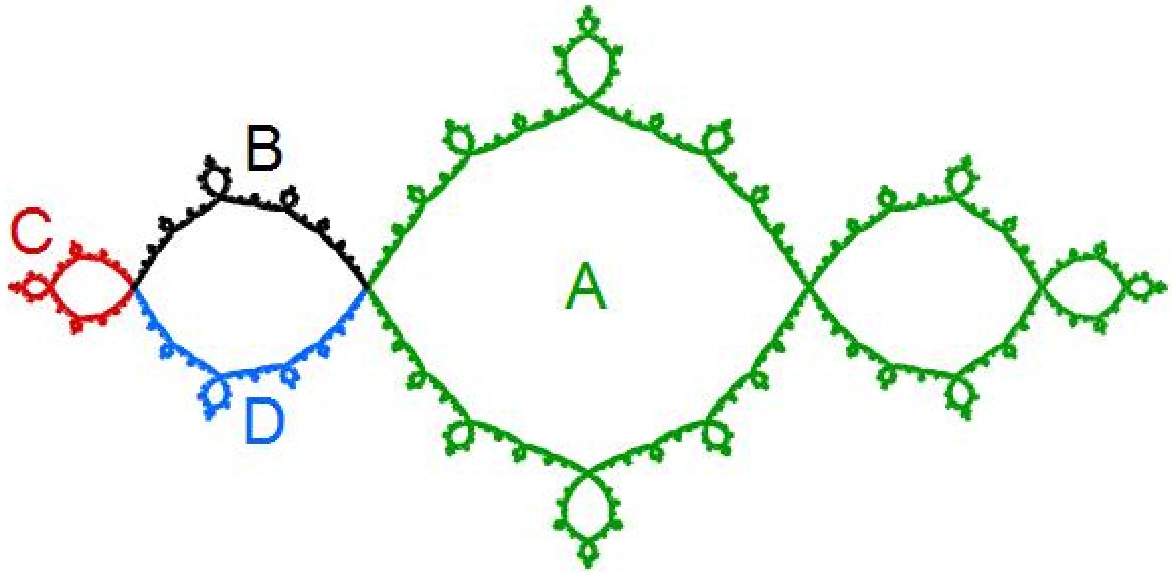
A ***rearrangement*** of the Basilica is a homeomorphism that maps conformally between the pieces of two allowed subdivisions.



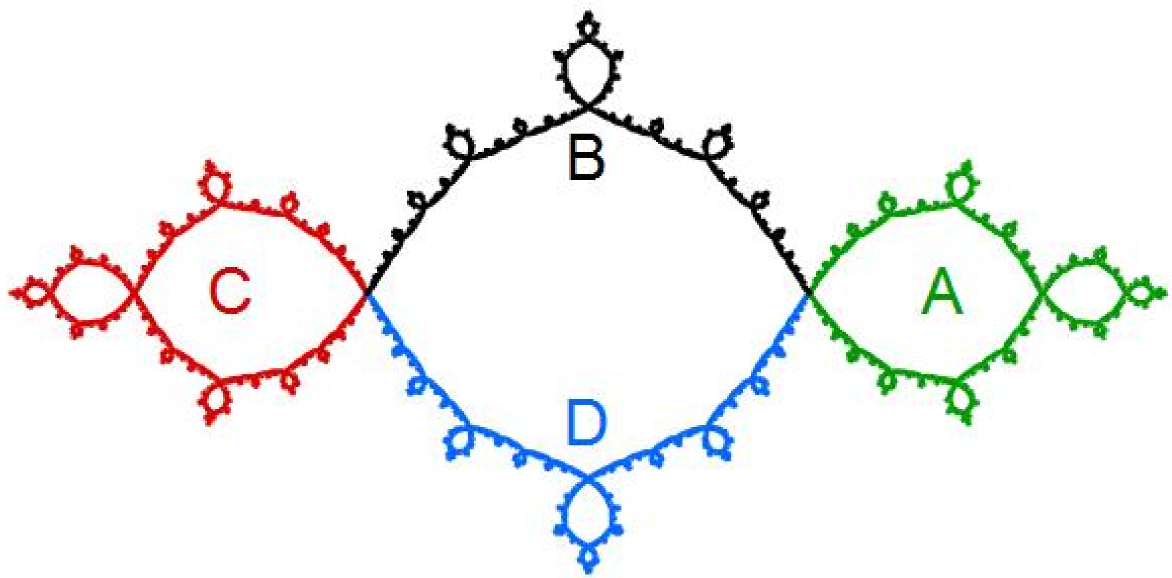
The group of all rearrangements is the ***Basilica Thompson group*** T_B.

Example Element

Domain:

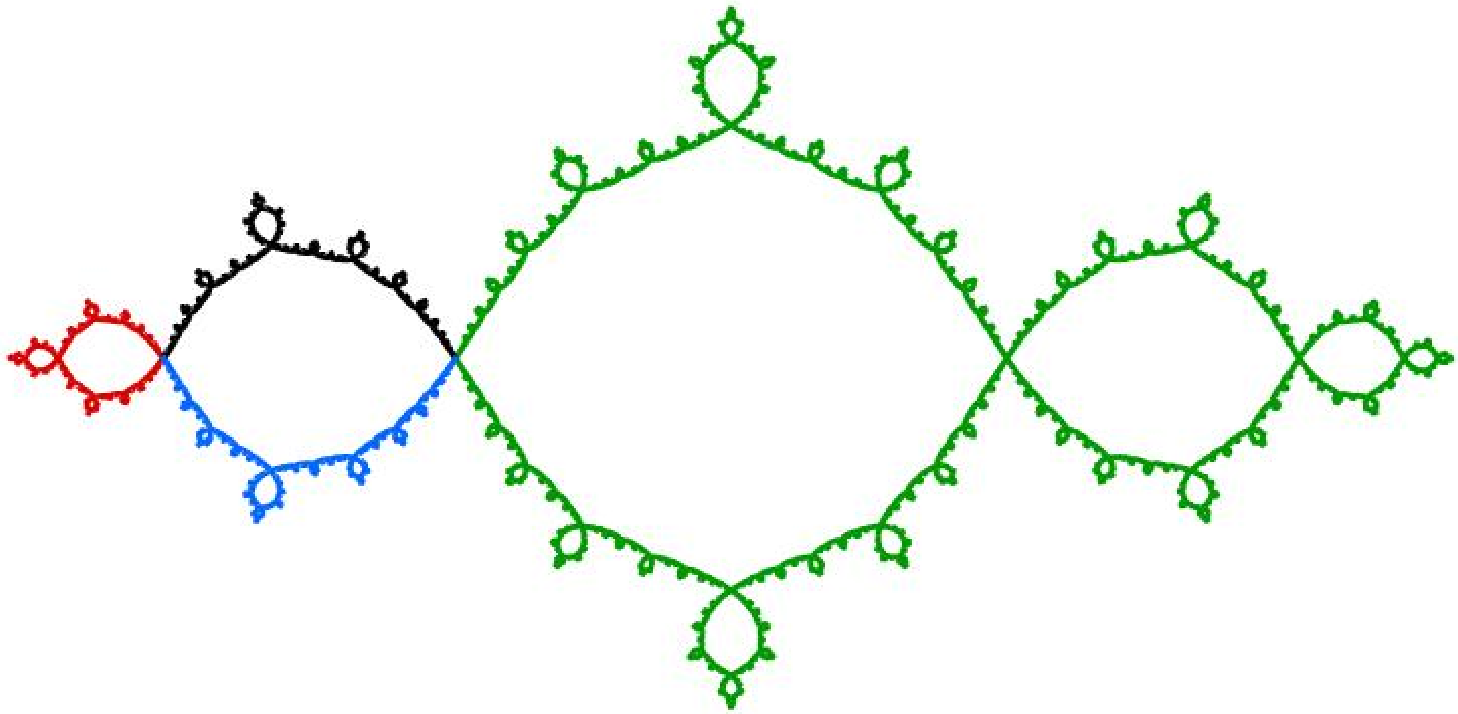


Range:



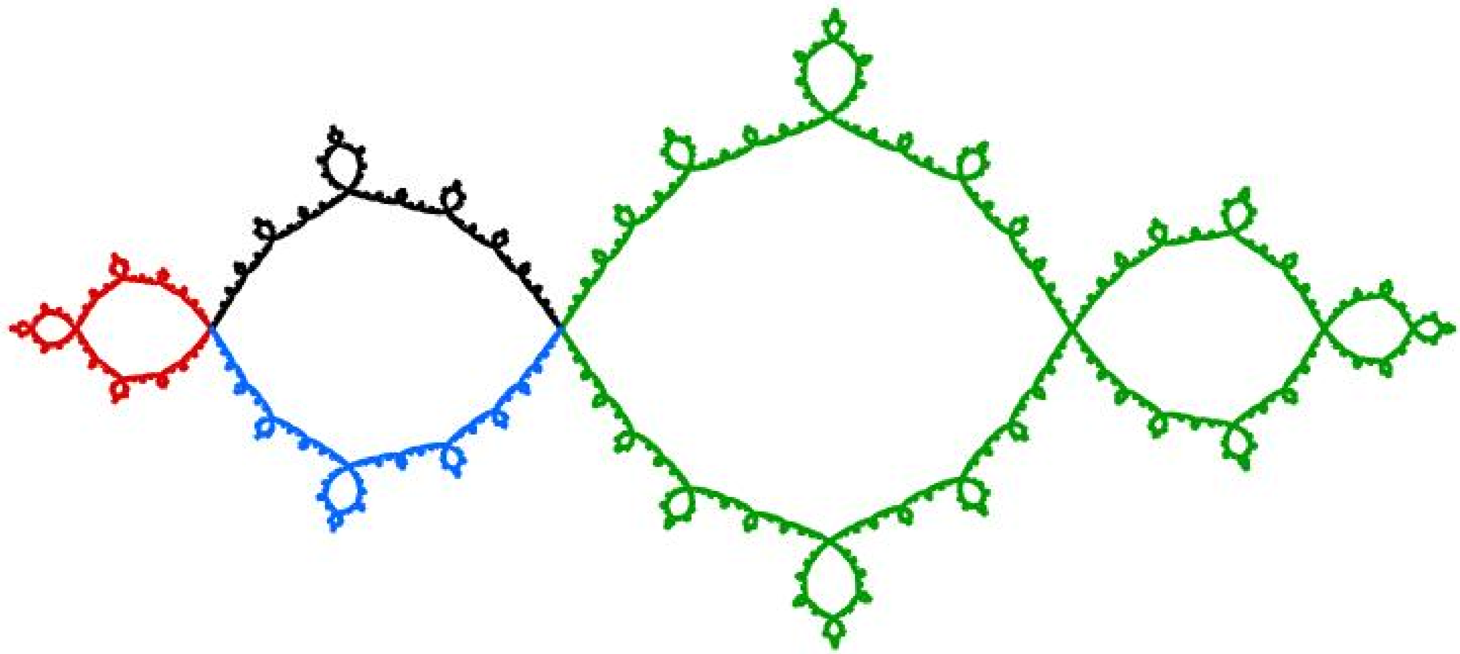
^

Example Element

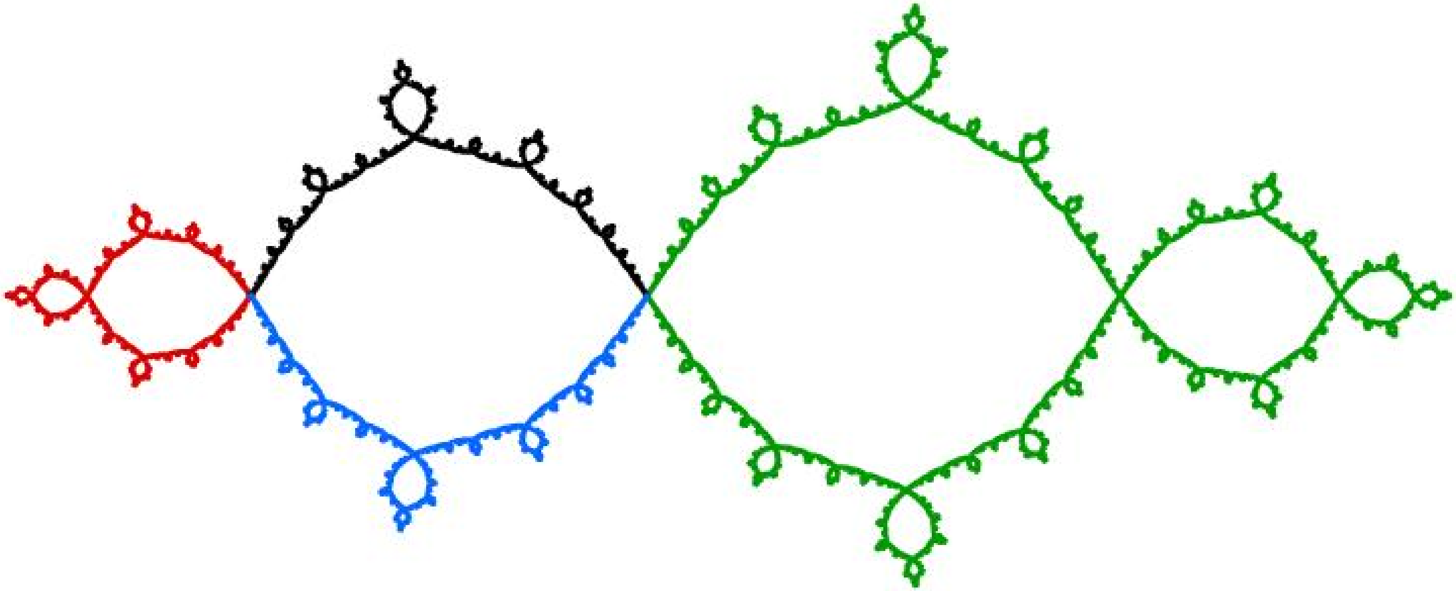


^

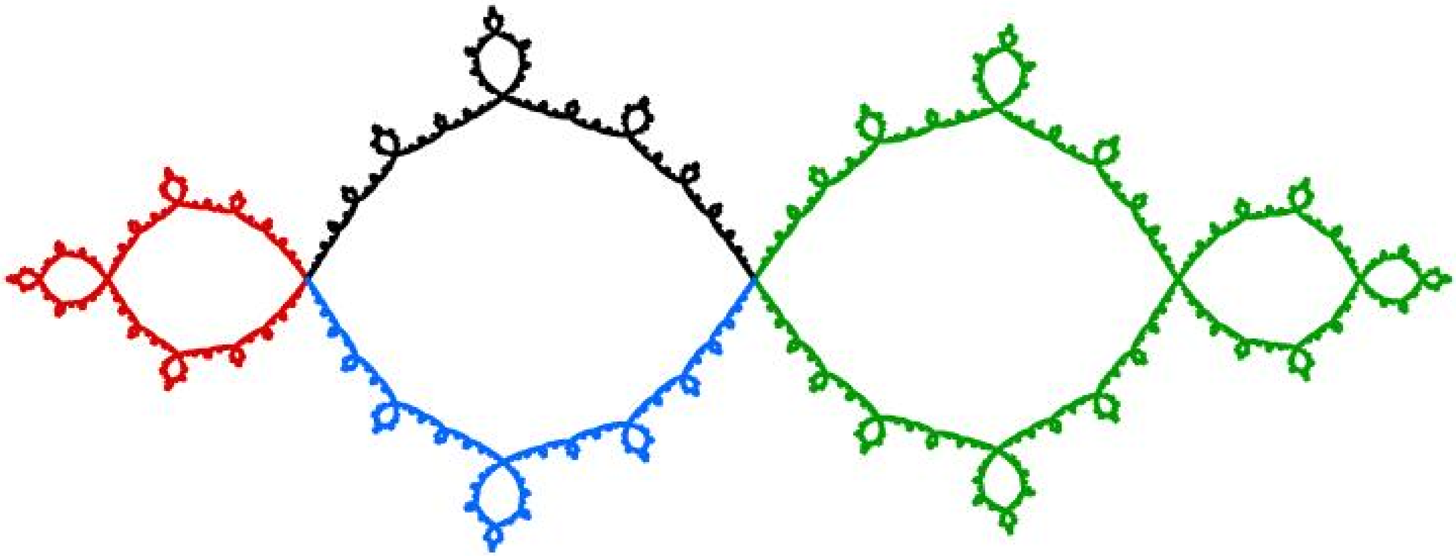
Example Element



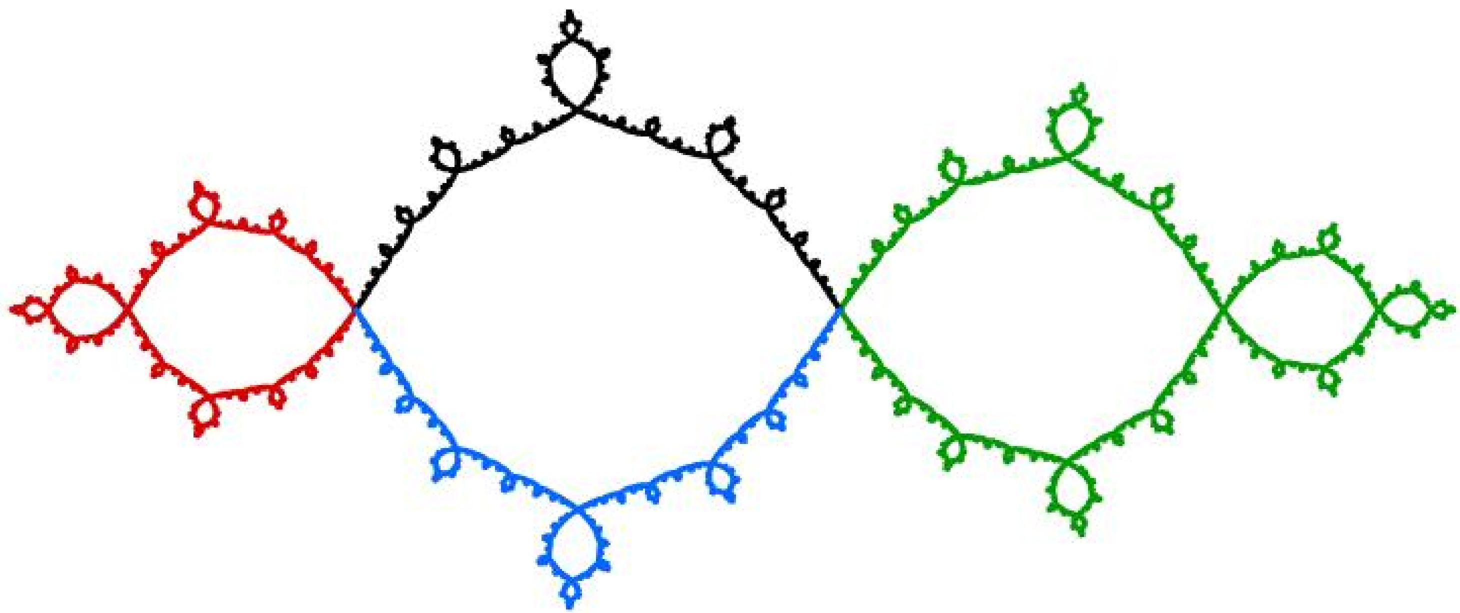
Example Element

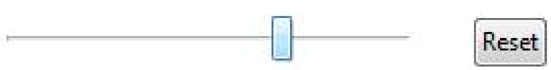


Example Element

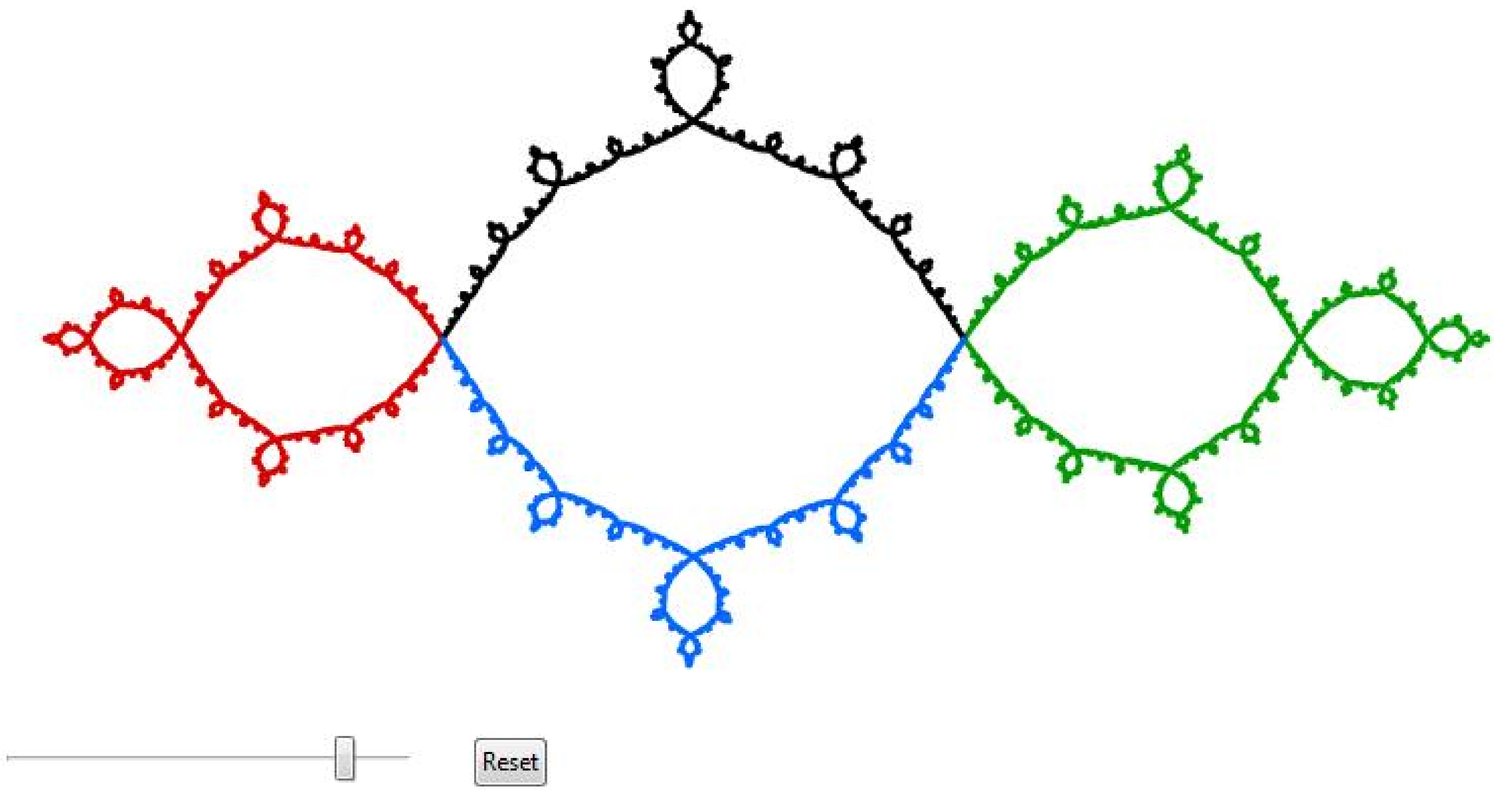


Example Element

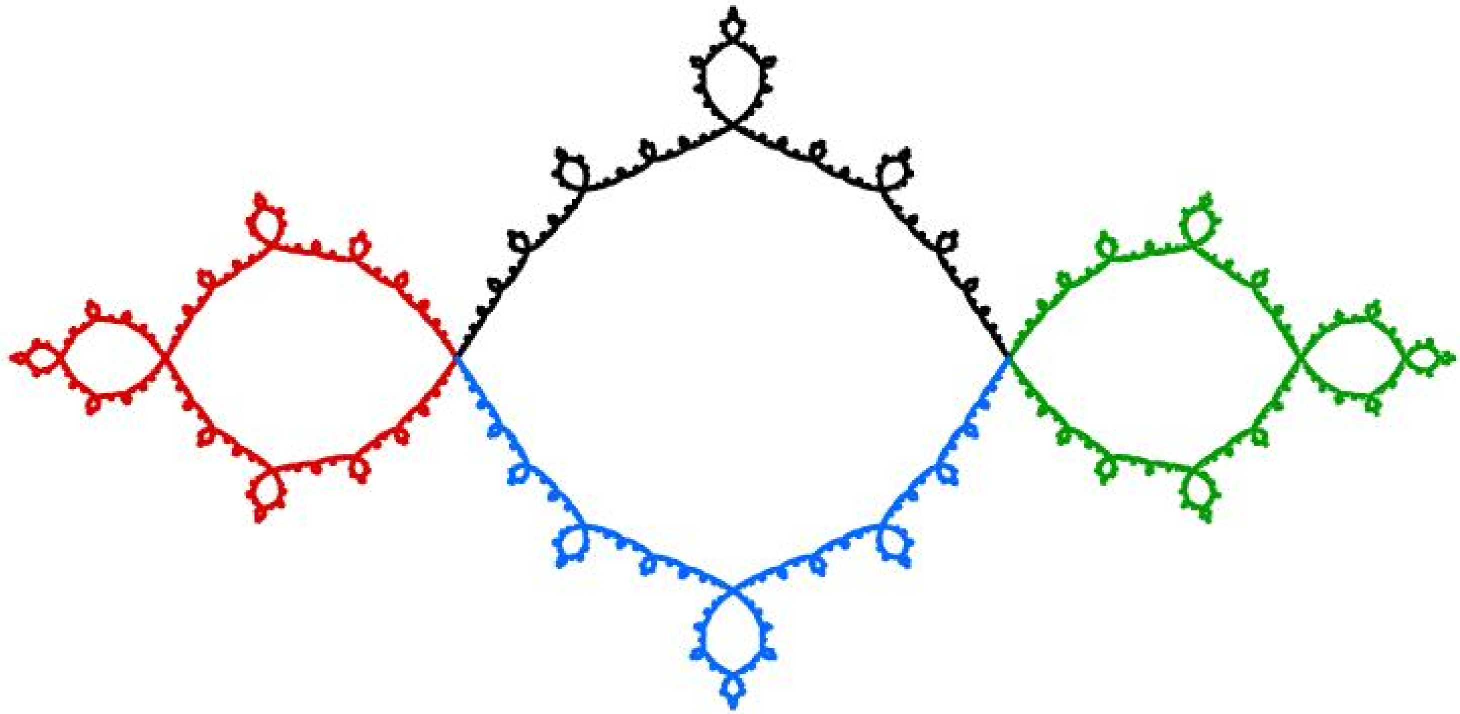




Example Element



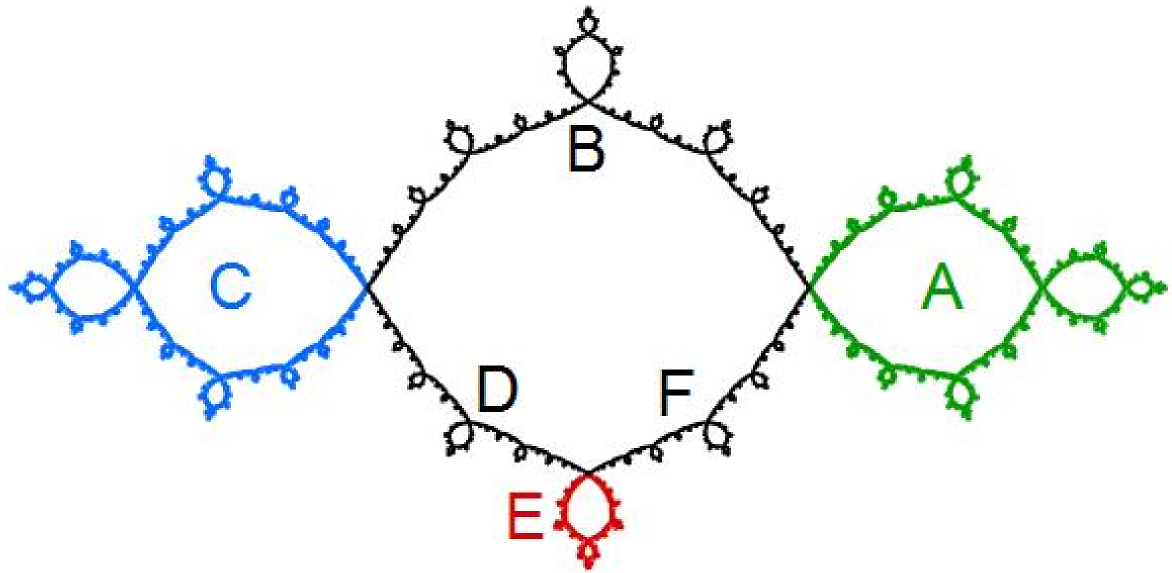
Example Element



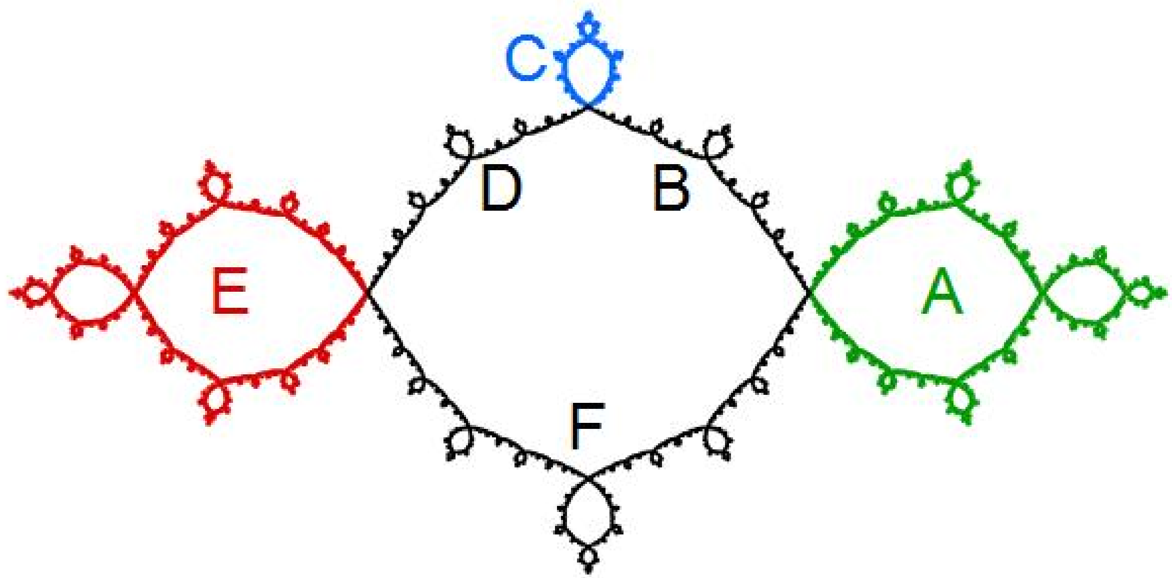
^

Example Element 2

Domain:

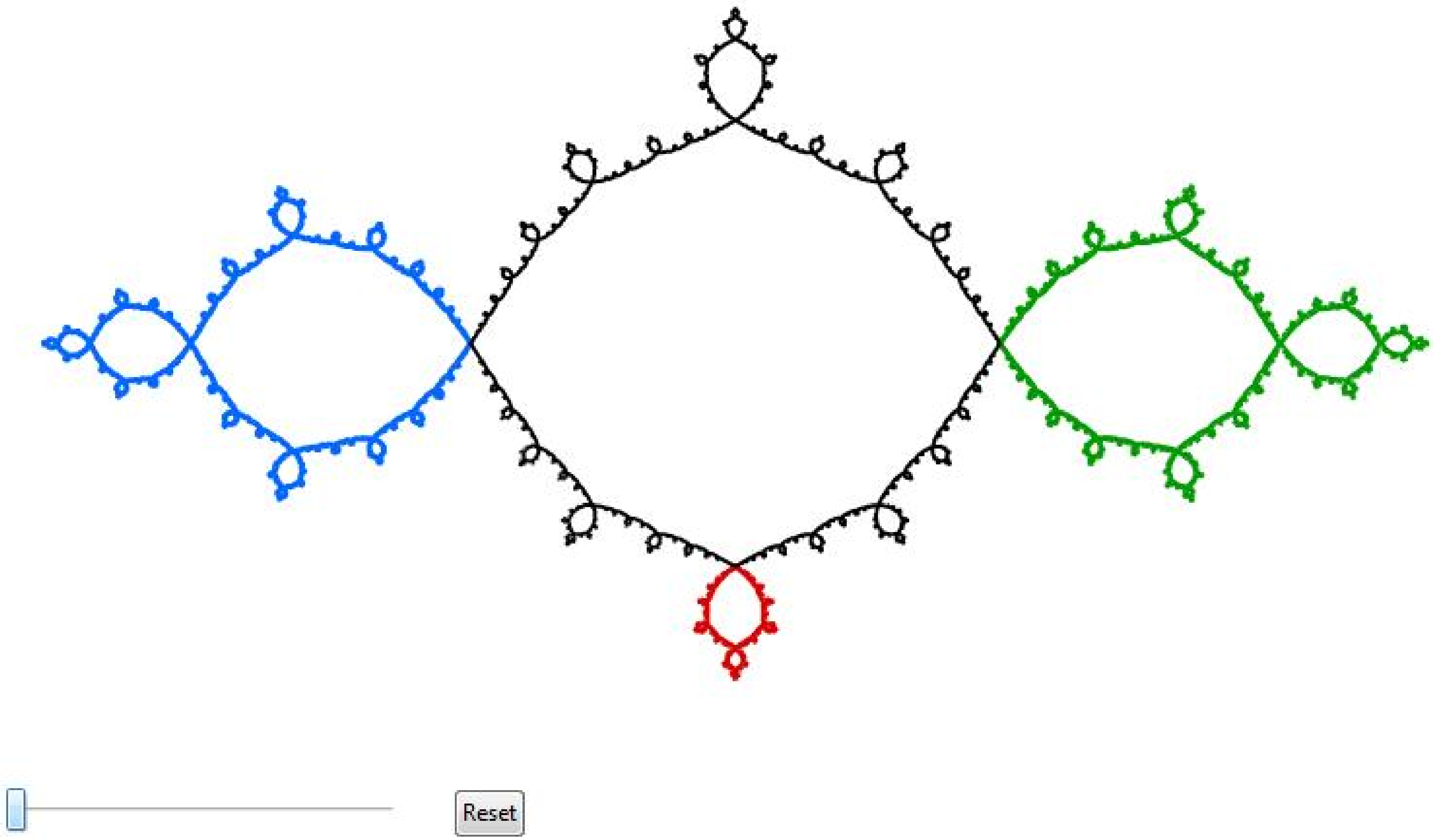


Range:

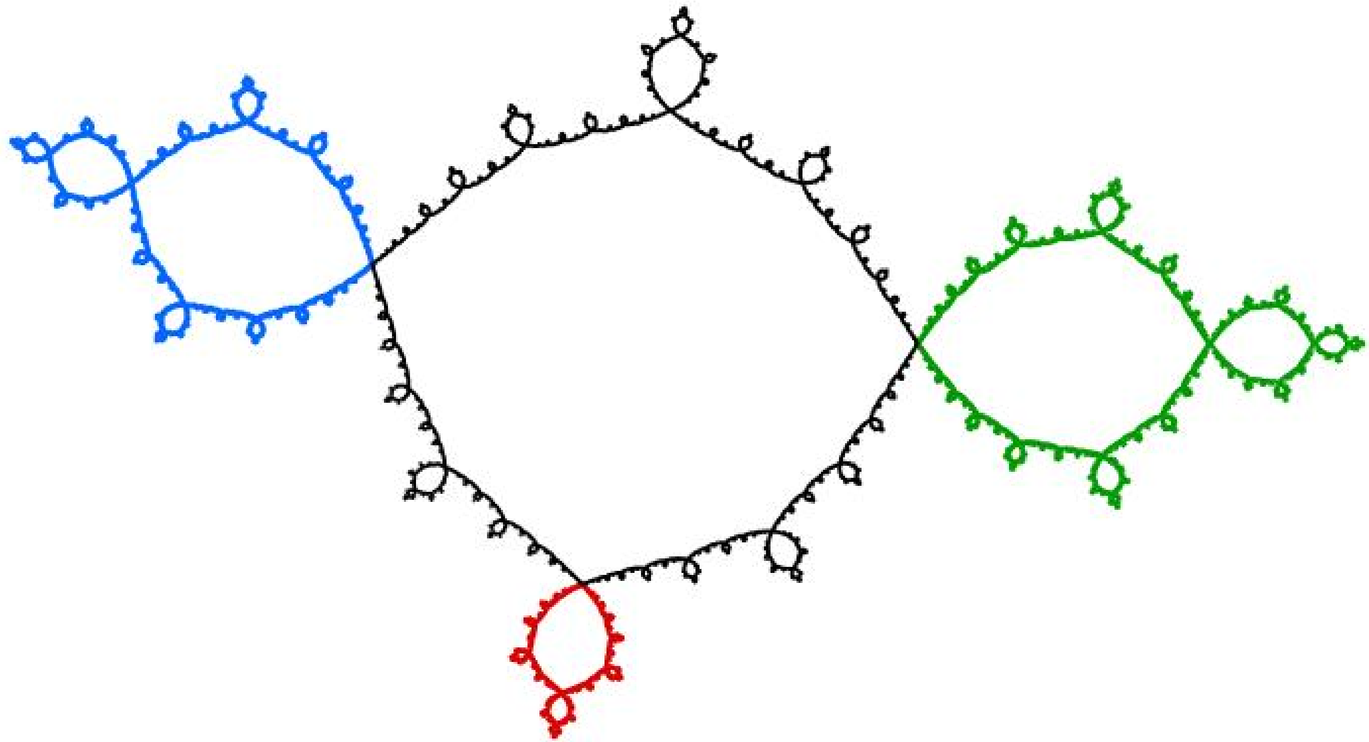


^

Example Element 2

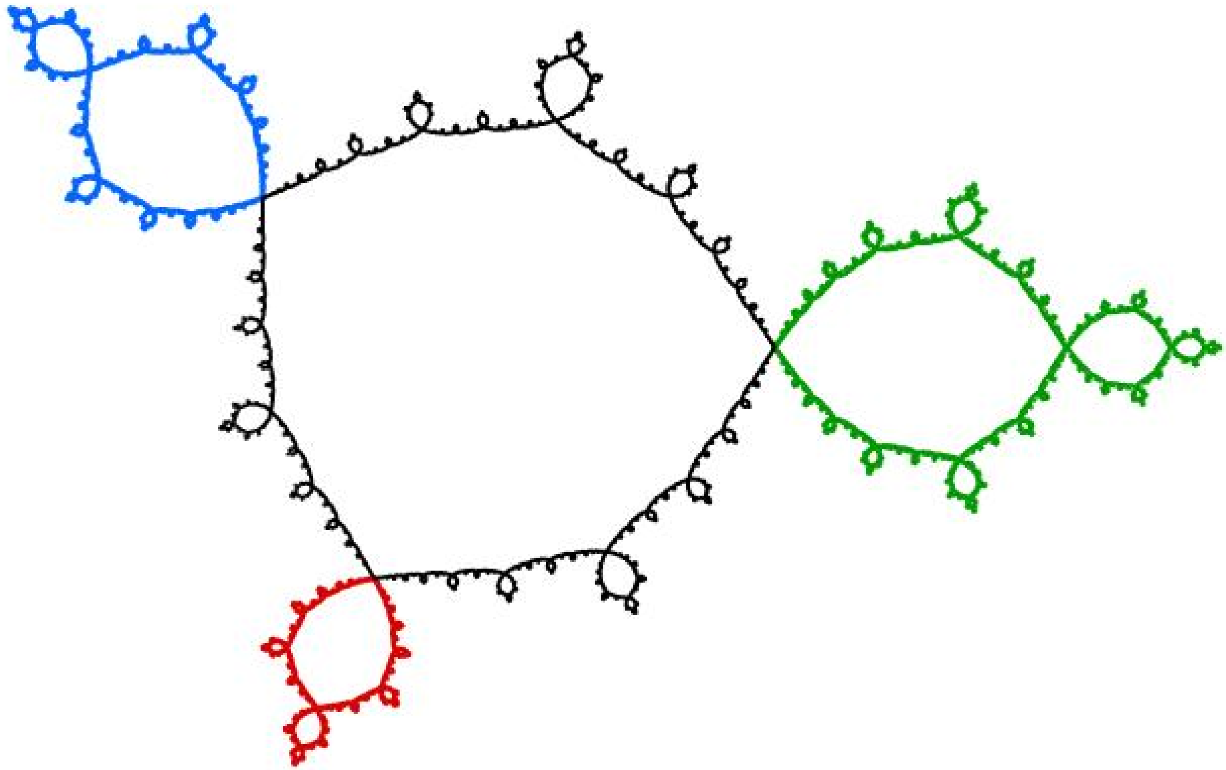


Example Element 2



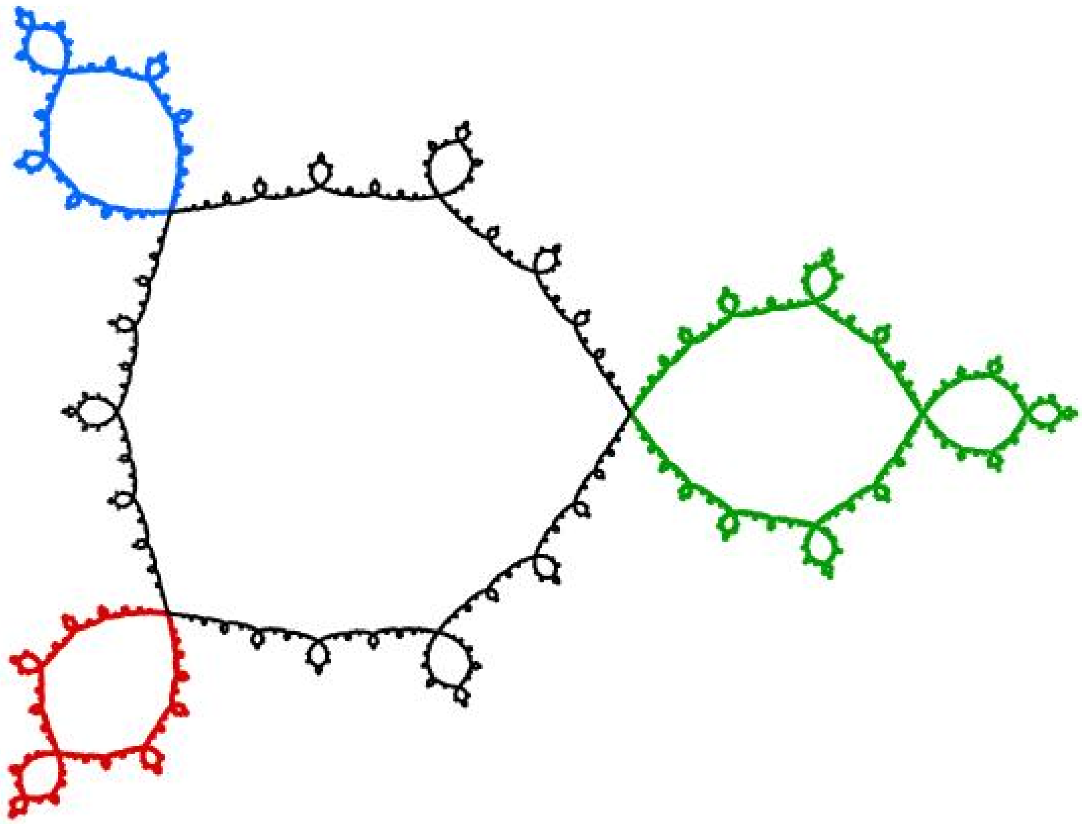
^

Example Element 2

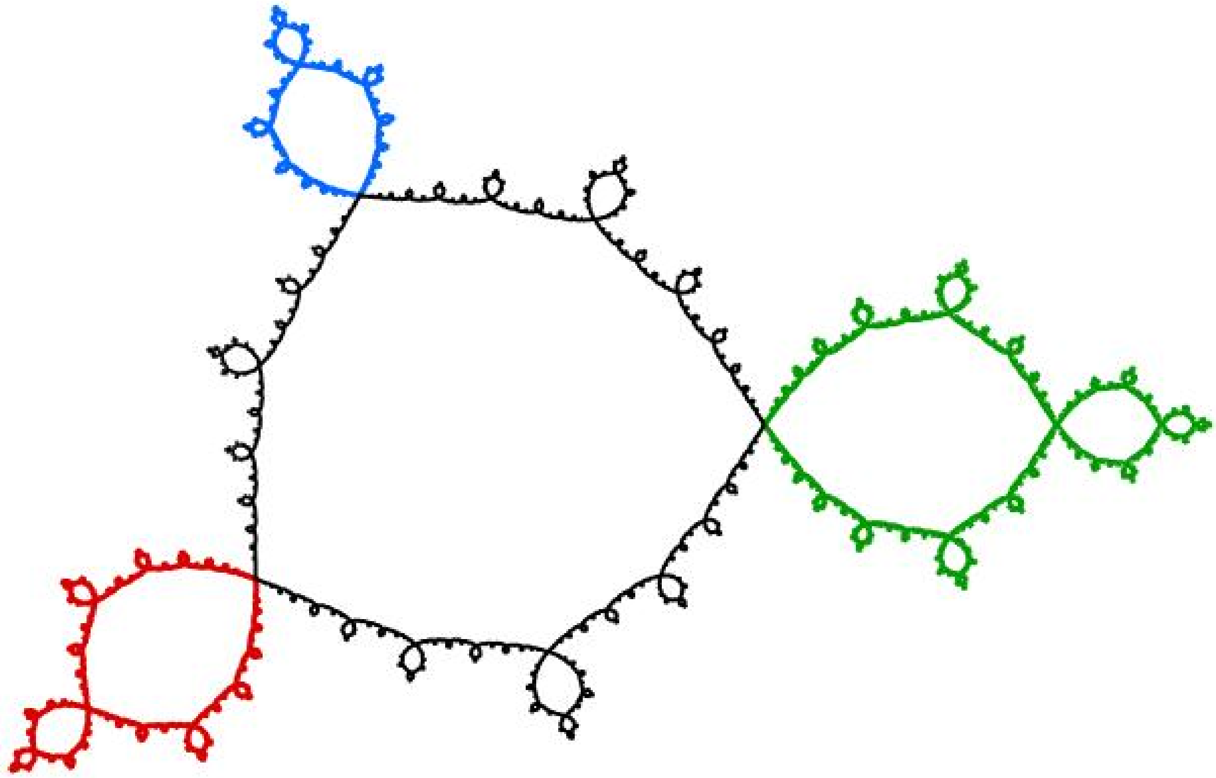


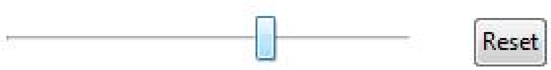
^

Example Element 2

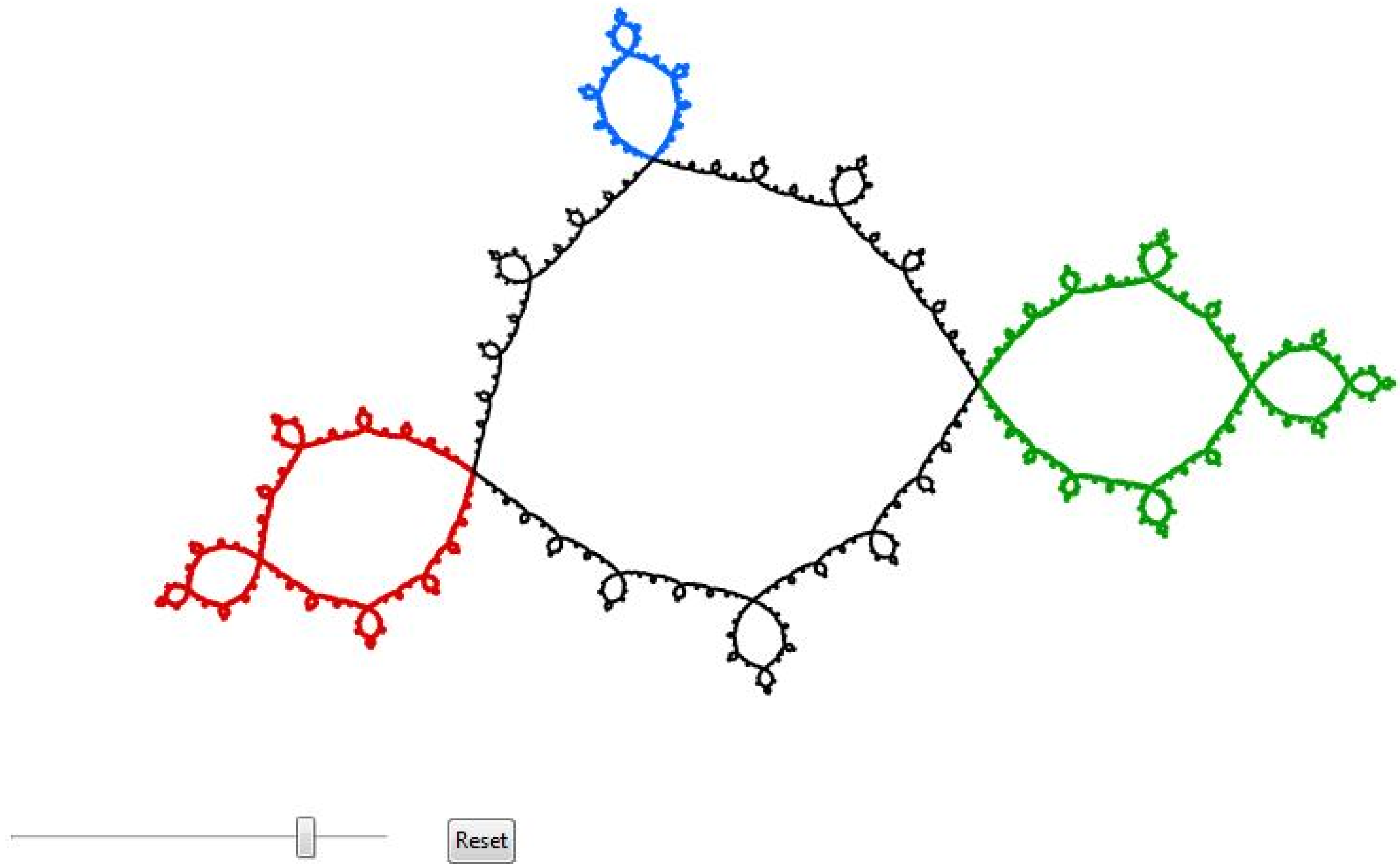


Example Element 2

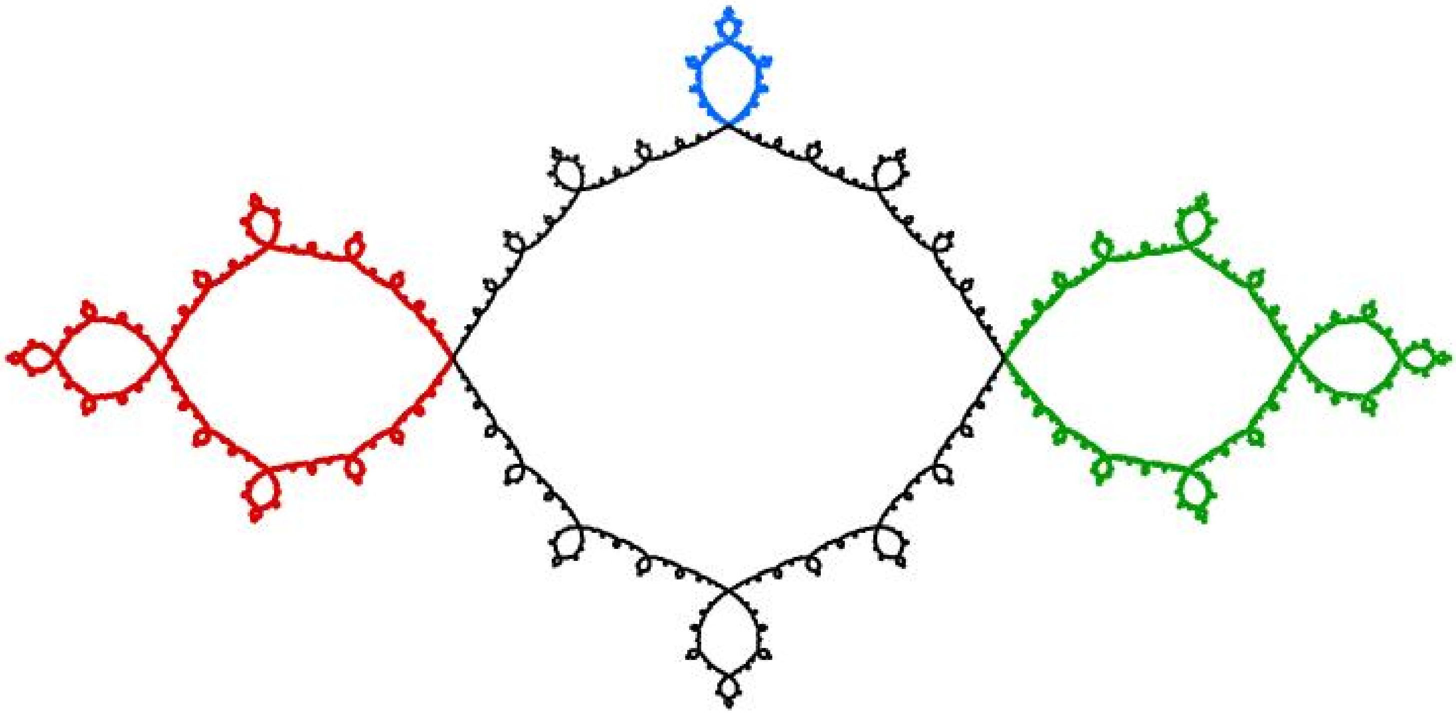




Example Element 2



Example Element 2



Properties of T_B

Properties of T_B

Theorem

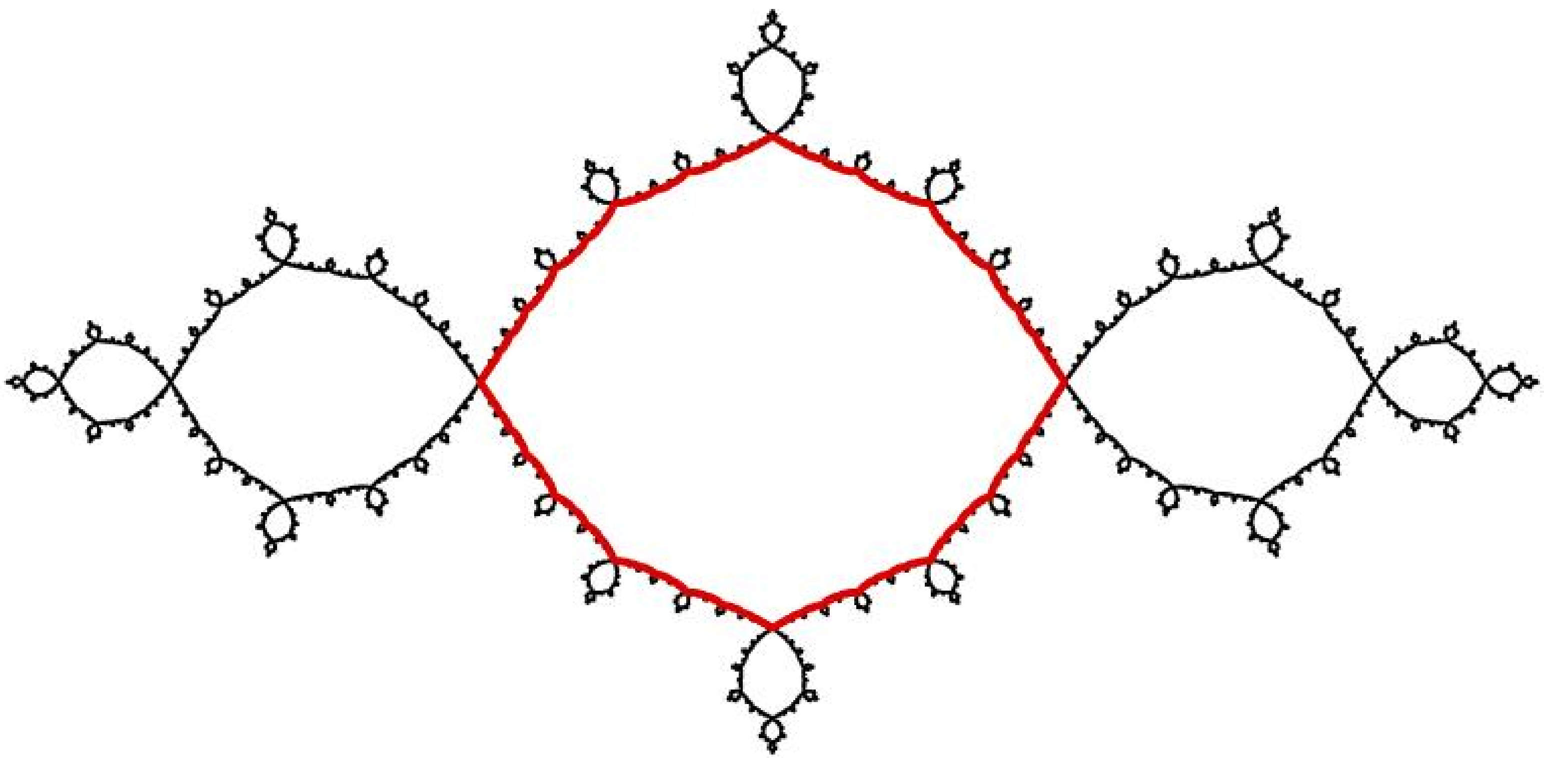
1. *T_B contains copies of Thompson's group T.*

^

Properties of T_B

Theorem

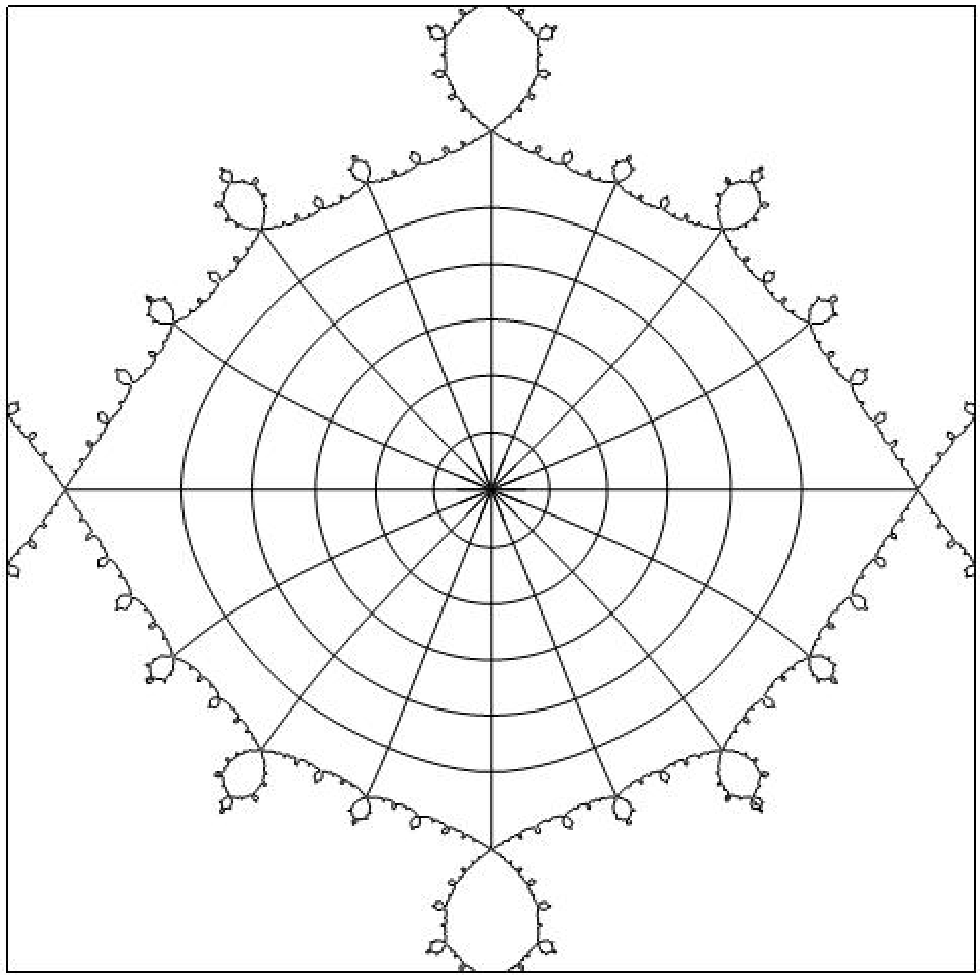
1. *T_B contains copies of Thompson's group T.*



Properties of T_B

Theorem

1. *T_B contains copies of Thompson's group T.*



Properties of T_B

Theorem

1. *T_B contains copies of Thompson's group T.*
2. *T_B is generated by four elements.*

Properties of T_B

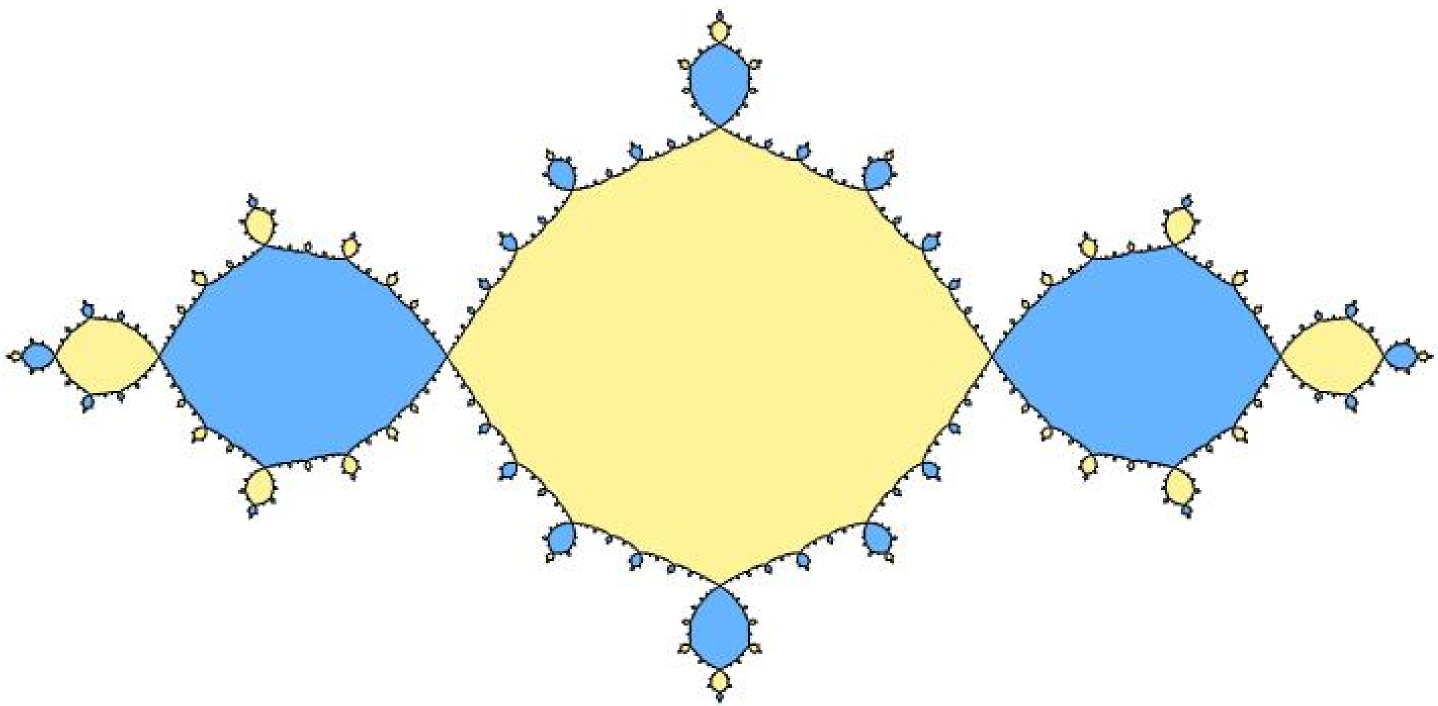
Theorem

1. T_B *contains copies of Thompson's group* T.
2. T_B *is generated by four elements.*
3. $[T_B, T_B]$ *has index two, and* $T_B = [T_B, T_B] \rtimes \mathbb{Z}_2$.

Properties of T_B

Theorem

1. *T_B contains copies of Thompson's group T.*
2. *T_B is generated by four elements.*
3. *$[T_B, T_B]$ has index two, and $T_B = [T_B, T_B] \rtimes \mathbb{Z}_2$.*



Properties of T_B

Theorem

1. *T_B contains copies of Thompson's group T.*
2. *T_B is generated by four elements.*
3. *$[T_B, T_B]$ has index two, and $T_B = [T_B, T_B] \rtimes \mathbb{Z}_2$.*
4. *$[T_B, T_B]$ is simple.*

^

Properties of T_B

Theorem

1. T_B *contains copies of Thompson's group* T.
2. T_B *is generated by four elements.*
3. $[T_B, T_B]$ *has index two, and* $T_B = [T_B, T_B] \rtimes \mathbb{Z}_2$.
4. $[T_B, T_B]$ *is simple*.
5. $[T_B, T_B]$ *is* ***not*** *isomorphic to Thompson's group* T.

Properties of T_B

Theorem

1. T_B *contains copies of Thompson's group T.*
2. T_B *is generated by four elements.*
3. $[T_B, T_B]$ *has index two, and* $T_B = [T_B, T_B] \rtimes \mathbb{Z}_2$.
4. $[T_B, T_B]$ *is simple.*
5. $[T_B, T_B]$ *is* ***not*** *isomorphic to Thompson's group T.*

Open Question

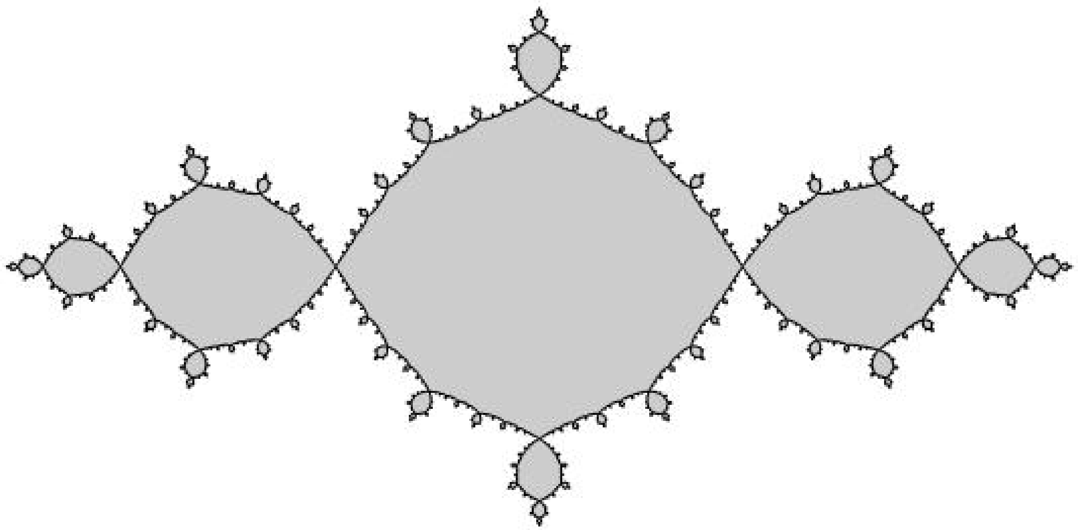
Is T_B finitely presented? (We suspect not.)

External Angles

^

External Angles

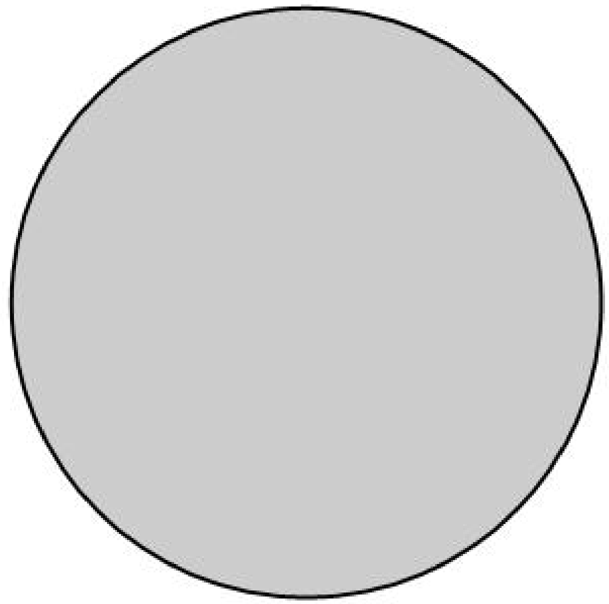
The exterior region for the Basilica is simply connected in the Riemann sphere:



^

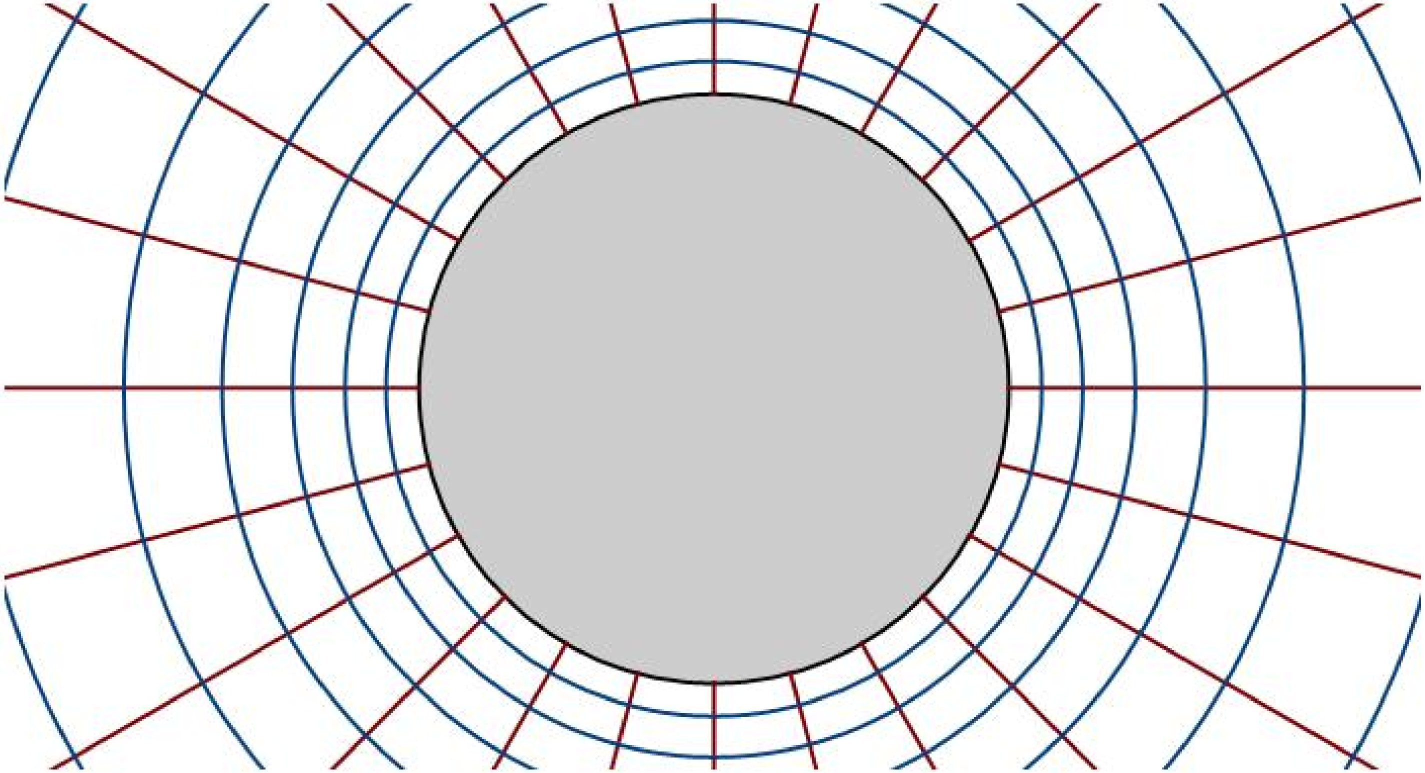
External Angles

The complement of a disk is also simply connected:



External Angles

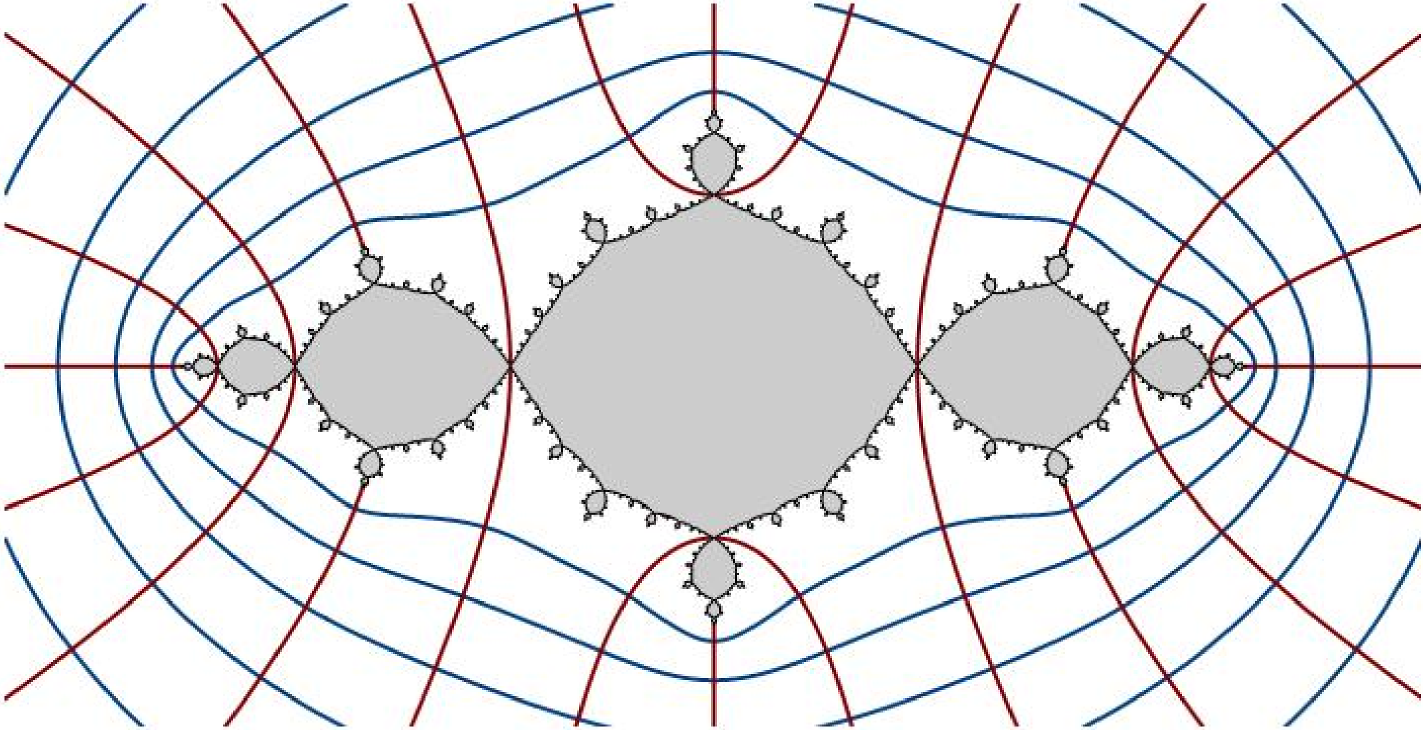
So there exists a Riemann map between the two regions.



^

External Angles

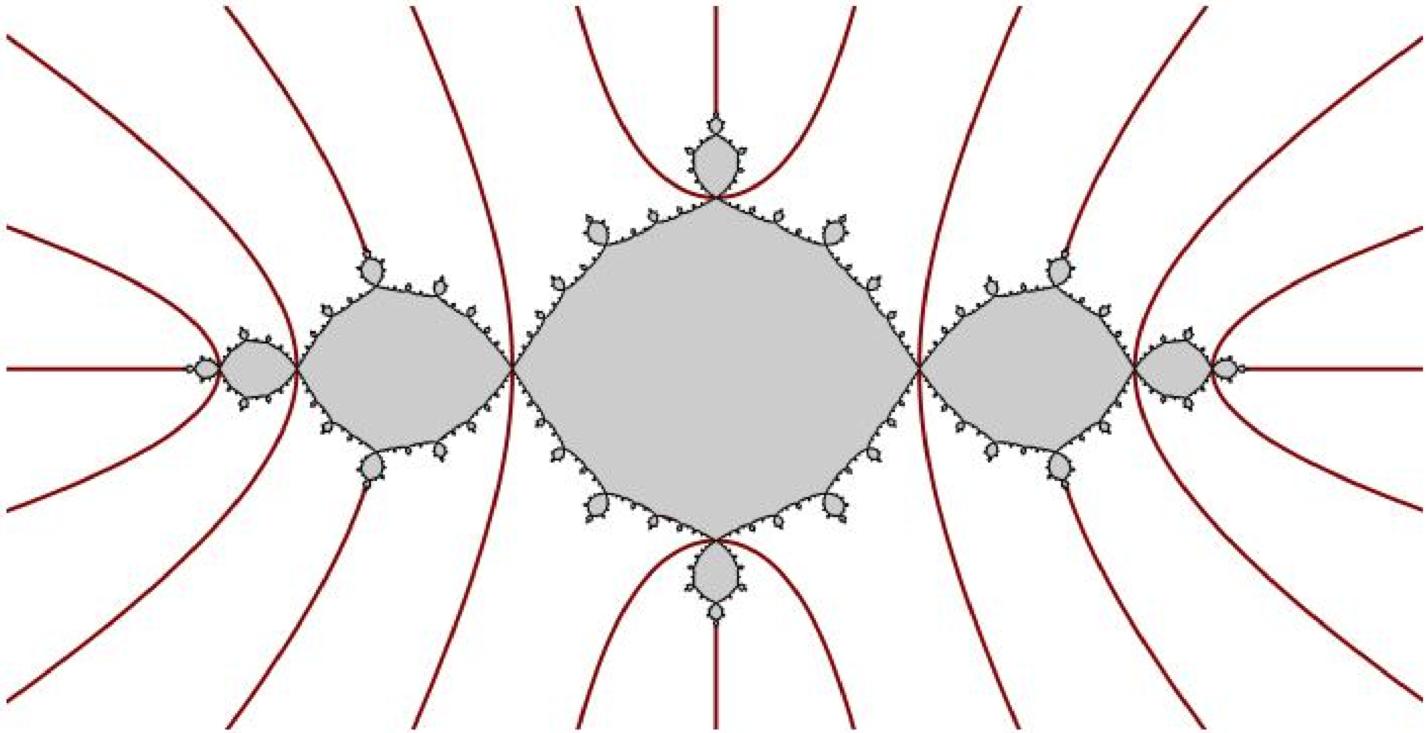
So there exists a Riemann map between the two regions.



^

External Angles

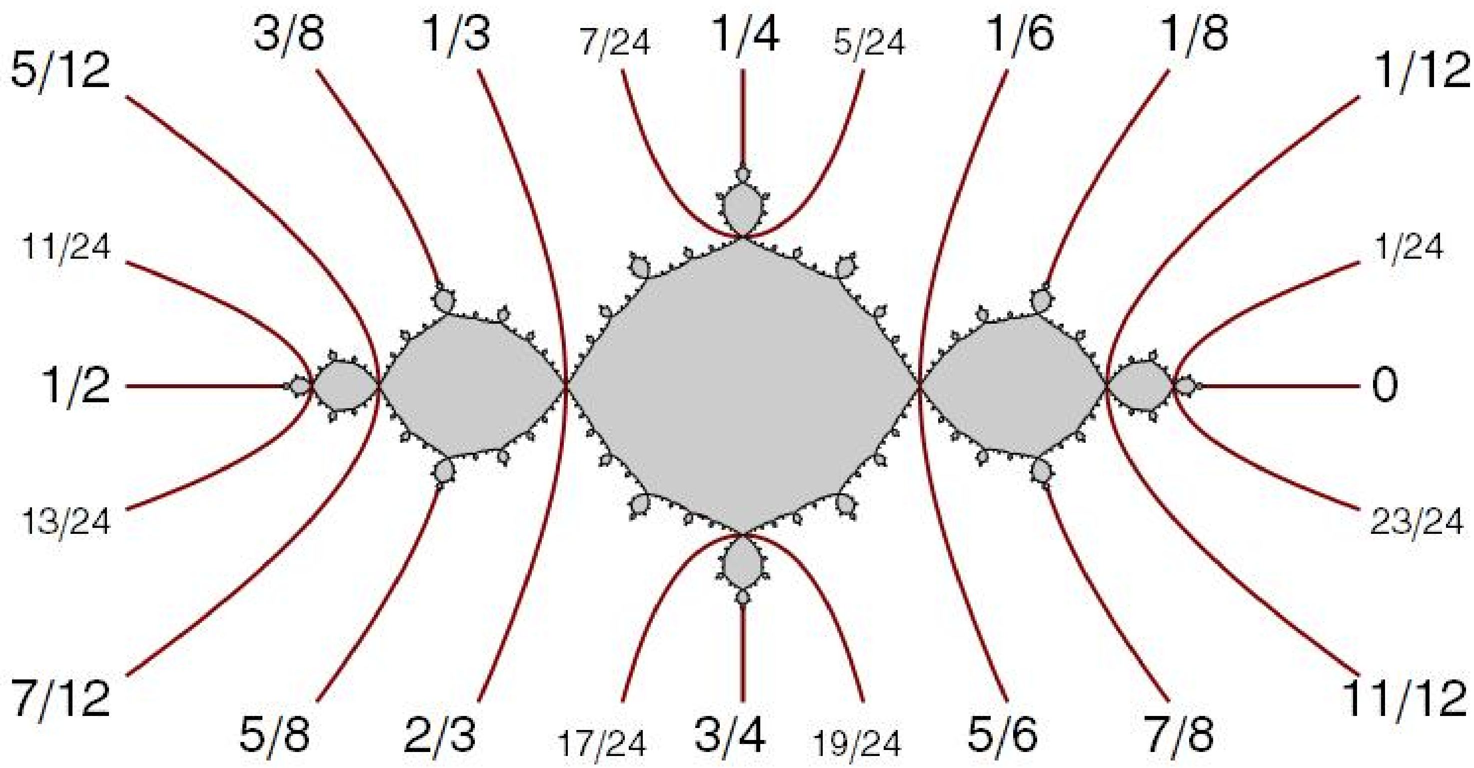
The radial lines are called ***external rays***.



^

External Angles

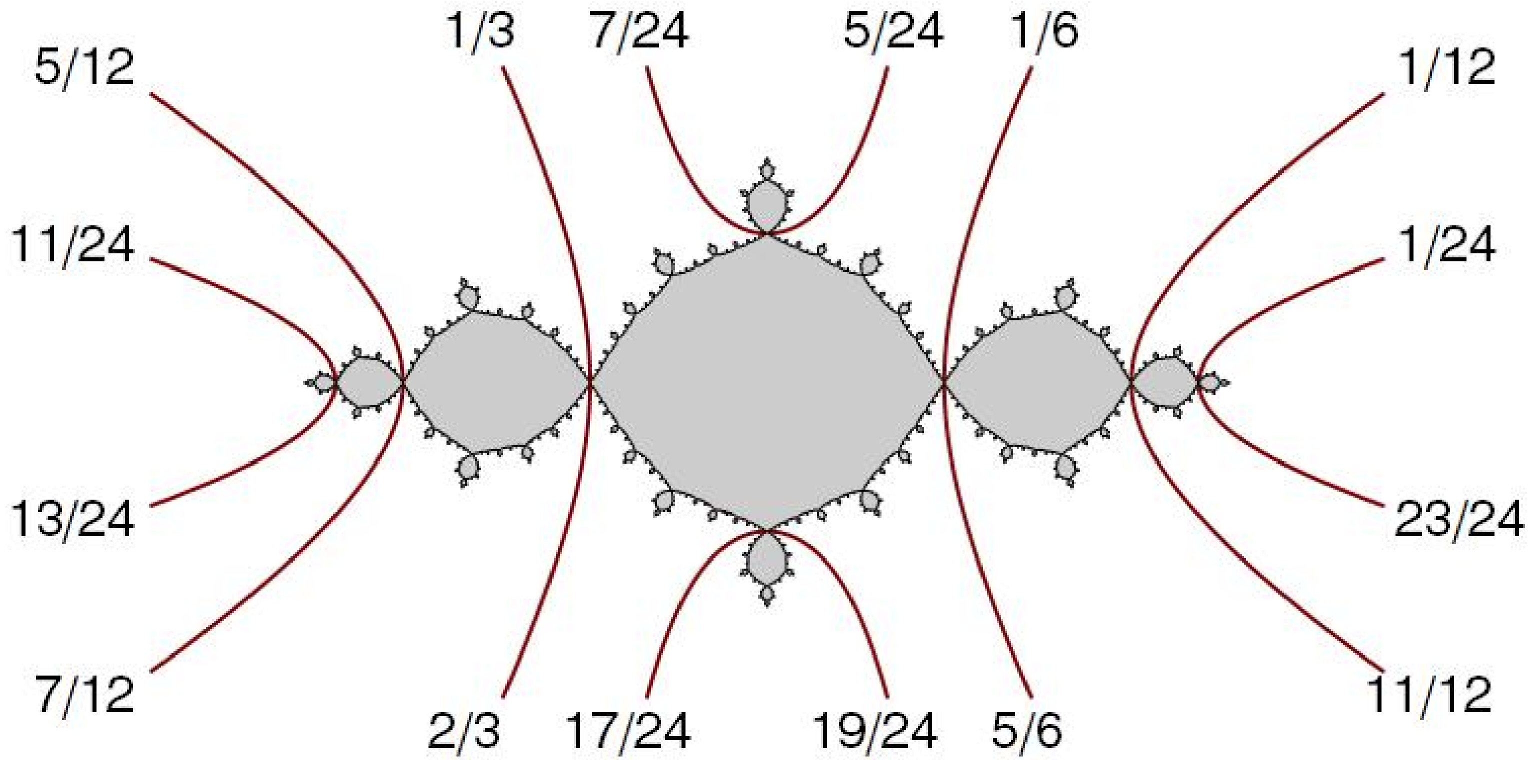
These define the ***external angles***.



^

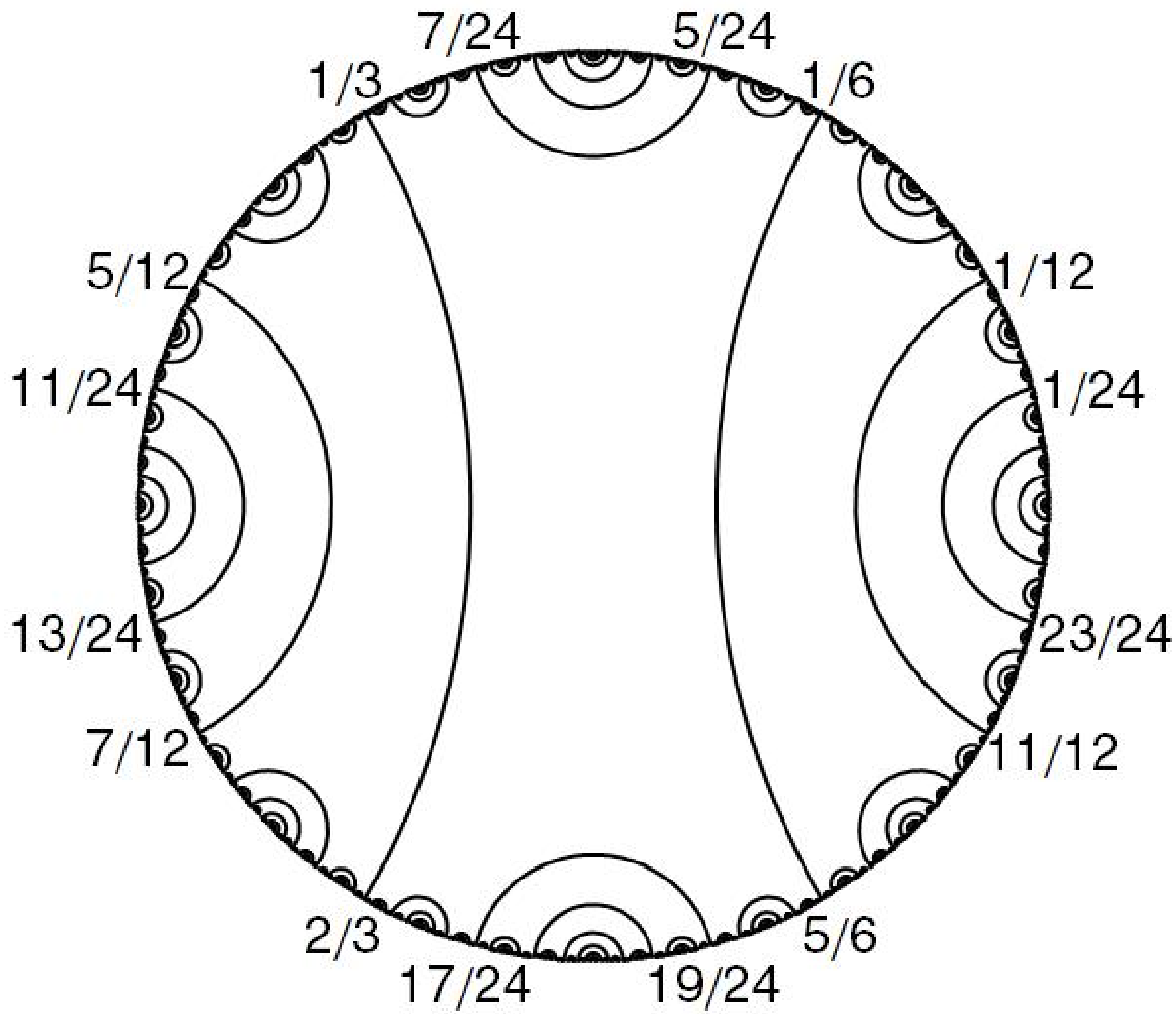
External Angles

Pinch Points: Denominator is $3 \cdot 2^n$



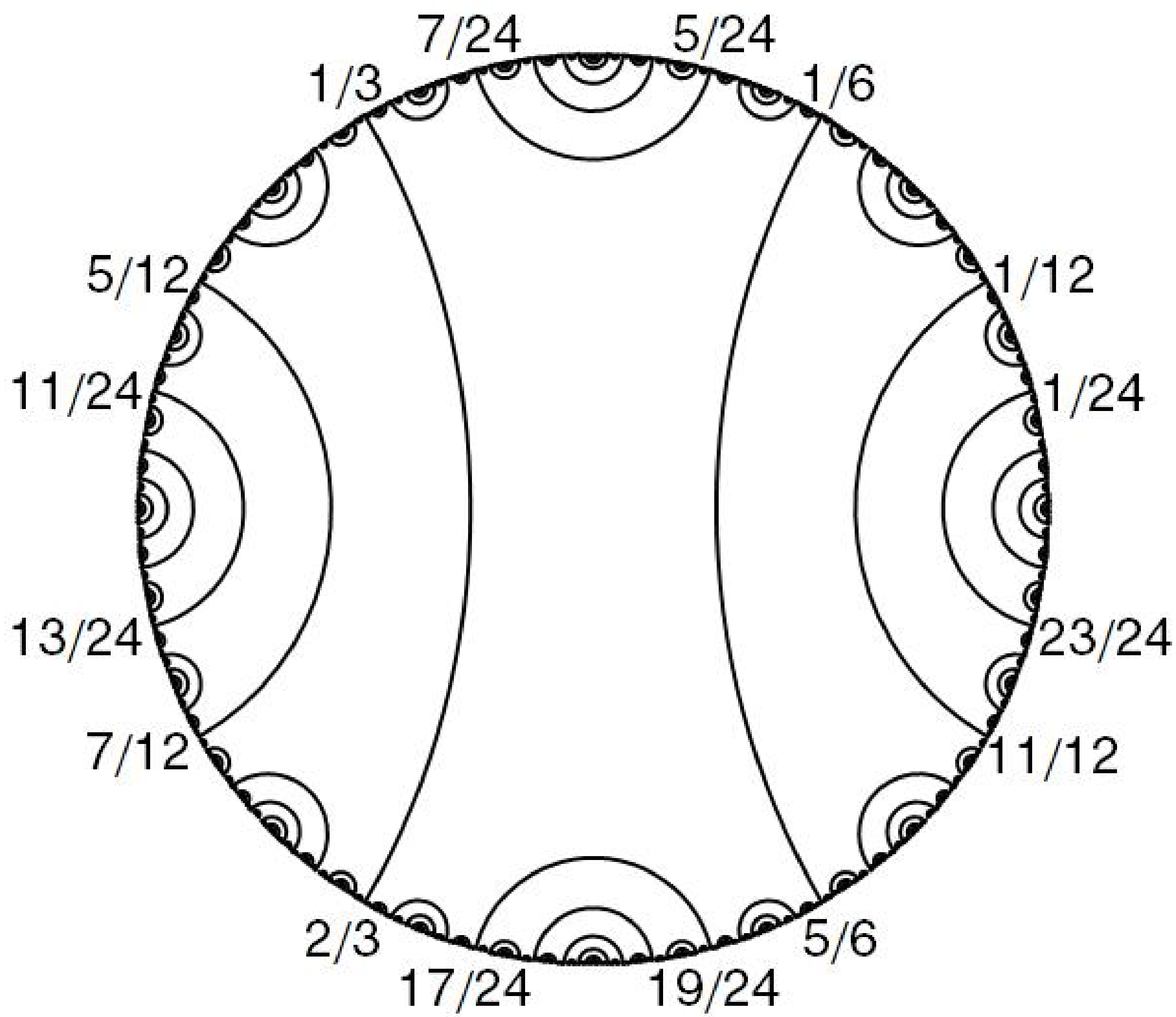
The Invariant Lamination

This picture shows an equivalence relation on the circle. It is called the ***invariant lamination*** for the Basilica.



The Invariant Lamination

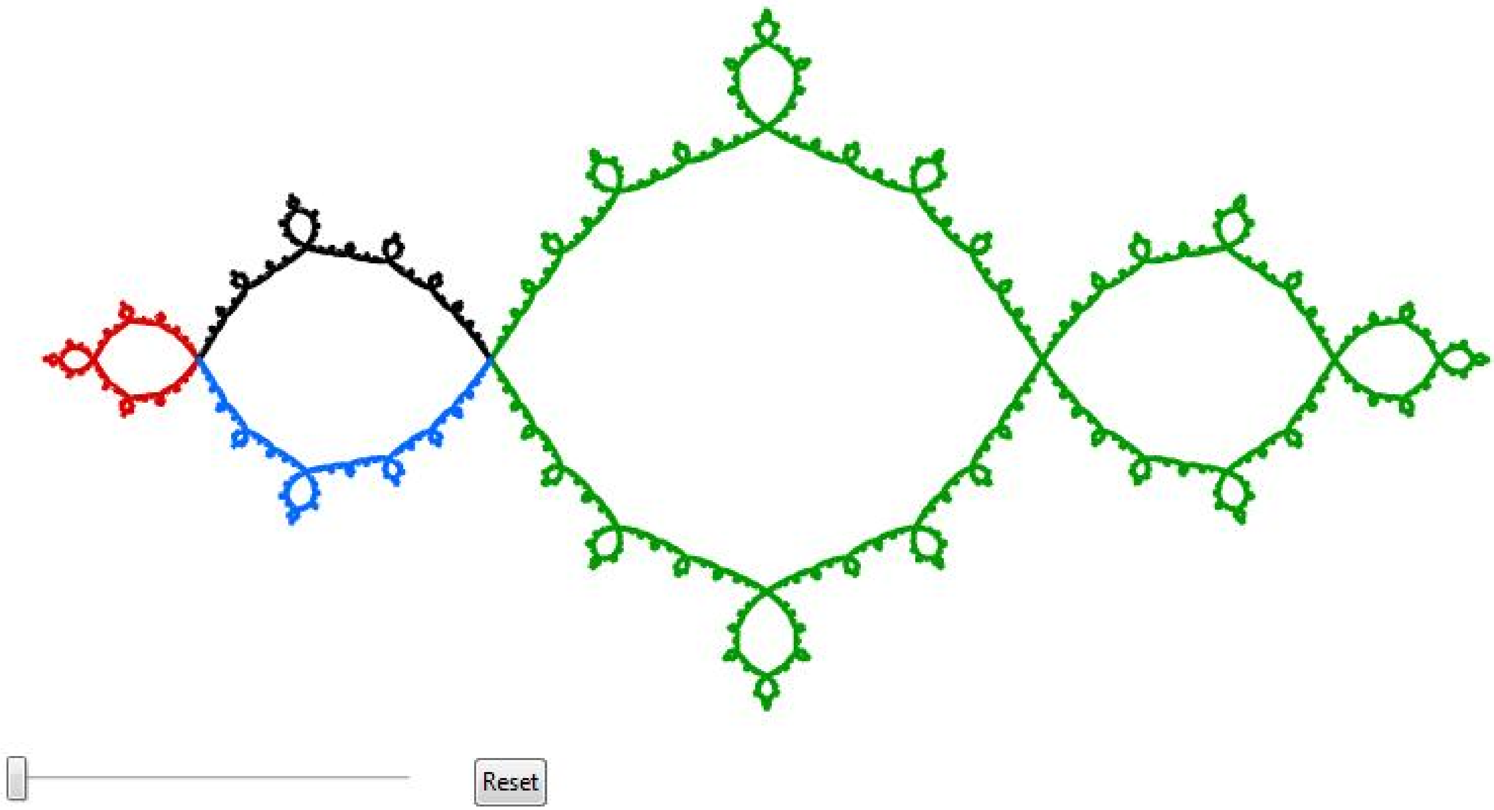
Homeomorphisms that preserve this equivalence relation descend to the Julia set.



^

Piecewise-Linear Homeomorphisms

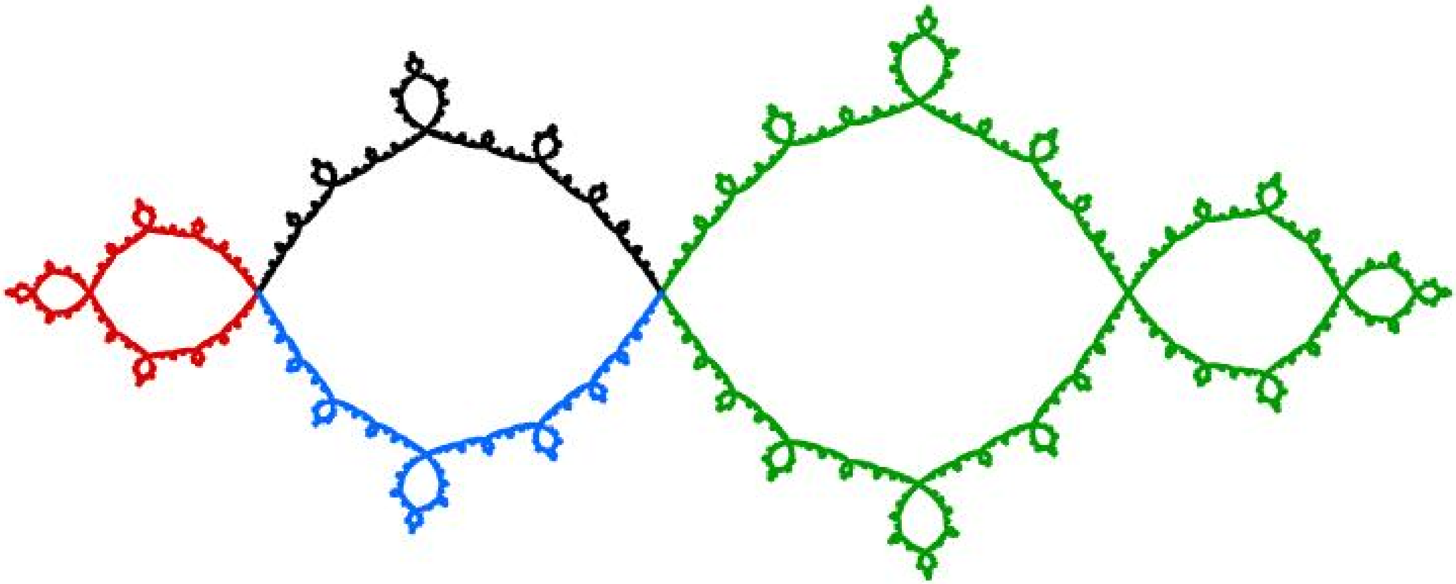
Elements of T_B act as piecewise-linear homeomorphisms.



^

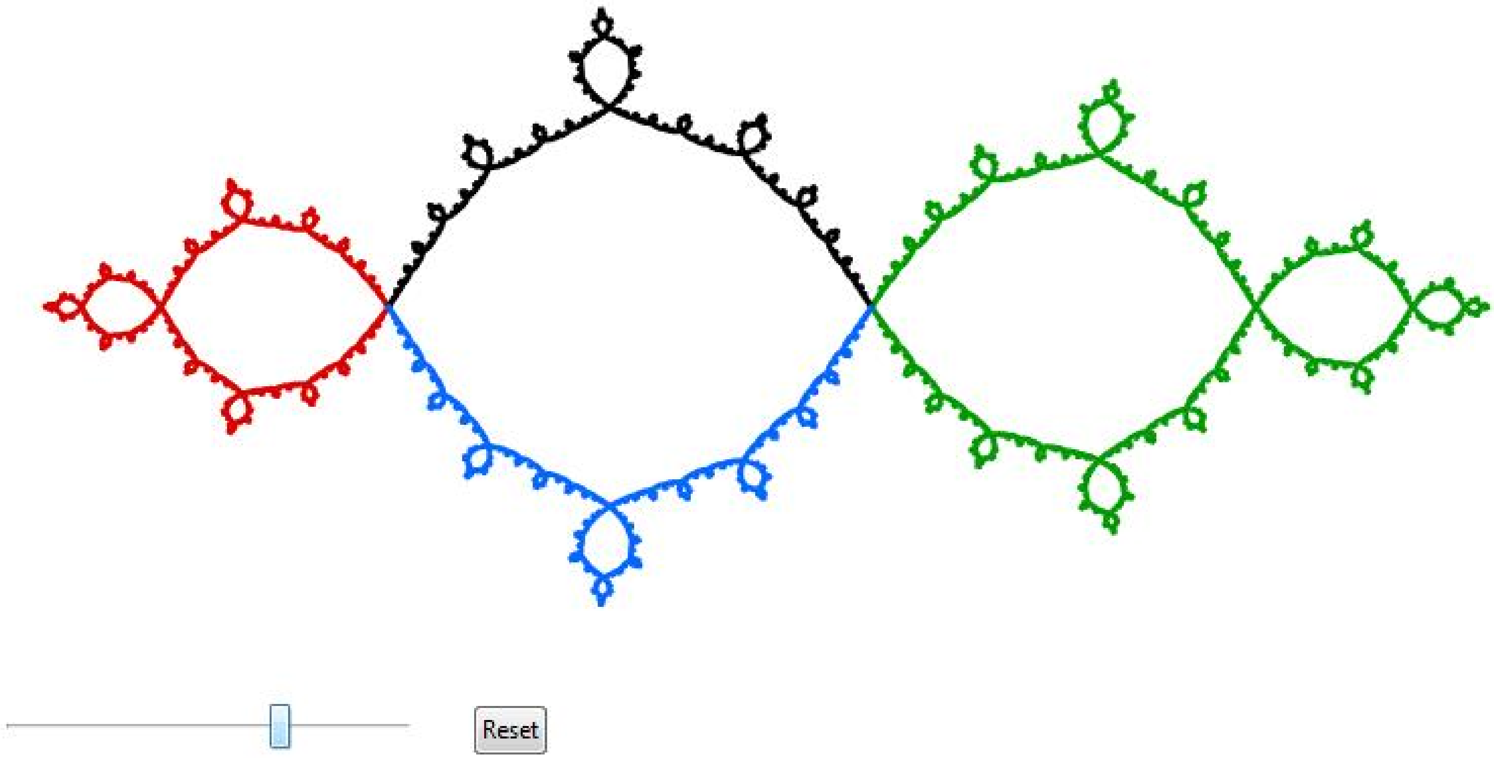
Piecewise-Linear Homeomorphisms

Elements of T_B act as piecewise-linear homeomorphisms.



Piecewise-Linear Homeomorphisms

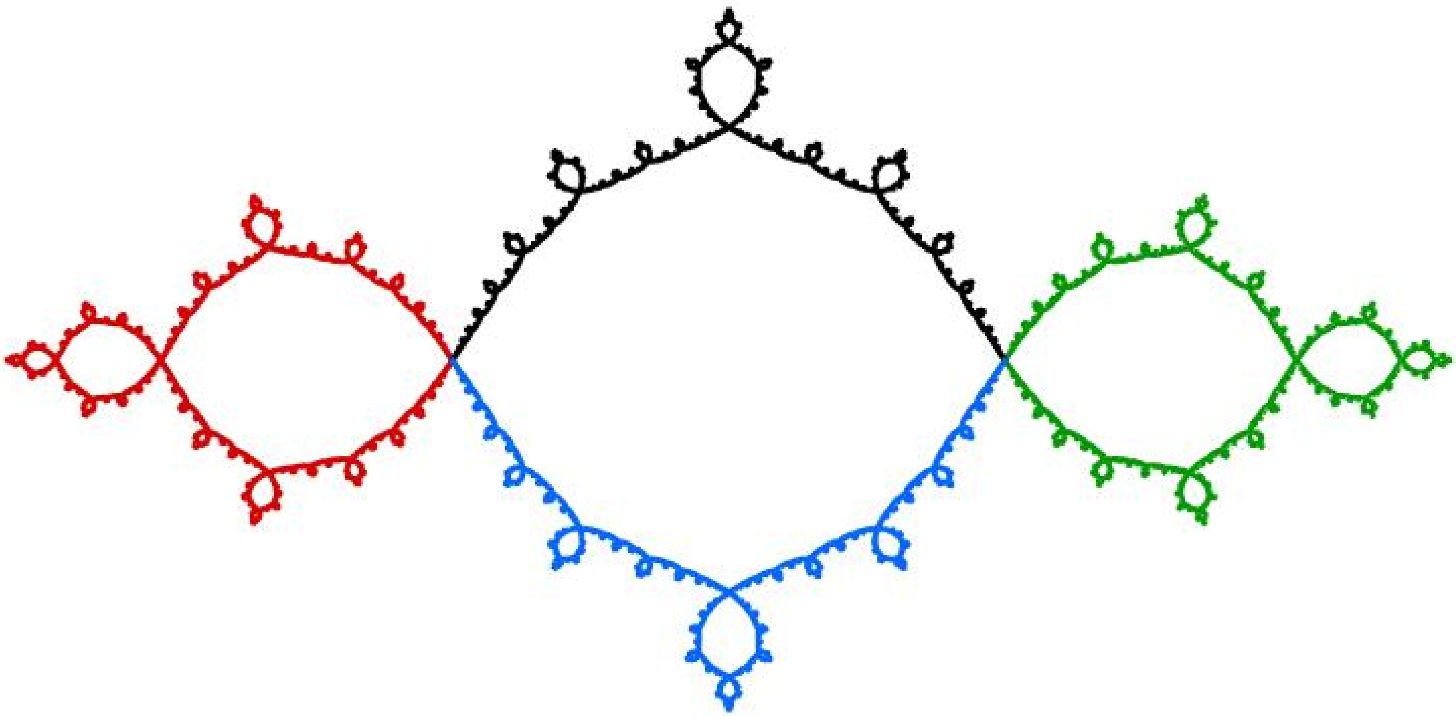
Elements of T_B act as piecewise-linear homeomorphisms.



^

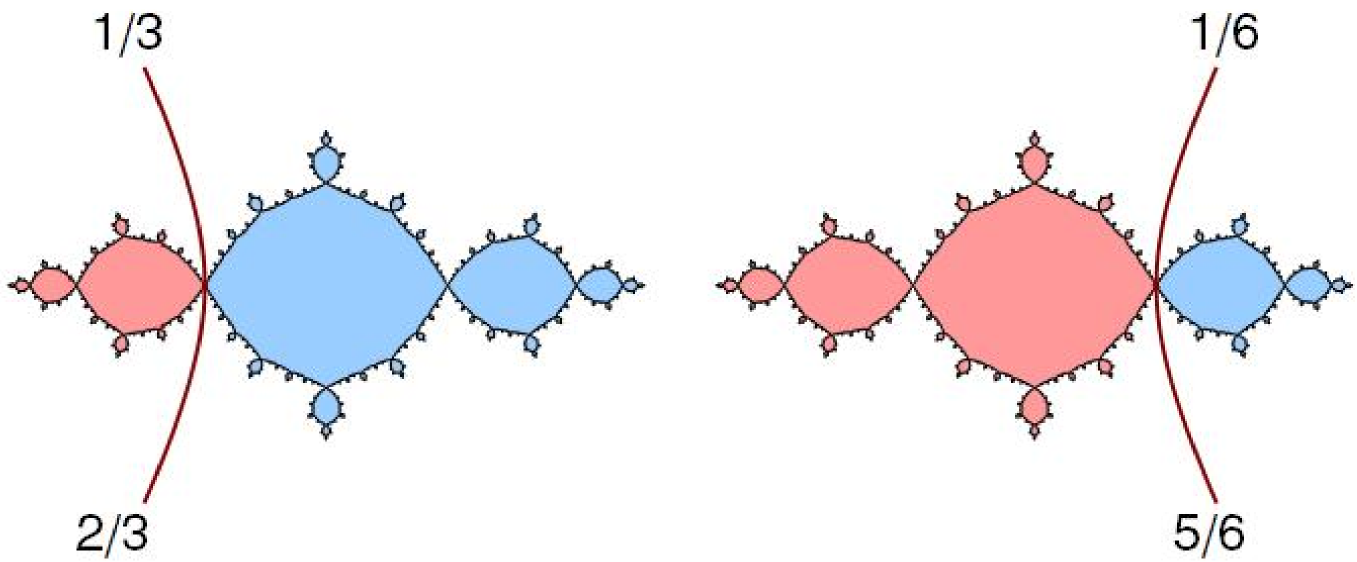
Piecewise-Linear Homeomorphisms

Elements of T_B act as piecewise-linear homeomorphisms.



Piecewise-Linear Homeomorphisms

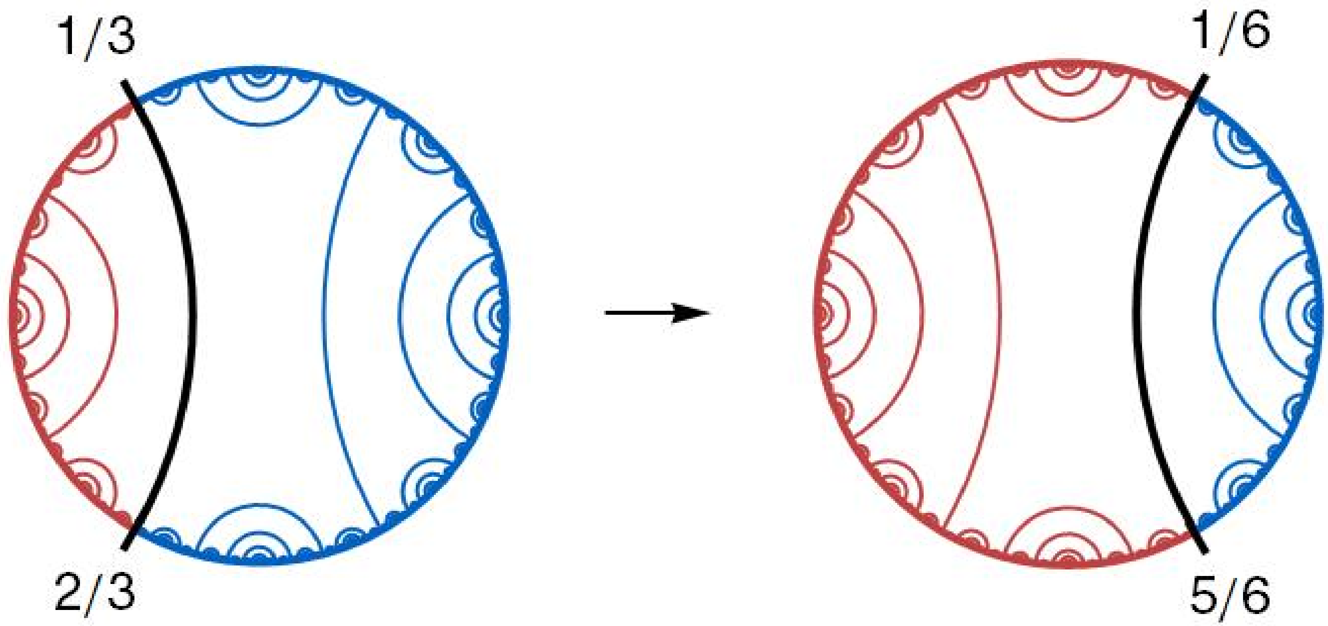
Elements of T_B act as piecewise-linear homeomorphisms.



$$f(\theta) = \begin{cases} \theta/2 & \text{if } -1/3 \leq \theta \leq 1/3 \\ 2\theta - 1/2 & \text{if } 1/3 \leq \theta \leq 2/3 \end{cases}$$

Piecewise-Linear Homeomorphisms

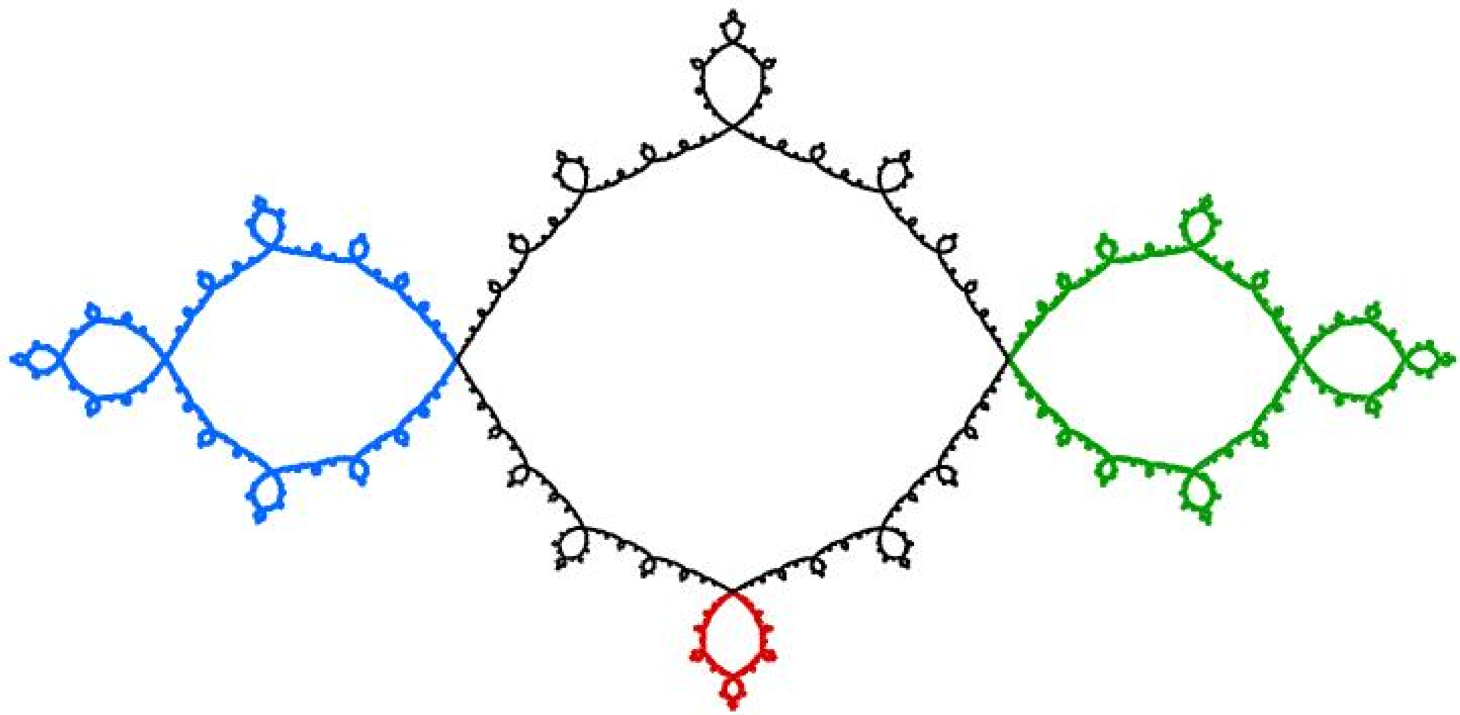
Elements of T_B act as piecewise-linear homeomorphisms.



$$f(\theta) = \begin{cases} \theta/2 & \text{if } -1/3 \le \theta \le 1/3 \\ 2\theta - 1/2 & \text{if } 1/3 \le \theta \le 2/3 \end{cases}$$

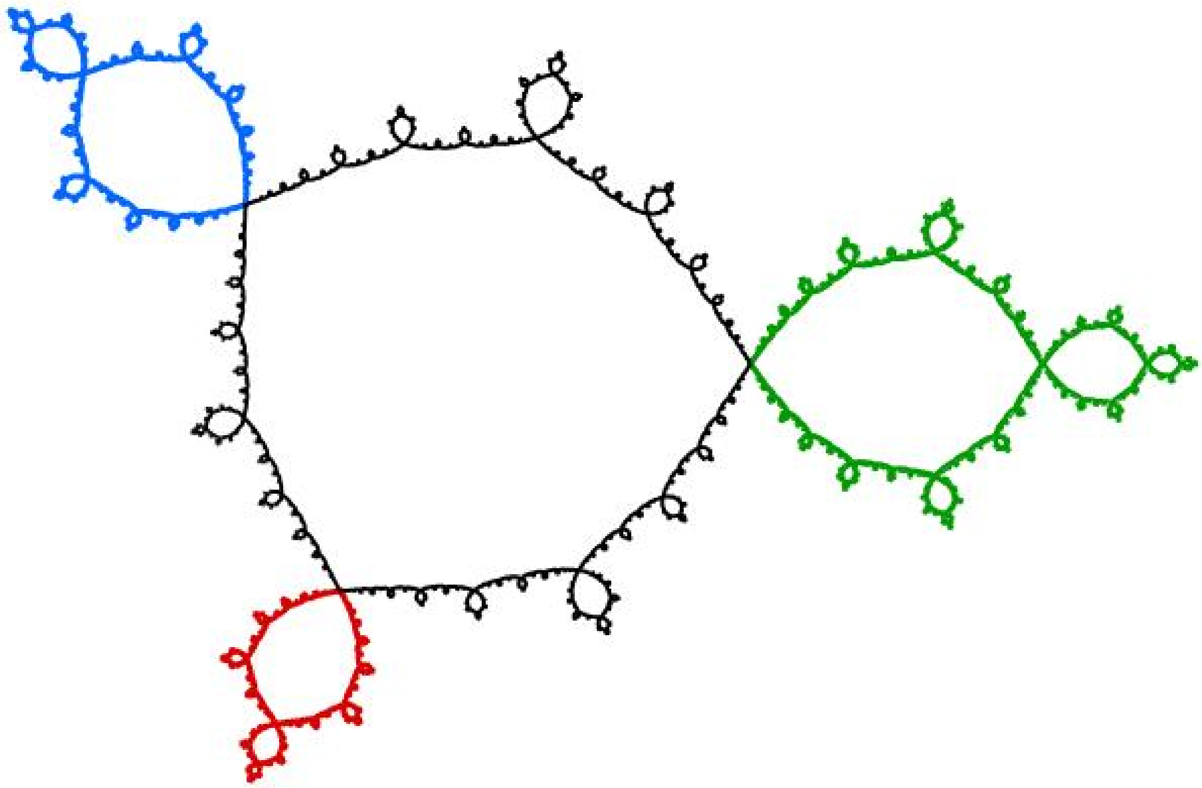
Piecewise-Linear Homeomorphisms

Elements of T_B act as piecewise-linear homeomorphisms.



Piecewise-Linear Homeomorphisms

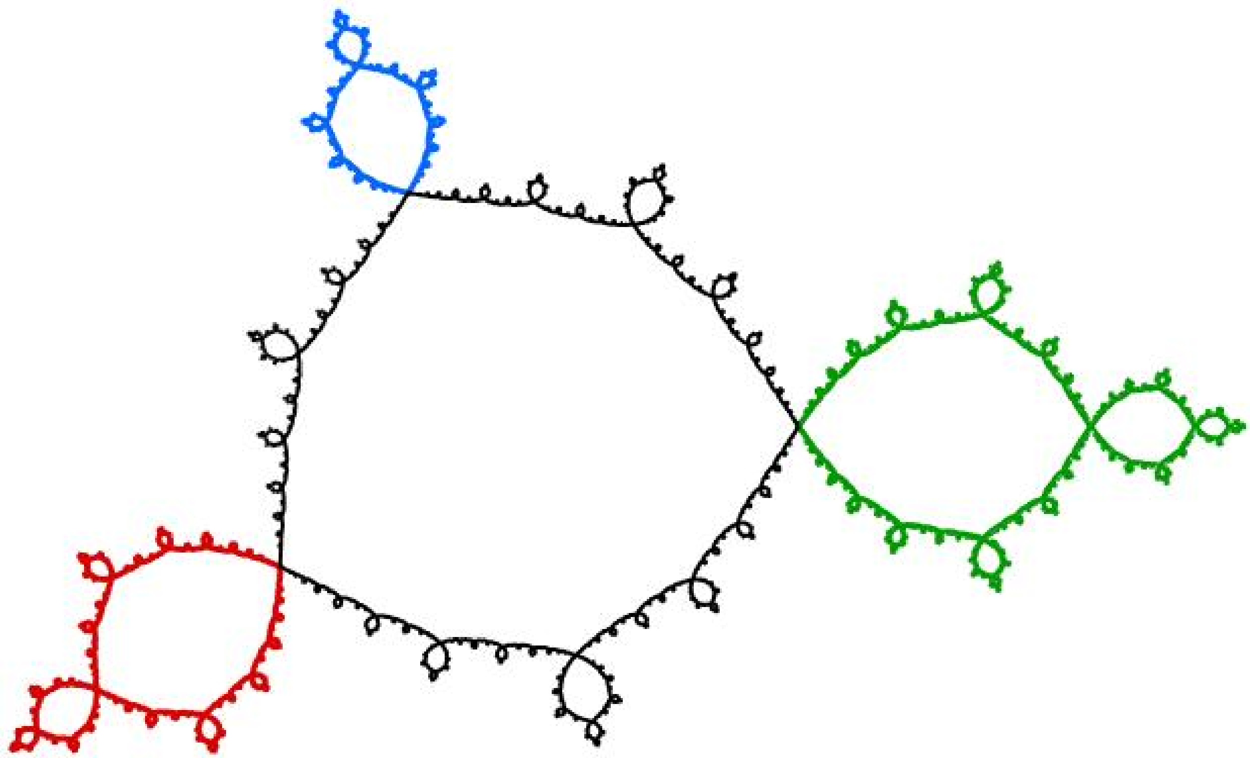
Elements of T_B act as piecewise-linear homeomorphisms.

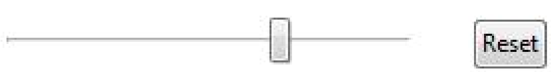


ˆ

Piecewise-Linear Homeomorphisms

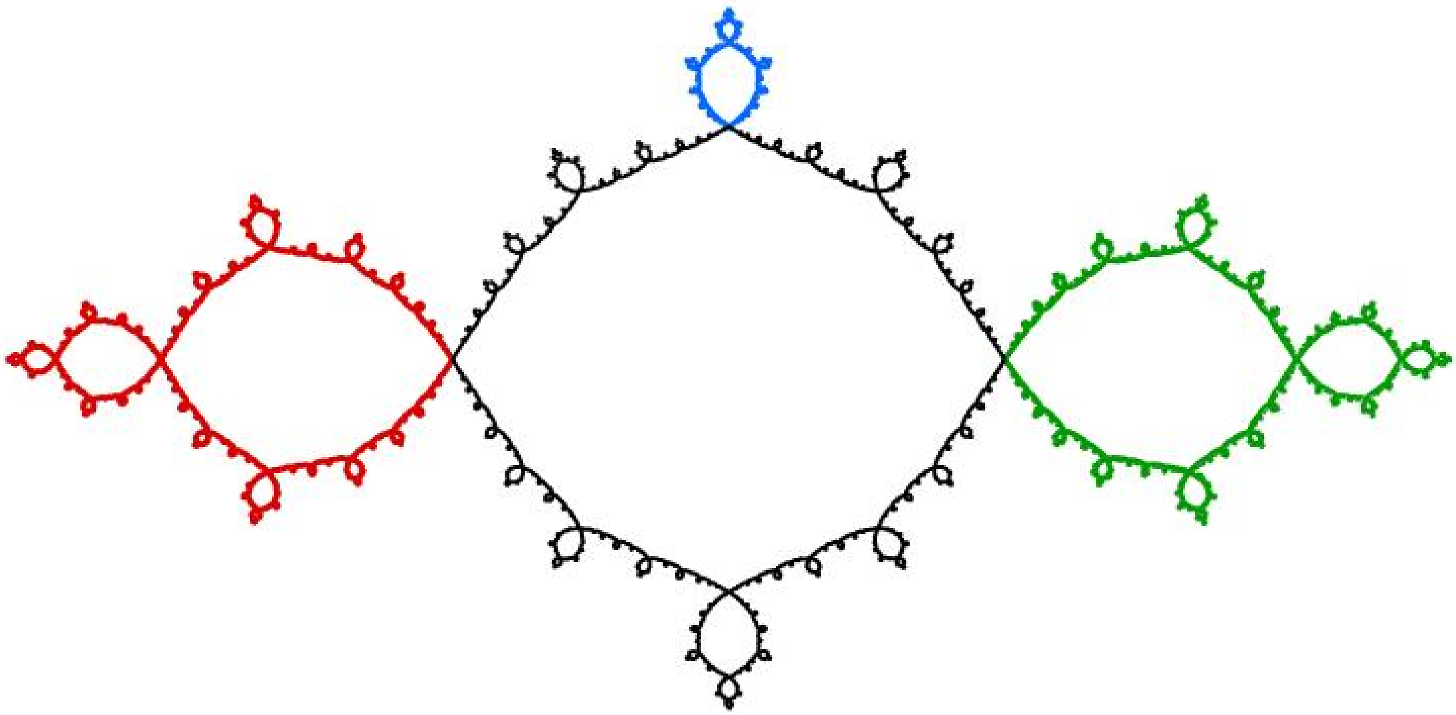
Elements of T_B act as piecewise-linear homeomorphisms.





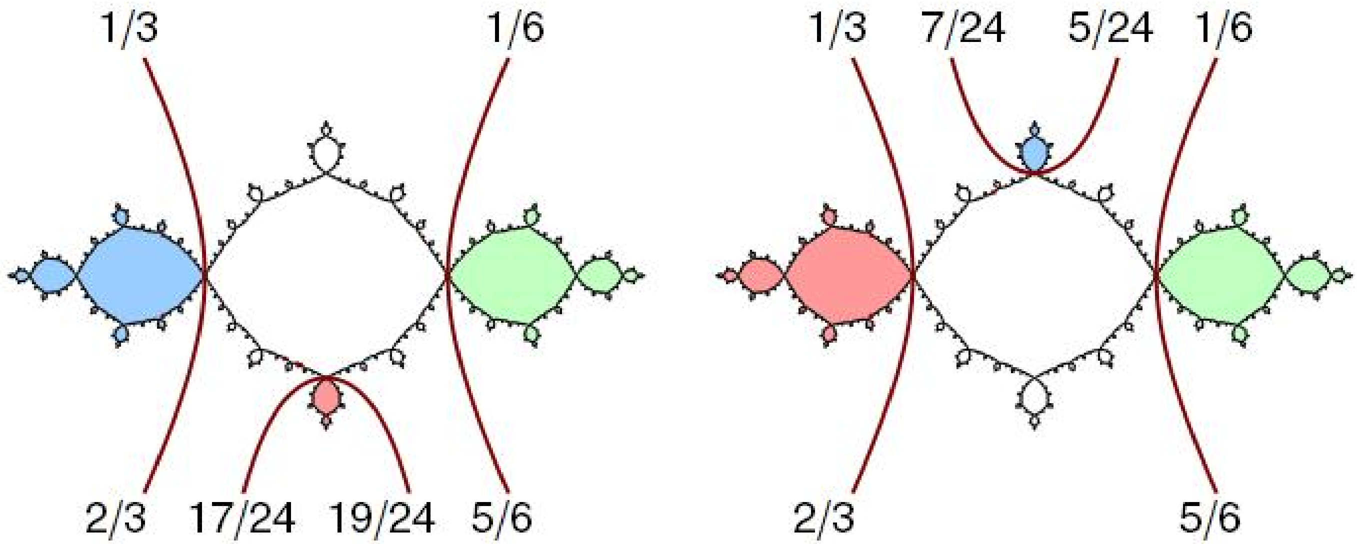
Piecewise-Linear Homeomorphisms

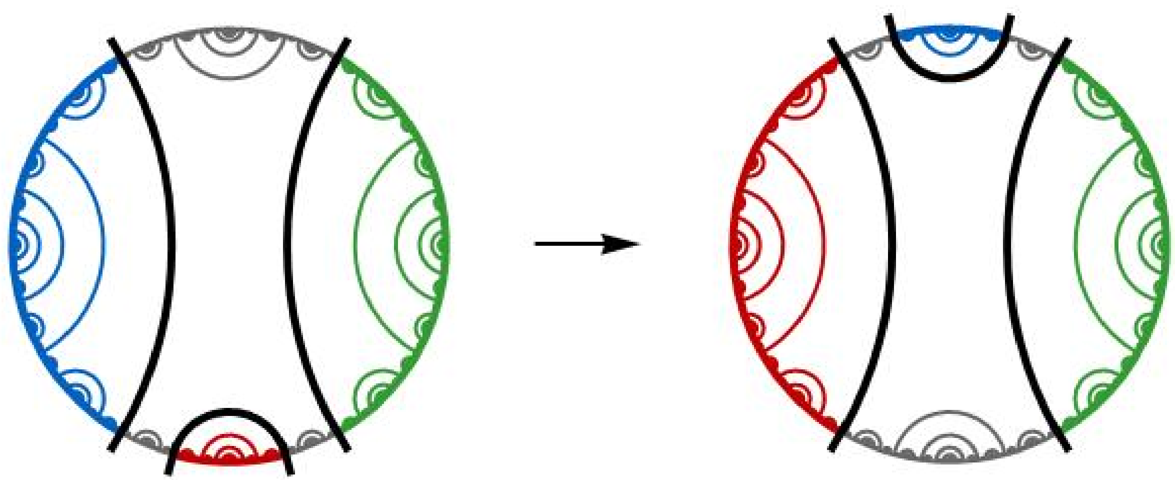
Elements of T_B act as piecewise-linear homeomorphisms.



Piecewise-Linear Homeomorphisms

Elements of T_B act as piecewise-linear homeomorphisms.





Piecewise-Linear Homeomorphisms

Theorem

T_B is isomorphic to the group of all PL homeomorphisms f of the circle for which:

1. *f preserves the invariant lamination, and*
2. *Each breakpoint of f is a pinch point.*

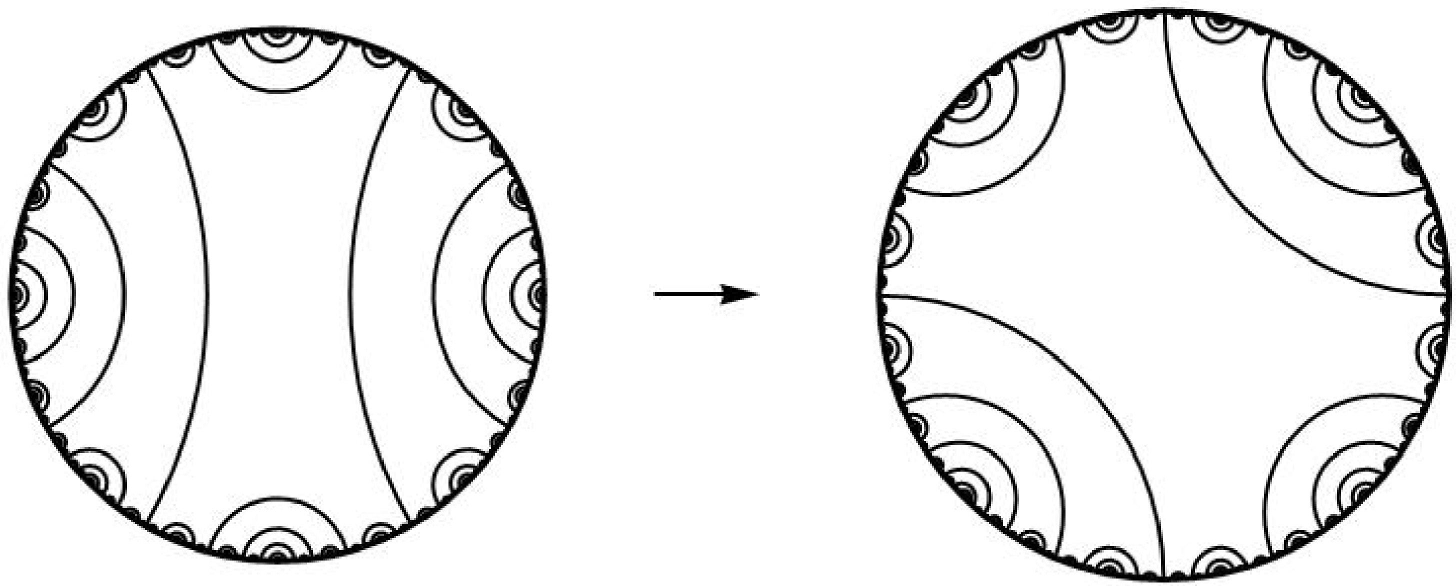
Observations

1. All slopes of elements of T_B are powers of two.
2. All breakpoints have denominators of the form $3 \cdot 2^n$.

Piecewise-Linear Homeomorphisms

Theorem

T_B is isomorphic to a subgroup of Thompson's group T.



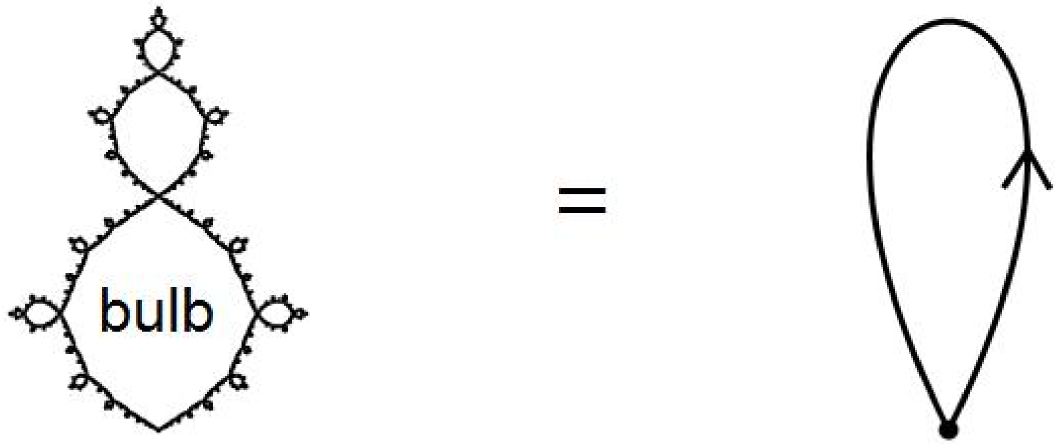
Question

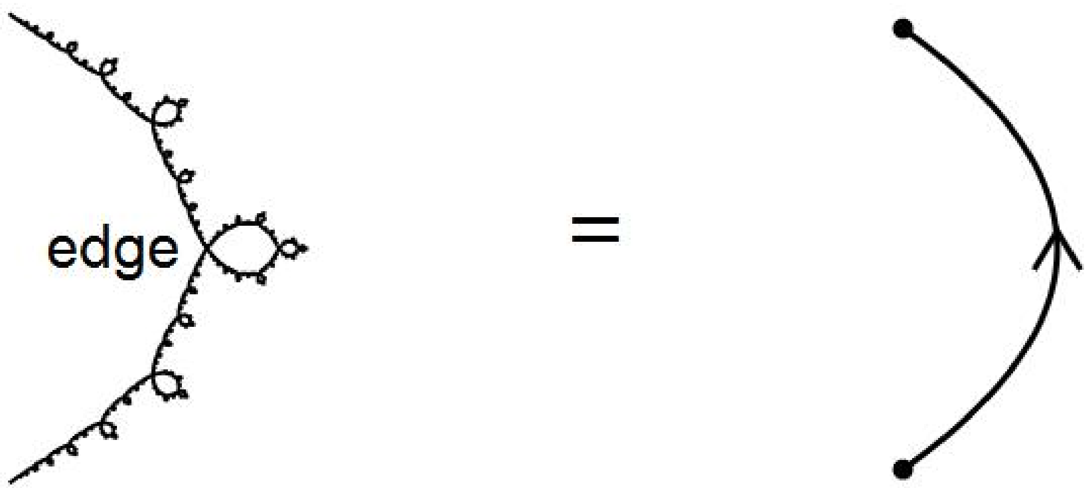
What can be said in general about subgroups of T defined by laminations?

Edge Replacement Systems

Edge Replacement

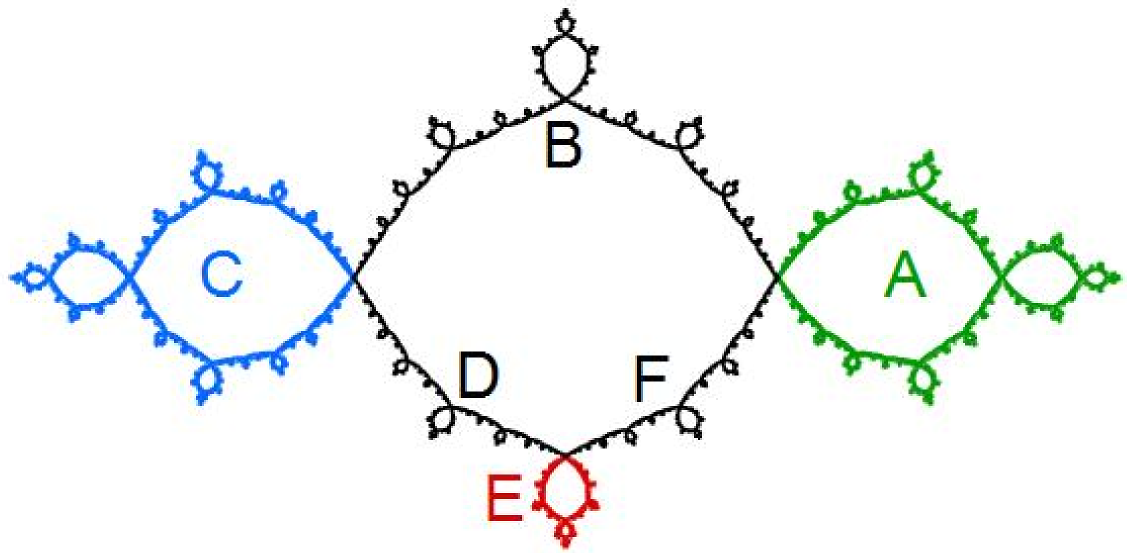
We can represent allowed subdivisions of the Basilica using finite graphs.

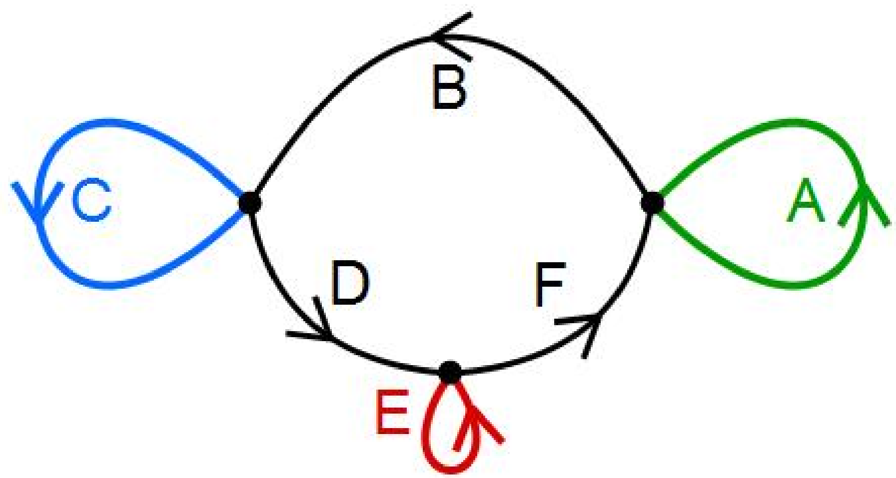




Edge Replacement

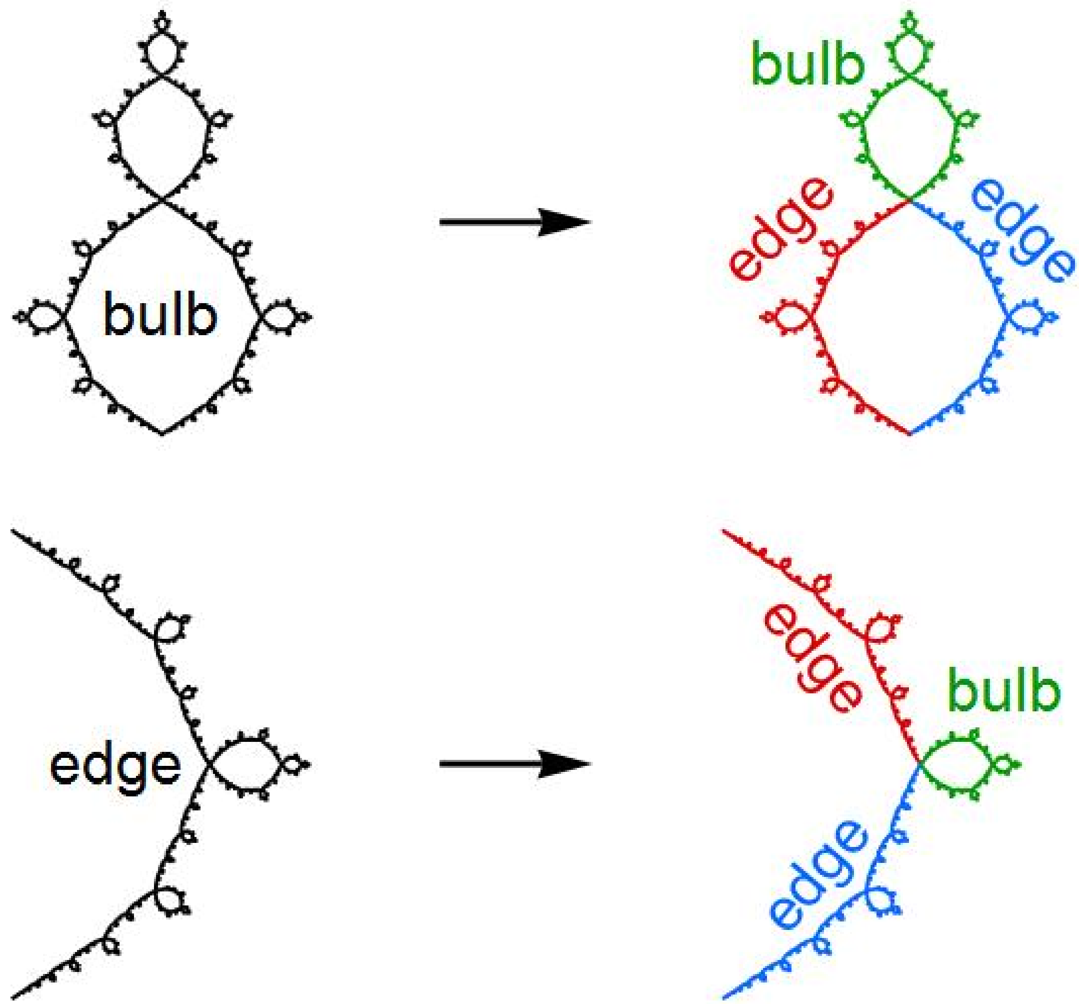
We can represent allowed subdivisions of the Basilica using finite graphs.





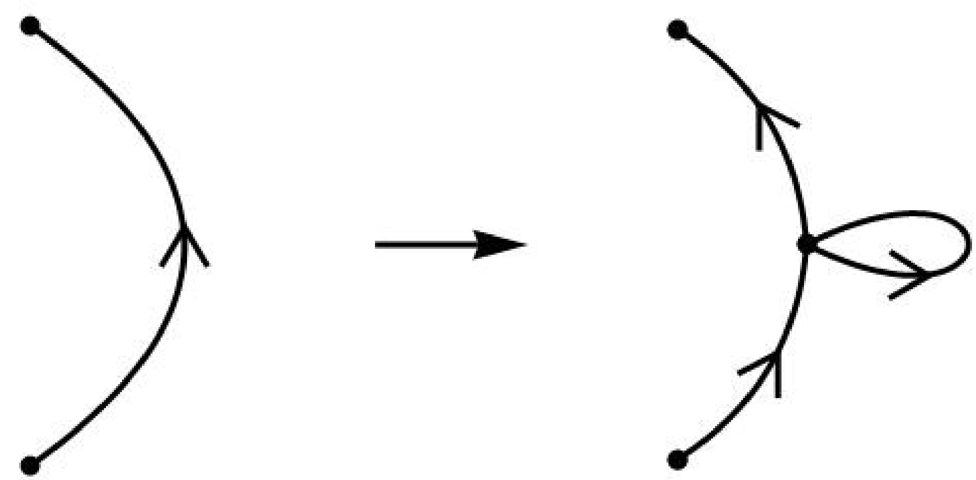
Edge Replacement

Instead of subdivision rules, we have an ***edge replacement rule***.



Edge Replacement

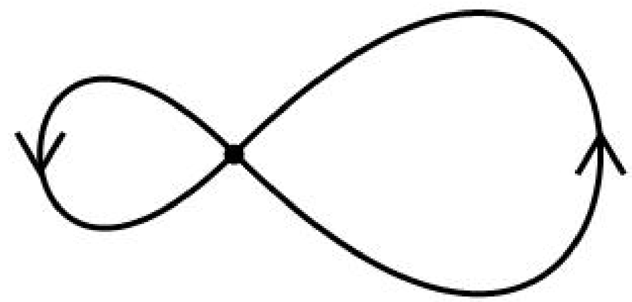
Instead of subdivision rules, we have an ***edge replacement rule***.



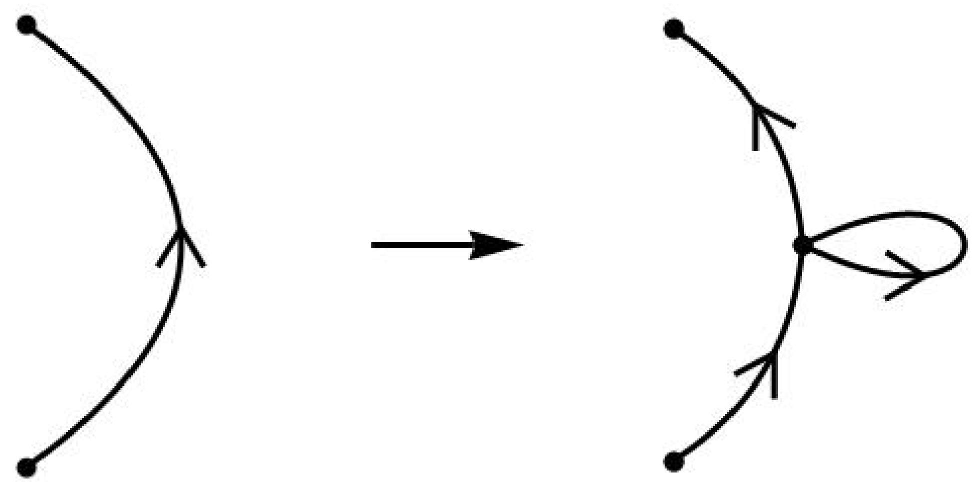
Edge Replacement

Together, these define an ***edge replacement system***.

Base Graph:



Replacement Rule:



Edge Replacement Systems

In general, an ***edge replacement*** system consists of:

1. A finite set of ***edge types***.
2. A ***base graph*** (directed graph, edges of the given types).
3. One ***replacement rule*** for each edge type.

Under certain conditions:

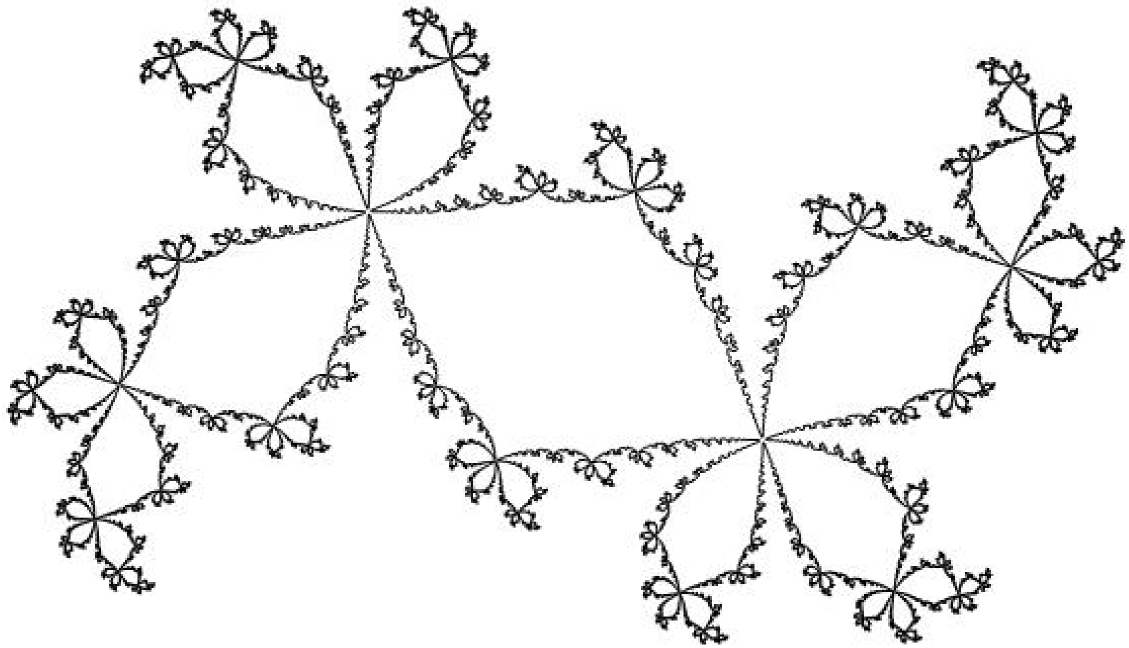
Edge Replacement System → Limit Space

Each edge corresponds to a portion of the limit space.

^

General Plan

Given: A Julia set (or other fractal).



Find: An edge replacement system that defines it.

Once we have an edge replacement system, it is easy to construct a Thompson-like group that acts on the fractal.

^

Diagram Groups

Actually, these groups *almost* fit into a well-known scheme.

Victor Guba and Mark Sapir defined ***diagram groups***:

- Generalization of Thompson's groups.
- Defined using a ***string rewriting system***.

Theorem (Farley)

Every diagram group over a finite string rewriting system acts properly by isometries on a CAT(0) *cubical complex.*

CAT(0) Cubical Complexes

An edge replacement system is a type of ***graph rewriting system***.

String Rewriting System → Graph Rewriting System

We have defined a class of groups called ***graph diagram groups***, which includes all diagram groups as well as all Julia set groups.

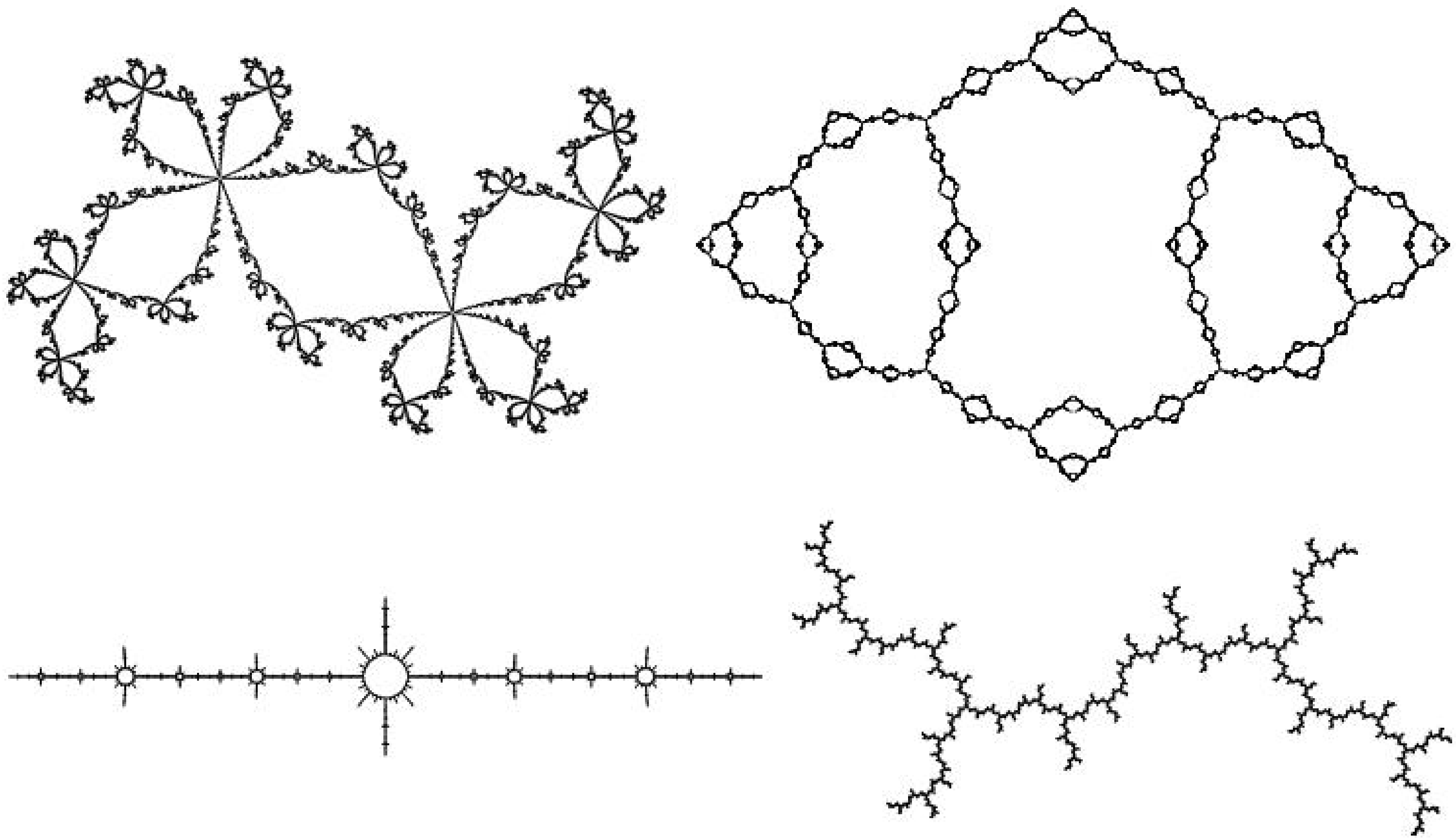
Theorem

Every graph diagram group over a finite graph rewriting system acts properly by isometries on a CAT(0) *cubical complex.*

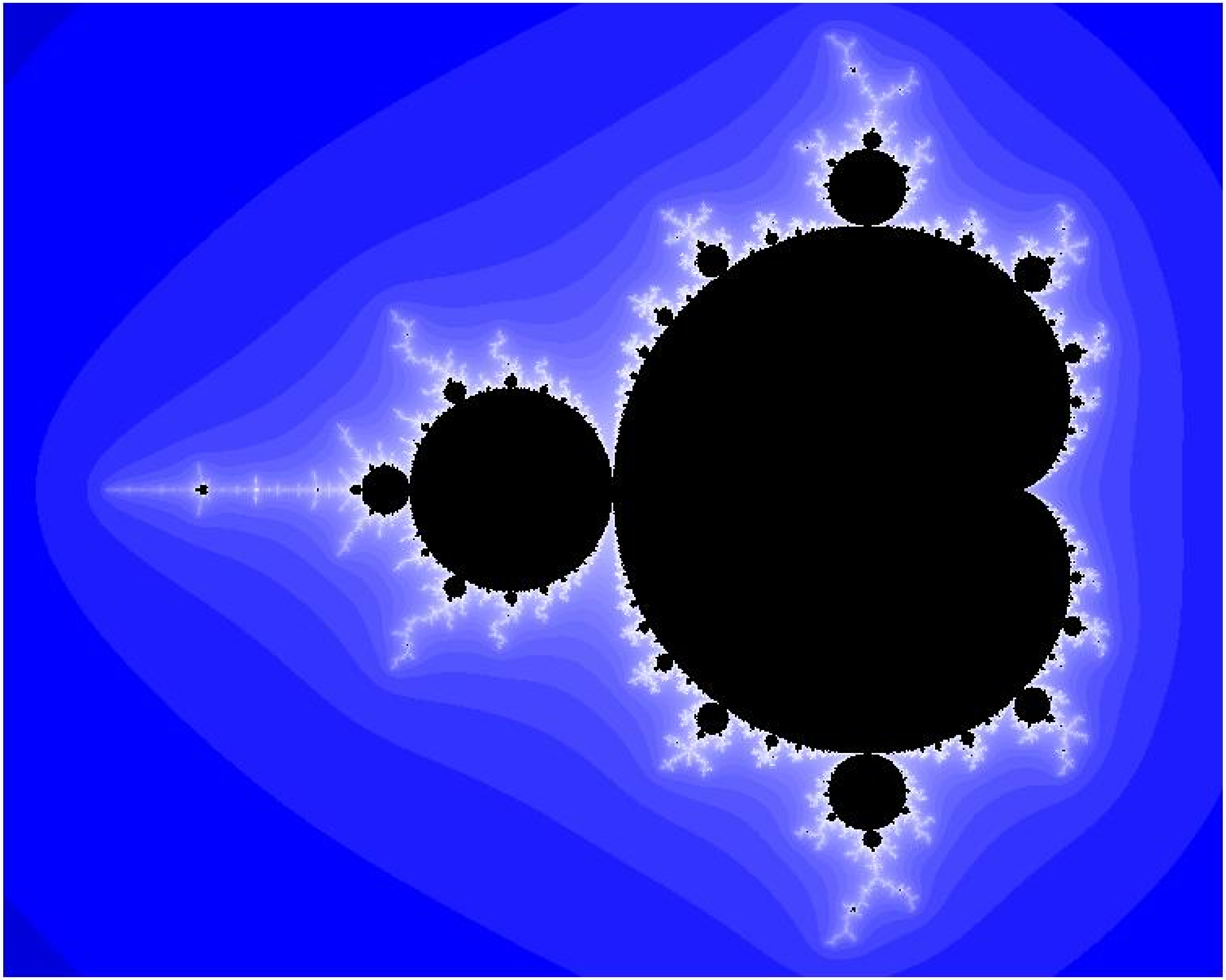
More Julia Sets

More Julia Sets

Every rational map on the Riemann sphere has an associated Julia set.

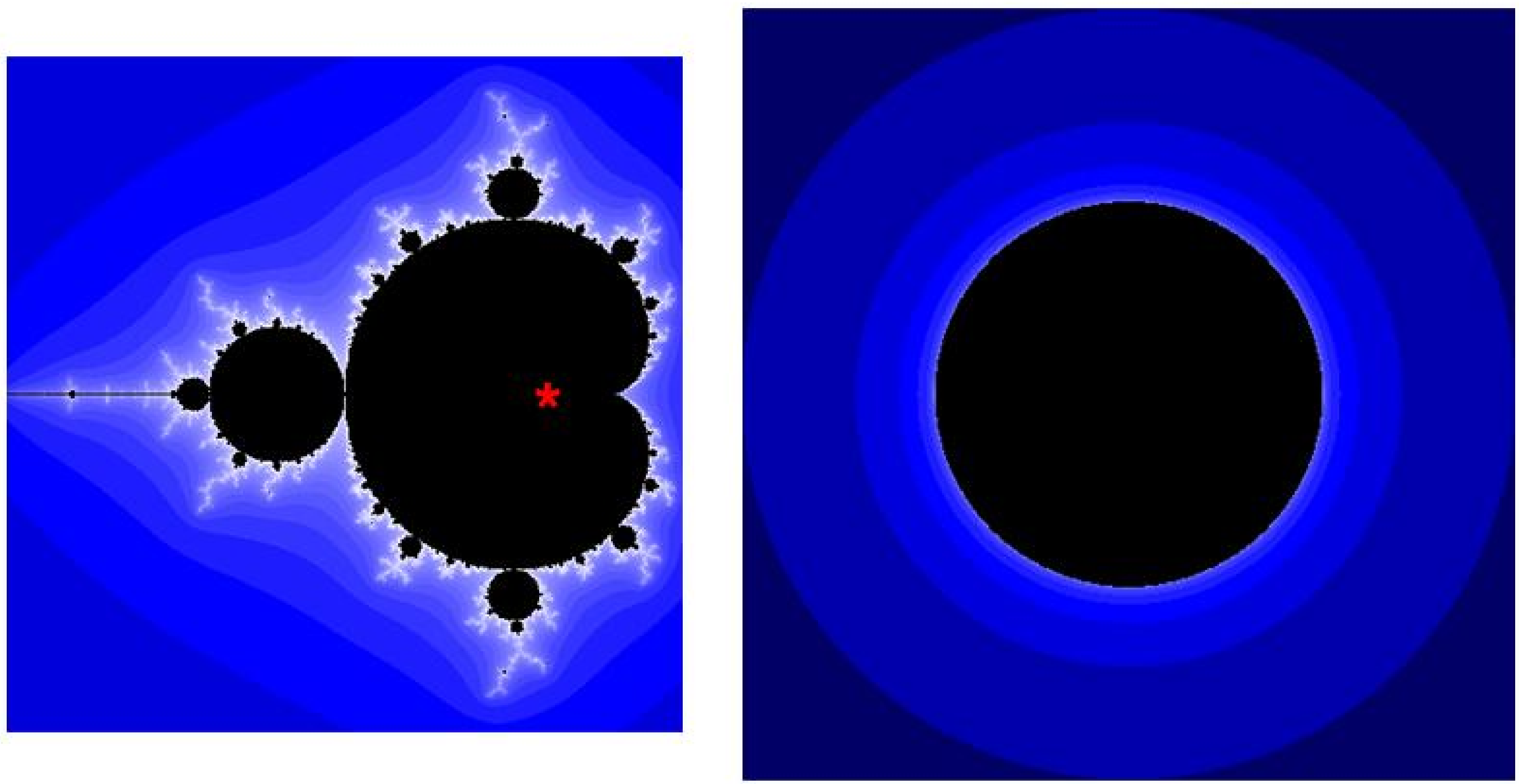


Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



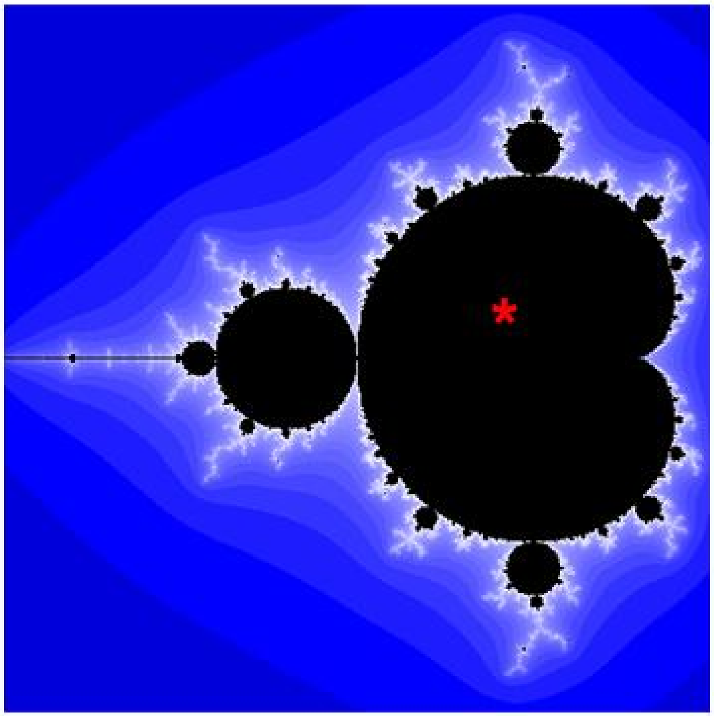
More Julia Sets

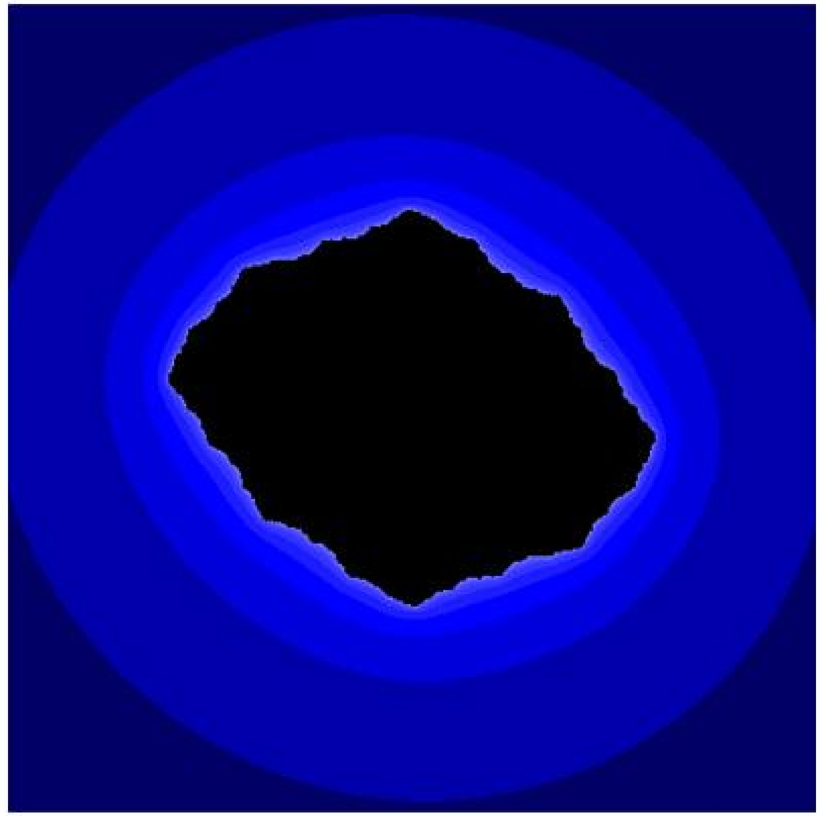
Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



More Julia Sets

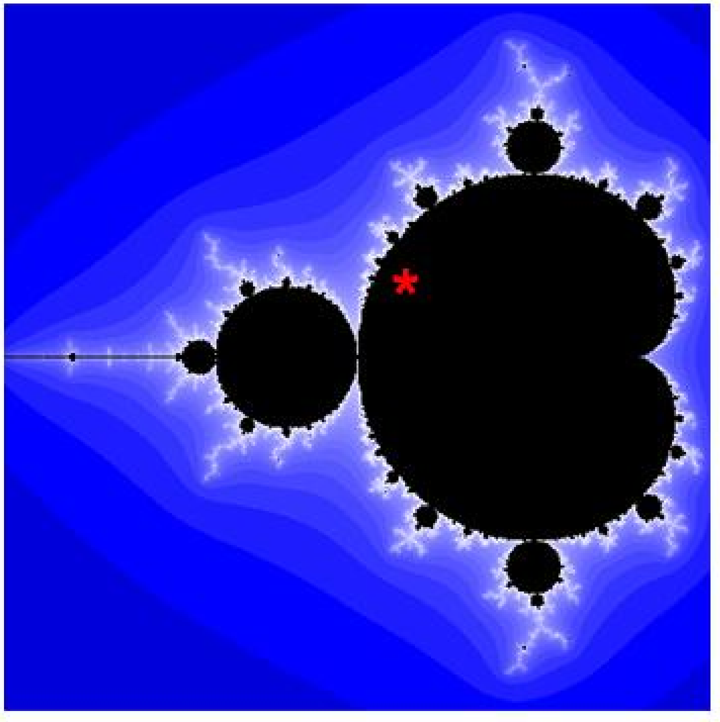
Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.

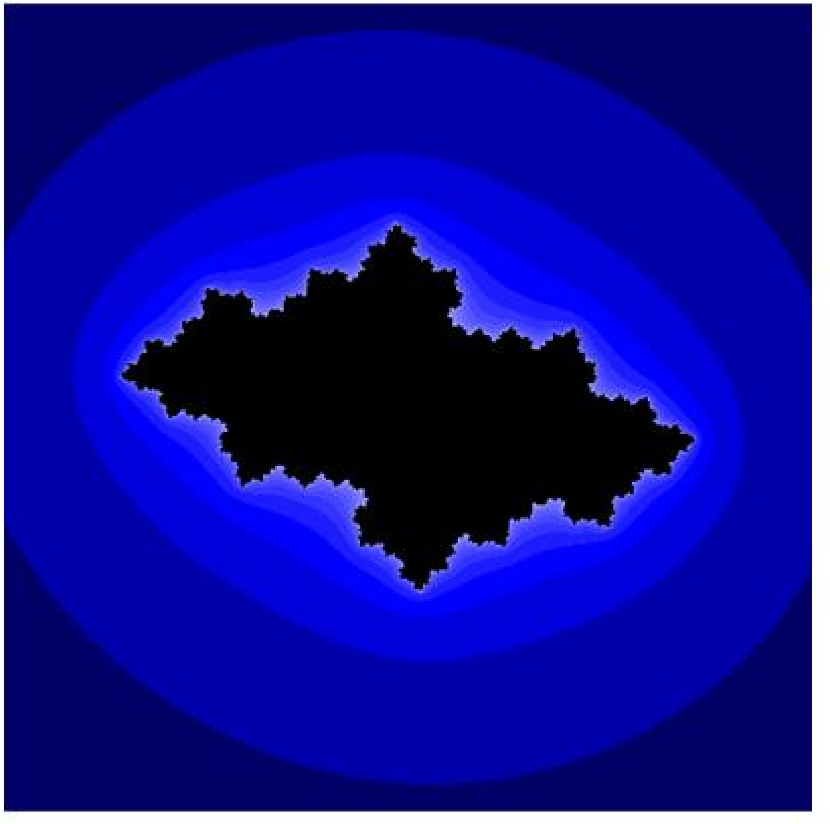




More Julia Sets

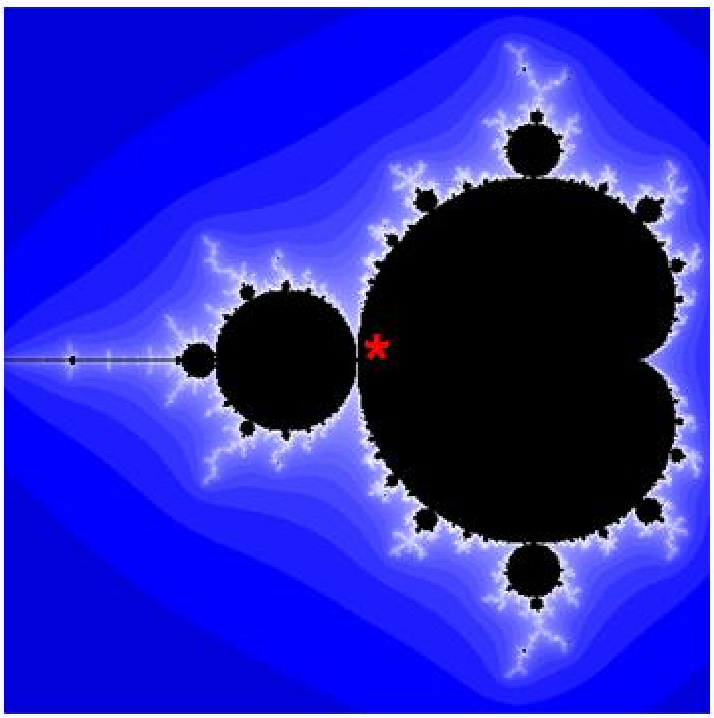
Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.

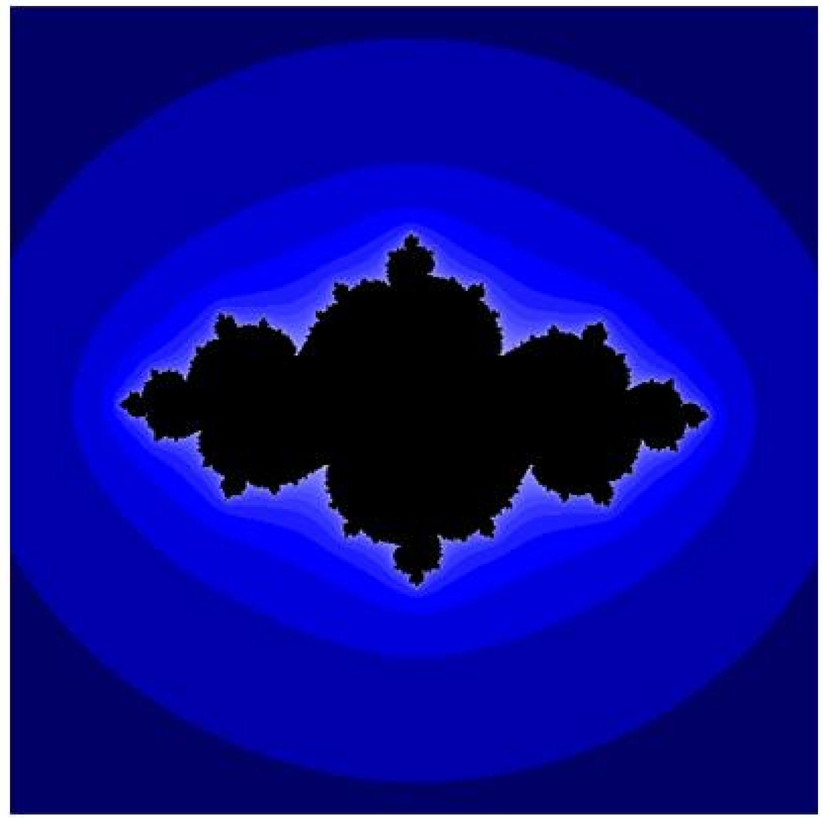




More Julia Sets

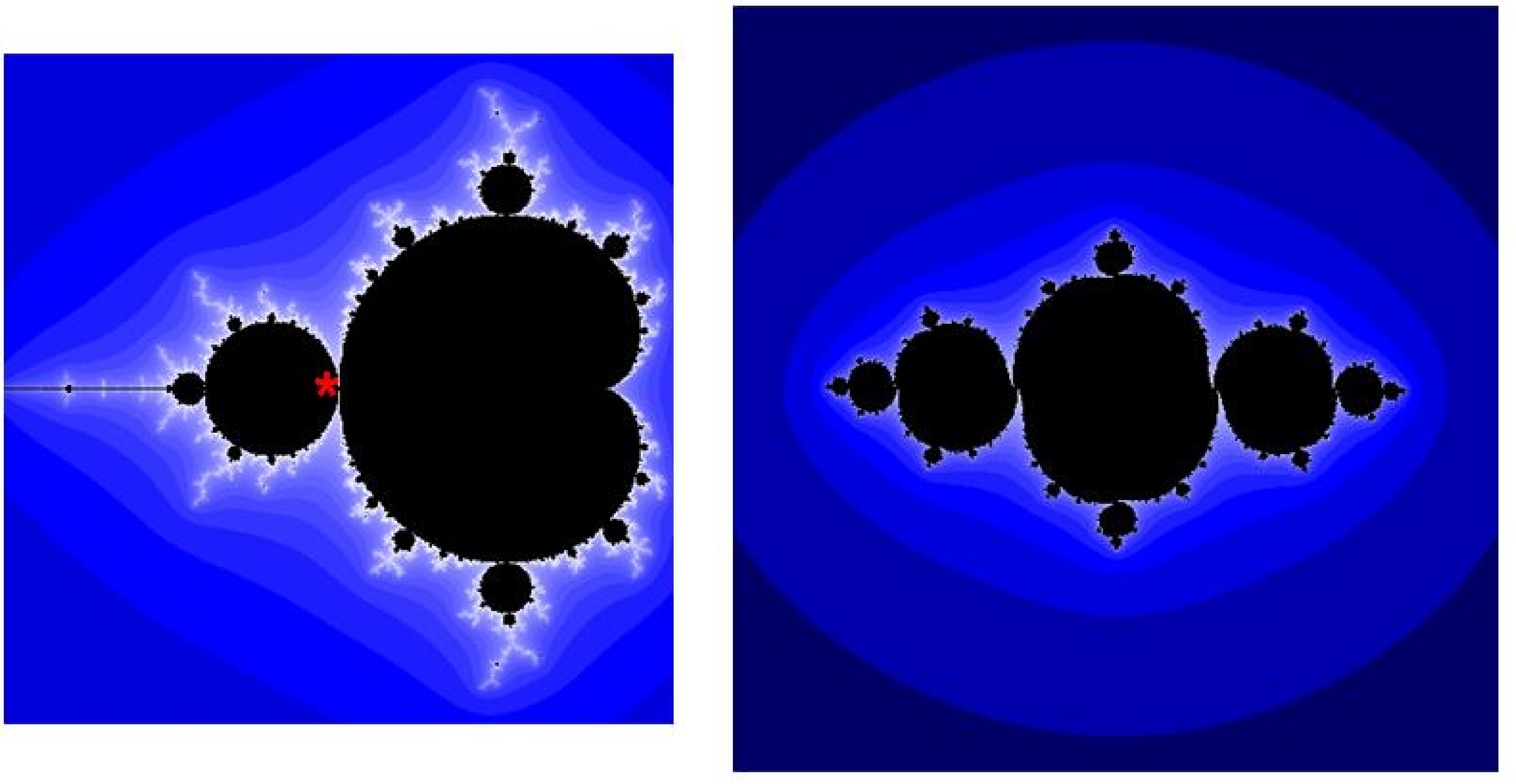
Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.





More Julia Sets

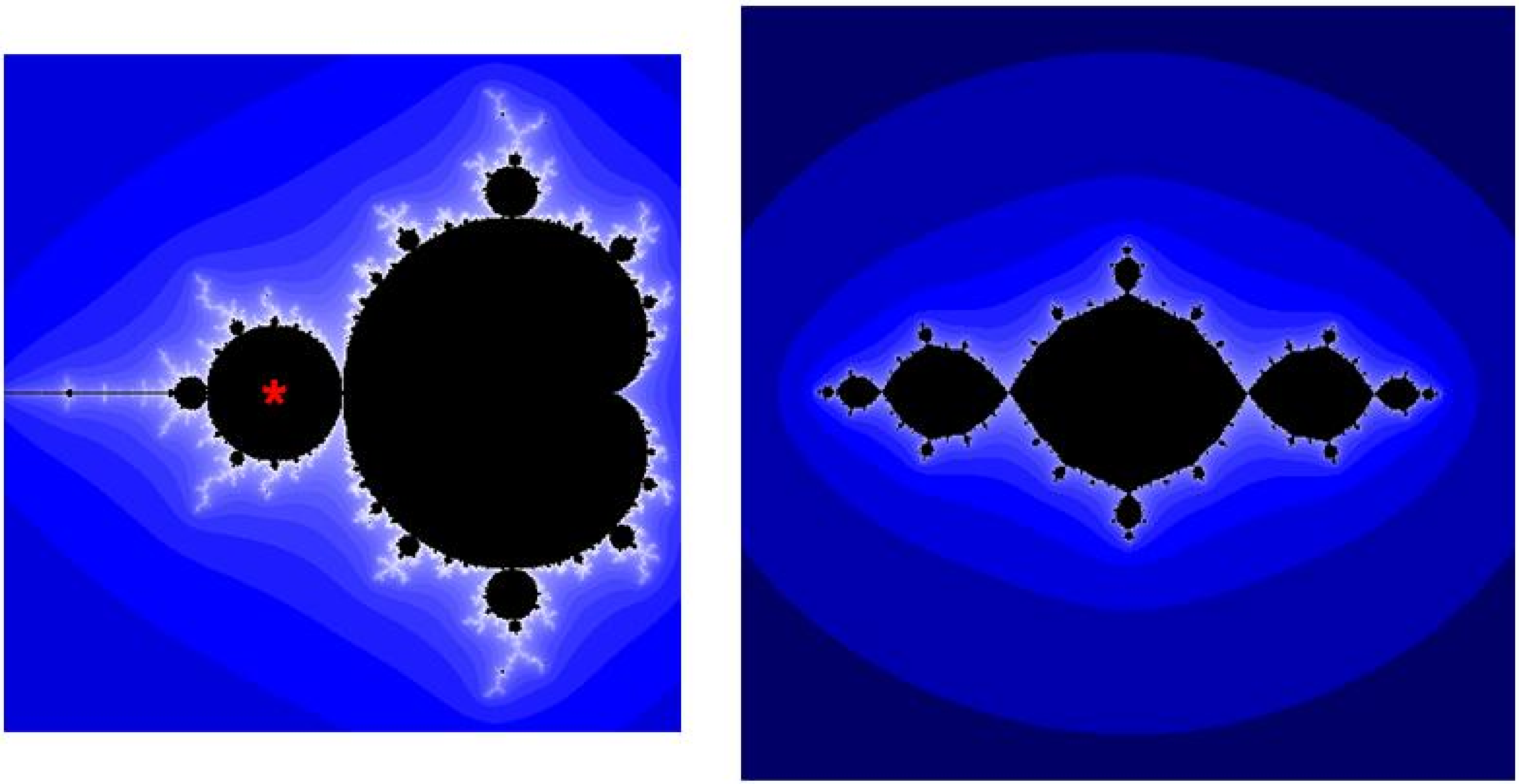
Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



^

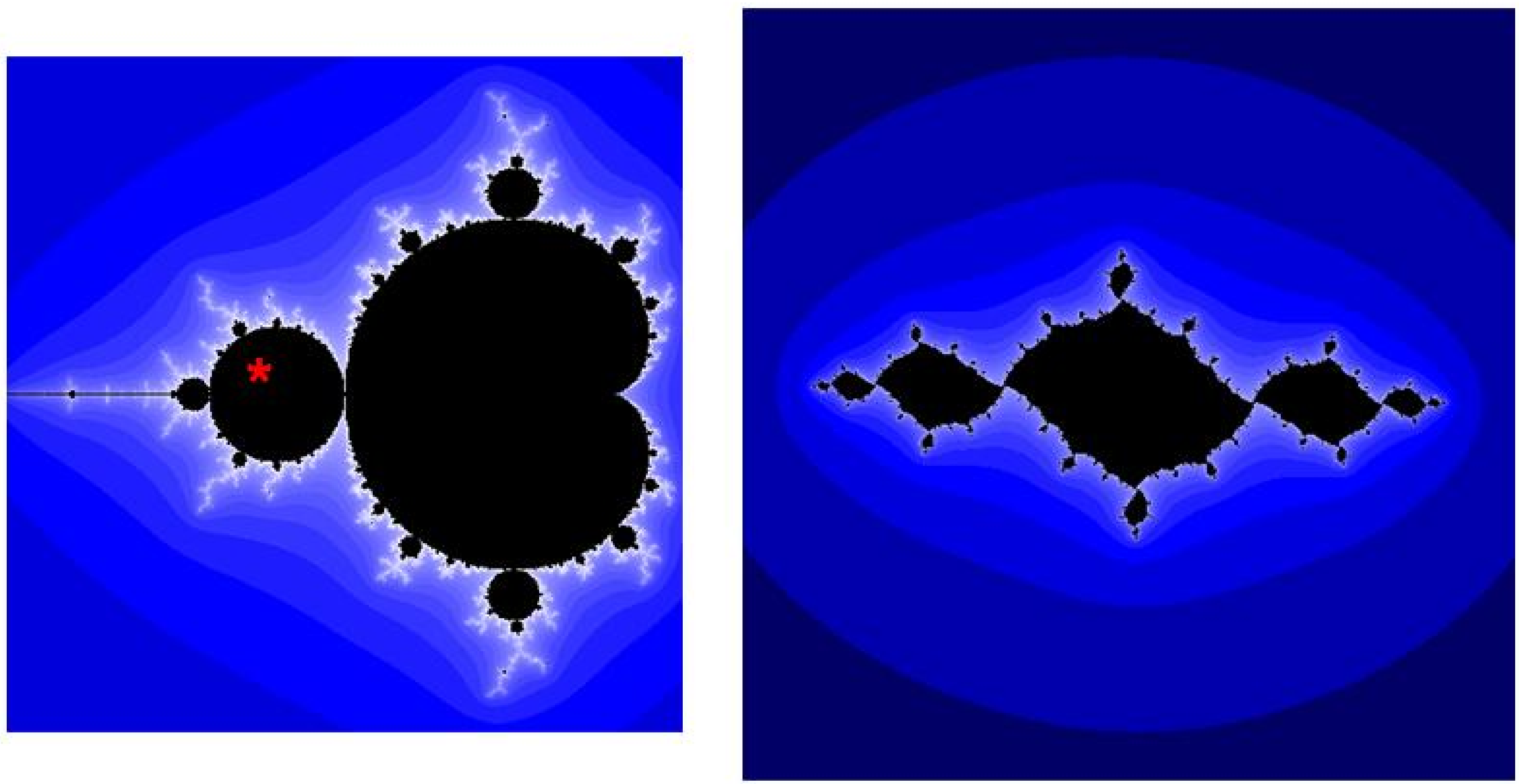
More Julia Sets

Julia sets for quadratic polynomials $z \longmapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



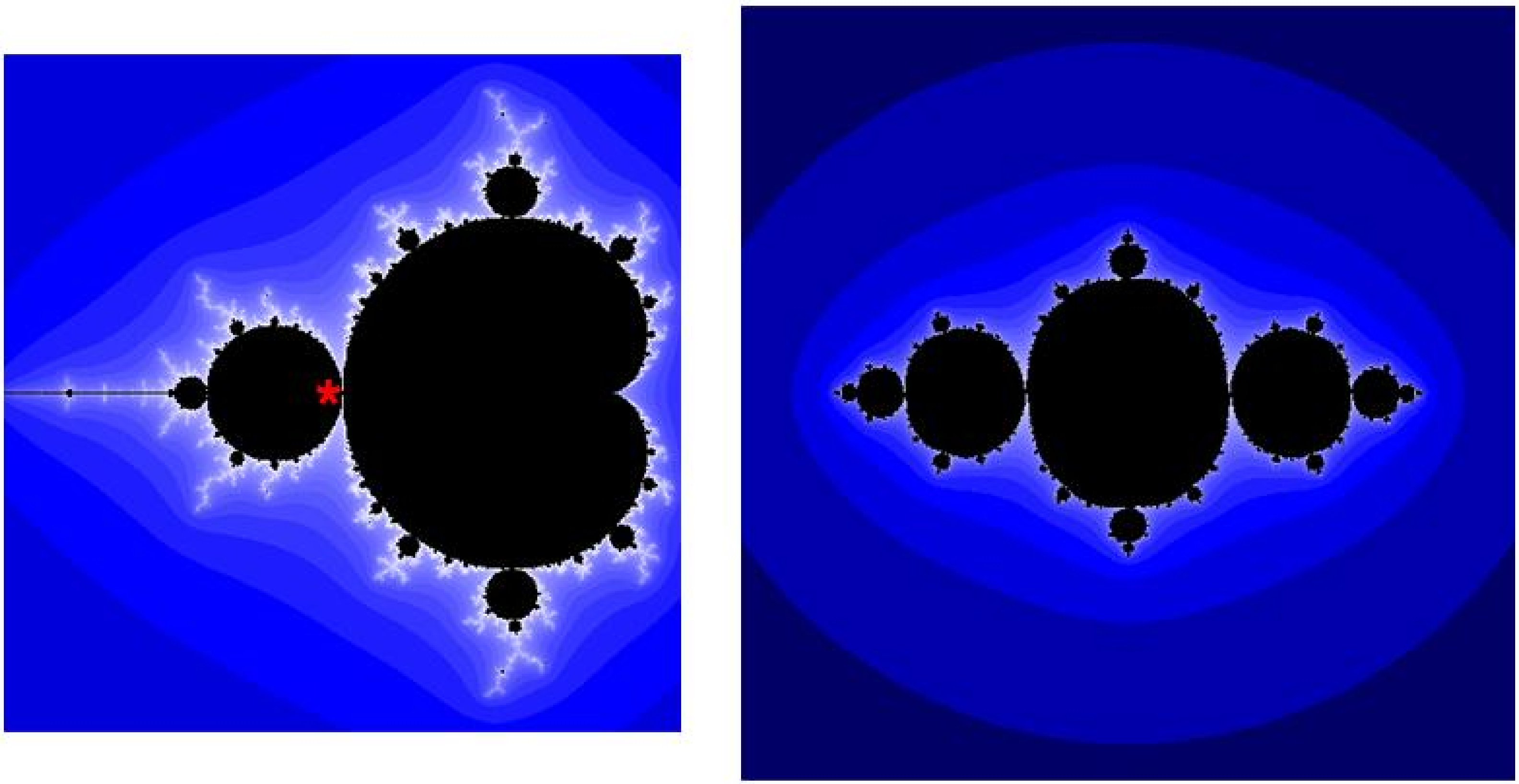
More Julia Sets

Julia sets for quadratic polynomials $z \longmapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



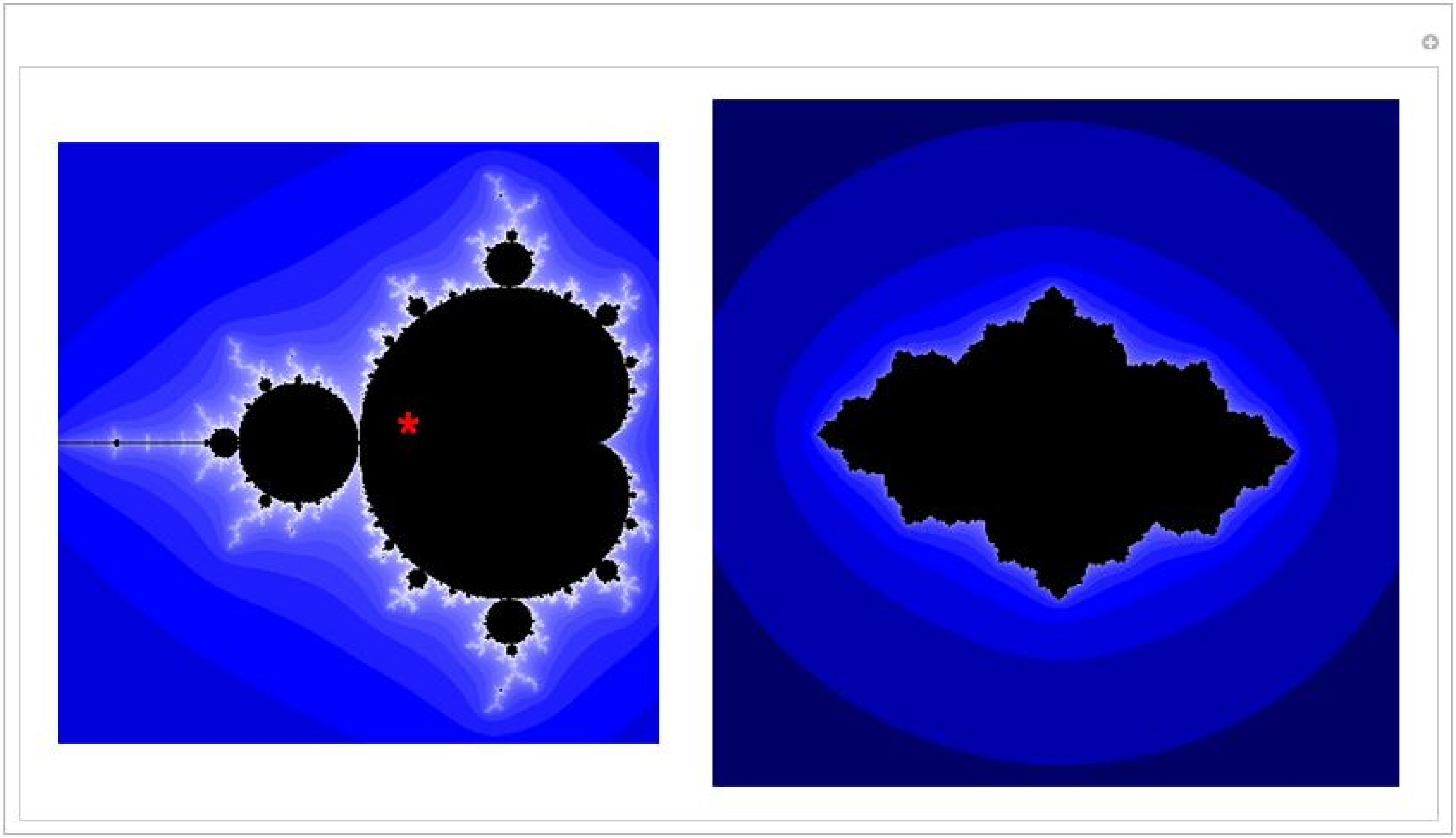
More Julia Sets

Julia sets for quadratic polynomials $z \longmapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



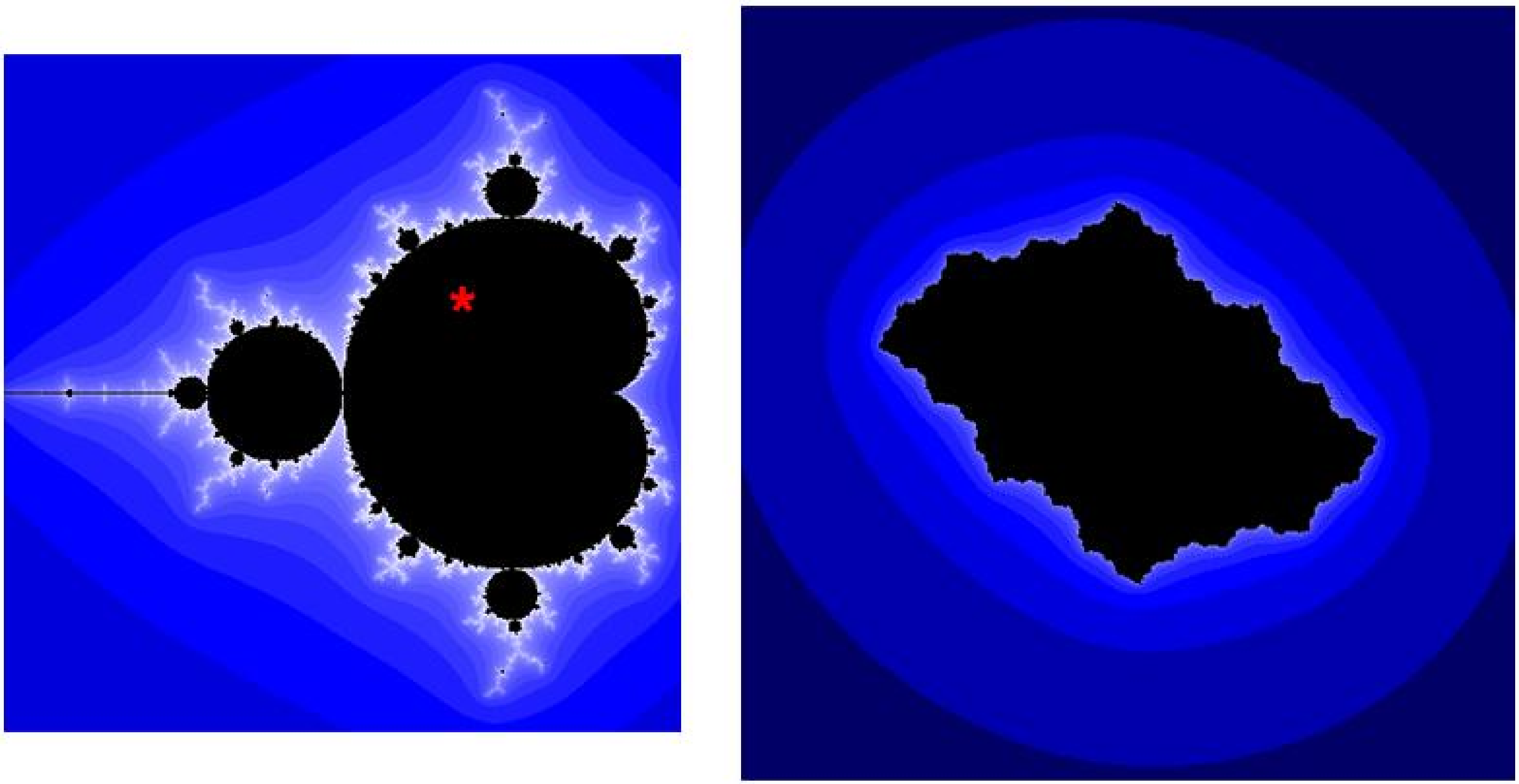
More Julia Sets

Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



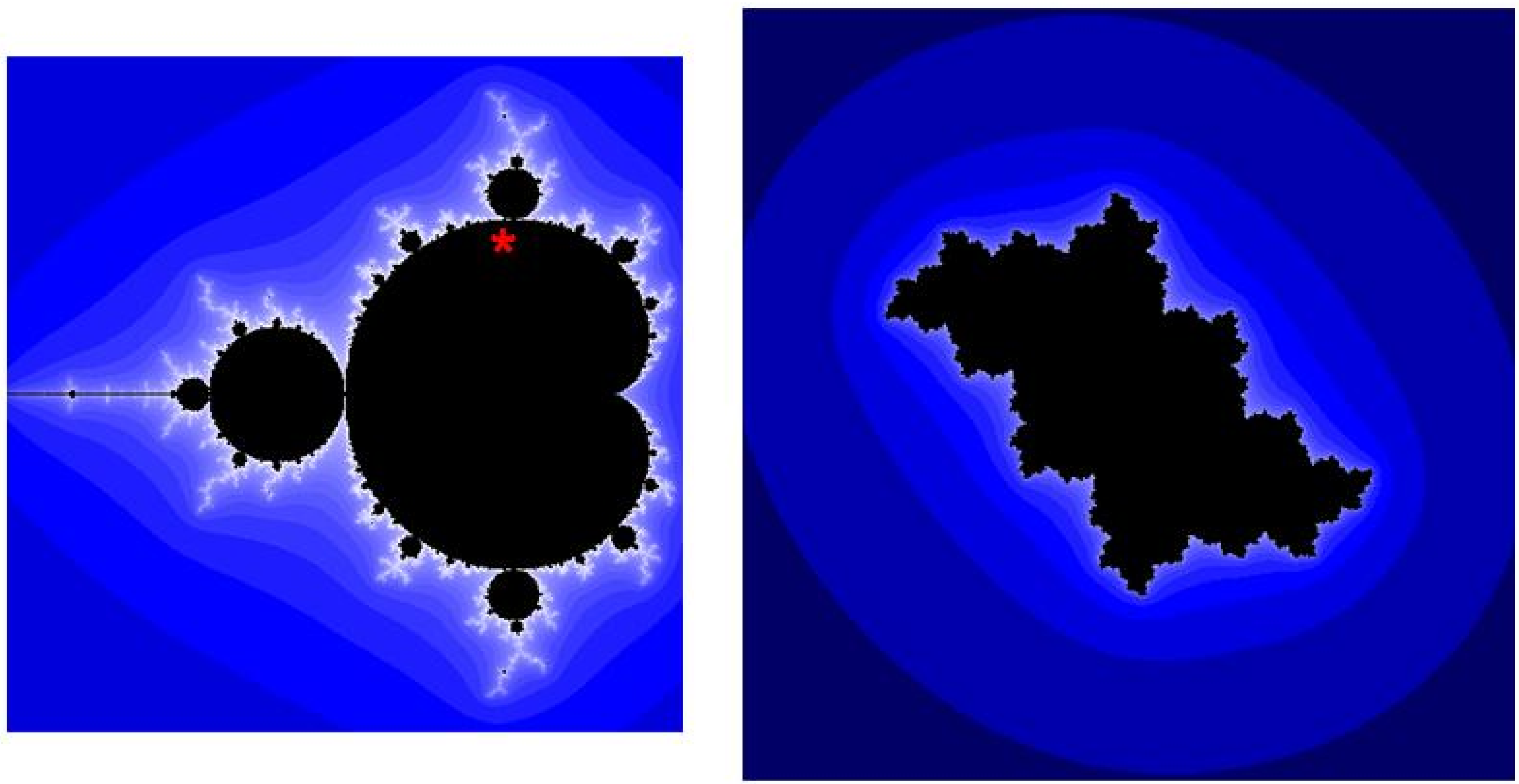
More Julia Sets

Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



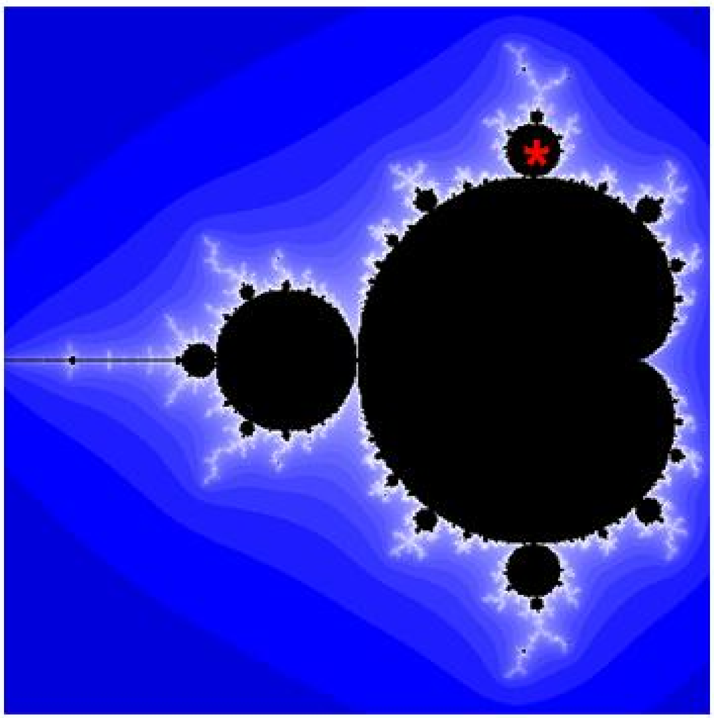
More Julia Sets

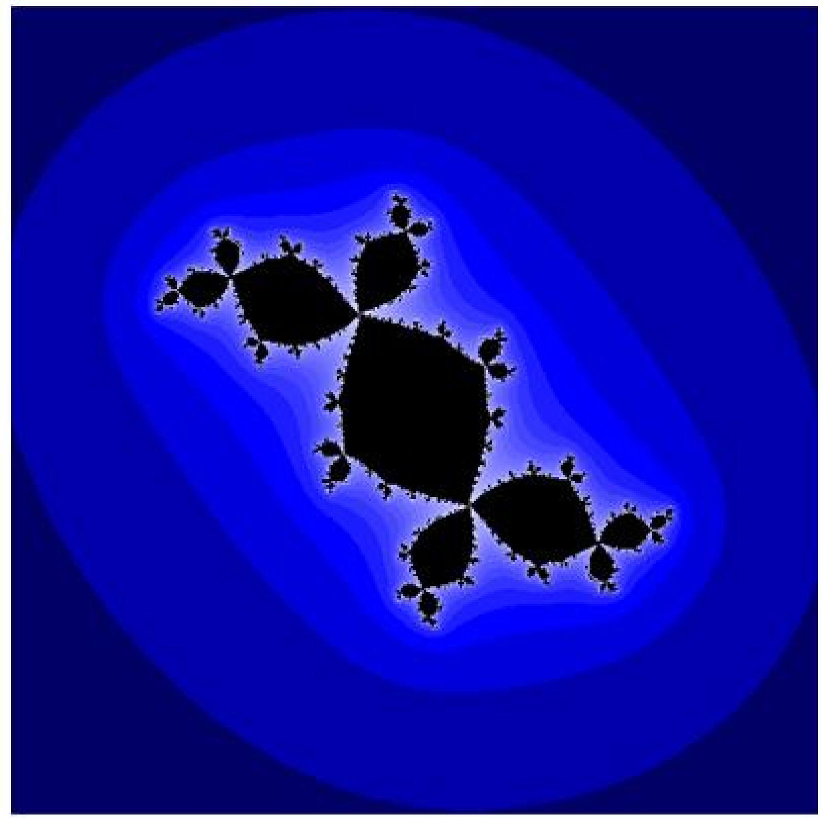
Julia sets for quadratic polynomials $z \mapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.



More Julia Sets

Julia sets for quadratic polynomials $z \longmapsto z^2 + c$ are parameterized by the ***Mandelbrot set***.





More Julia Sets

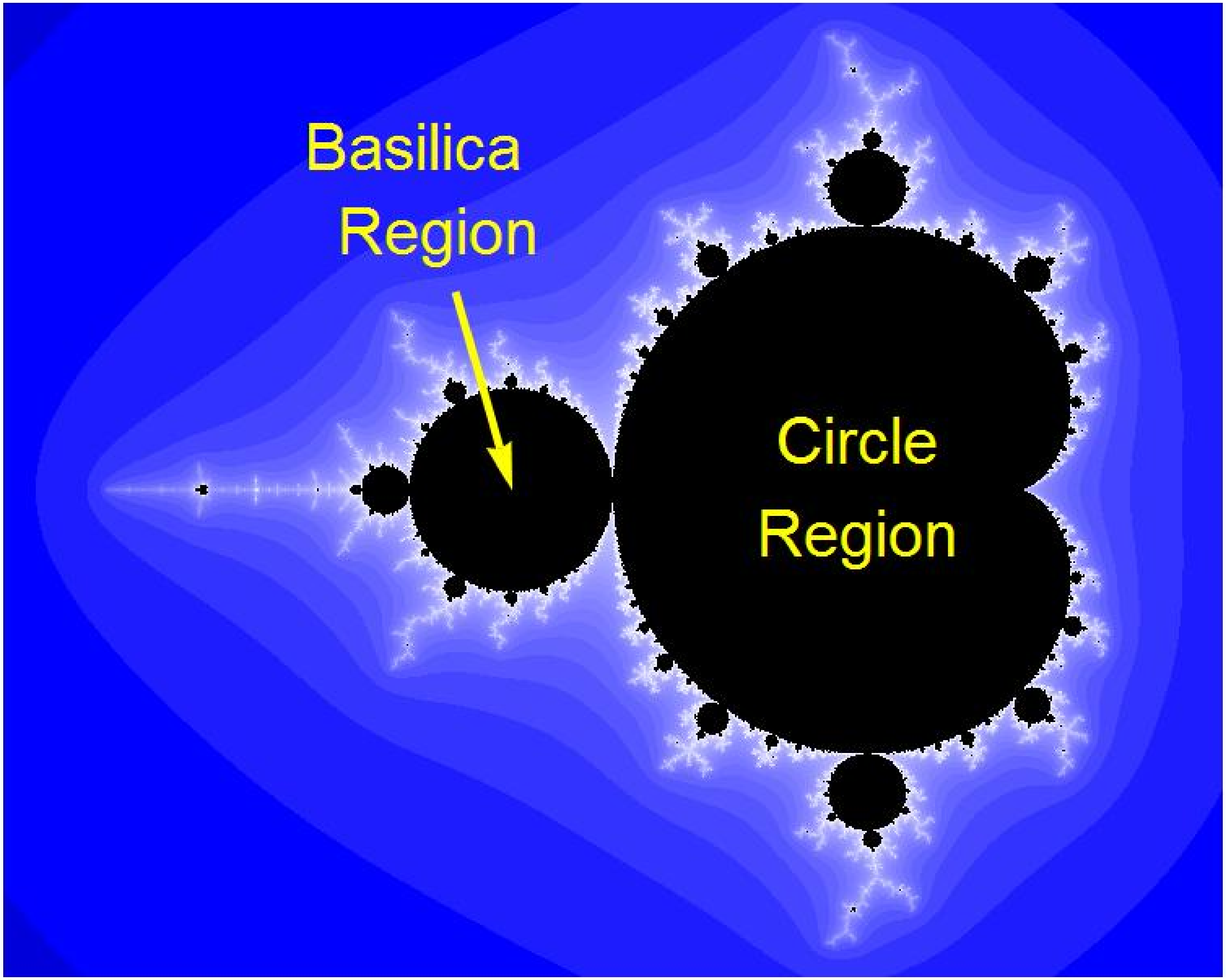
For "most" points in the Mandelbrot set, the structure of the Julia set is described by a ***Hubbard tree*.**

There exists a simple algorithm:

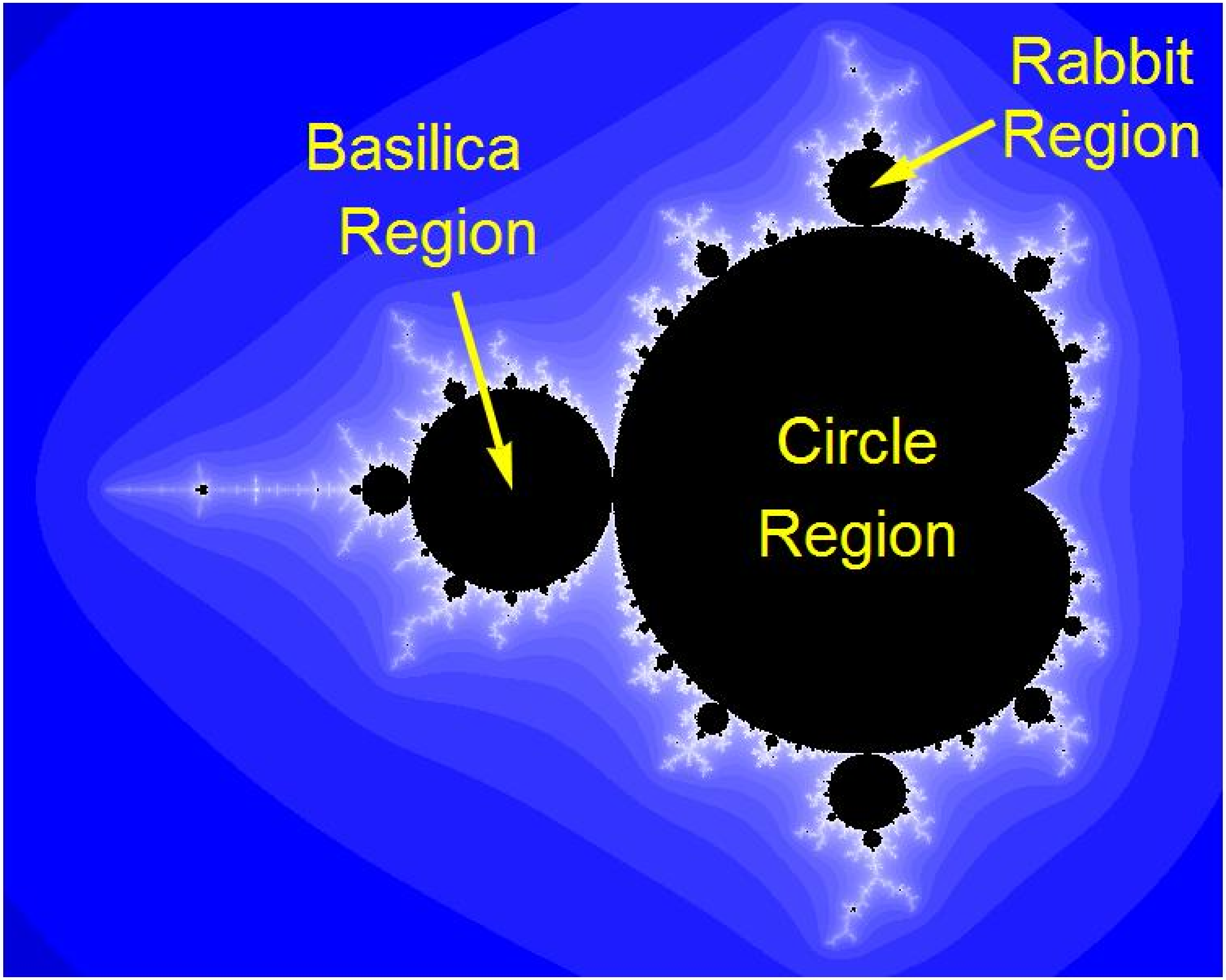
Hubbard Tree → Edge Replacement System

Conclusion

We can construct a Thompson-like group for "most" quadratic Julia sets.

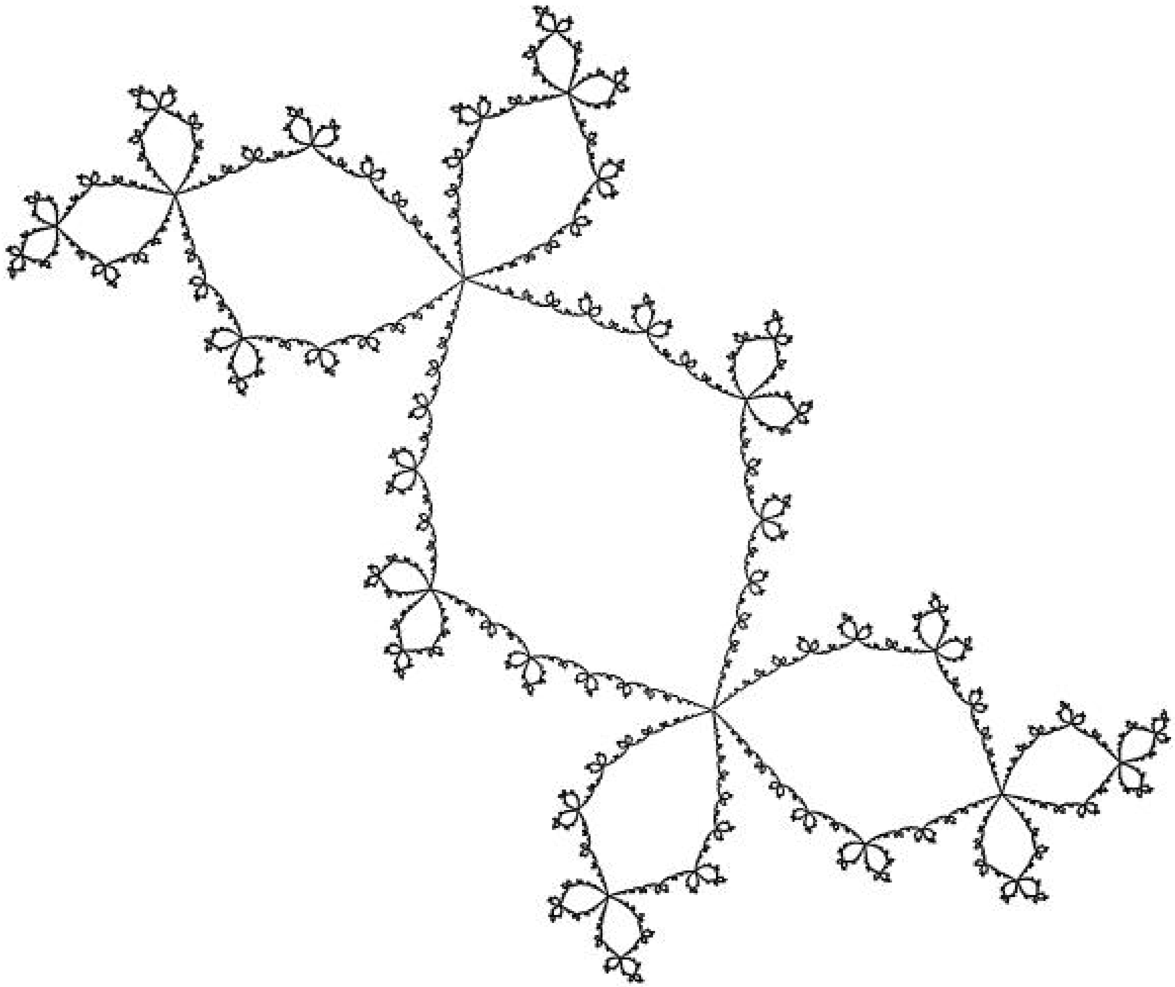

Basilica
Region
Circle
Region

^

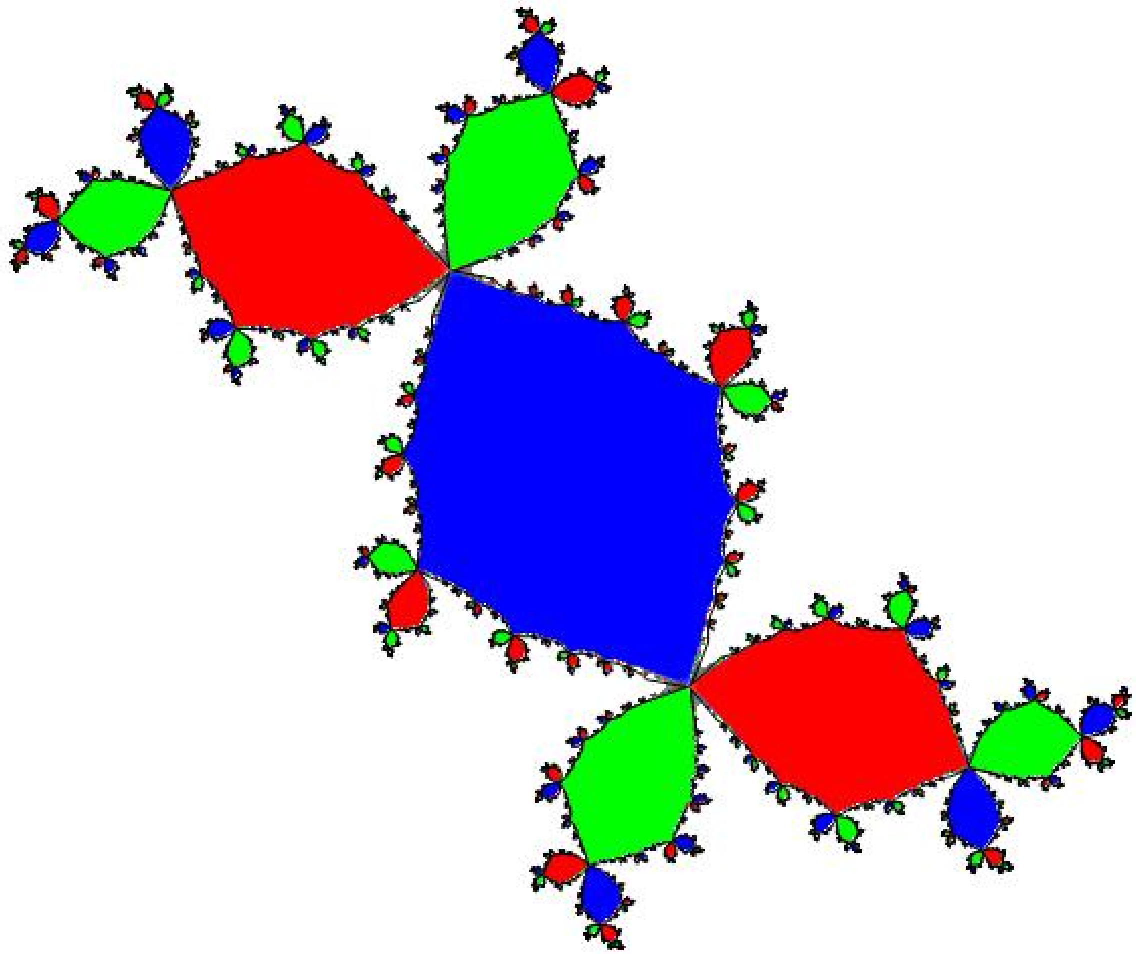

Rabbit
Region
Basilica
Region
Circle
Region

^

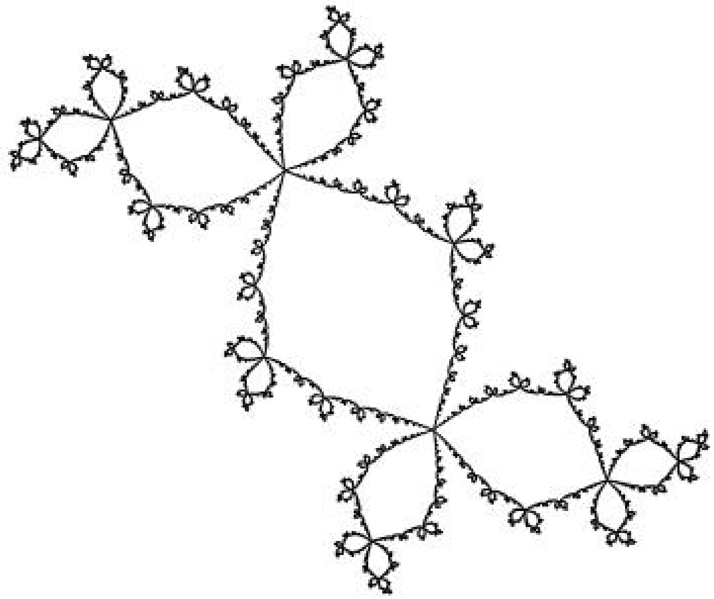
The Douady Rabbit



The Douady Rabbit



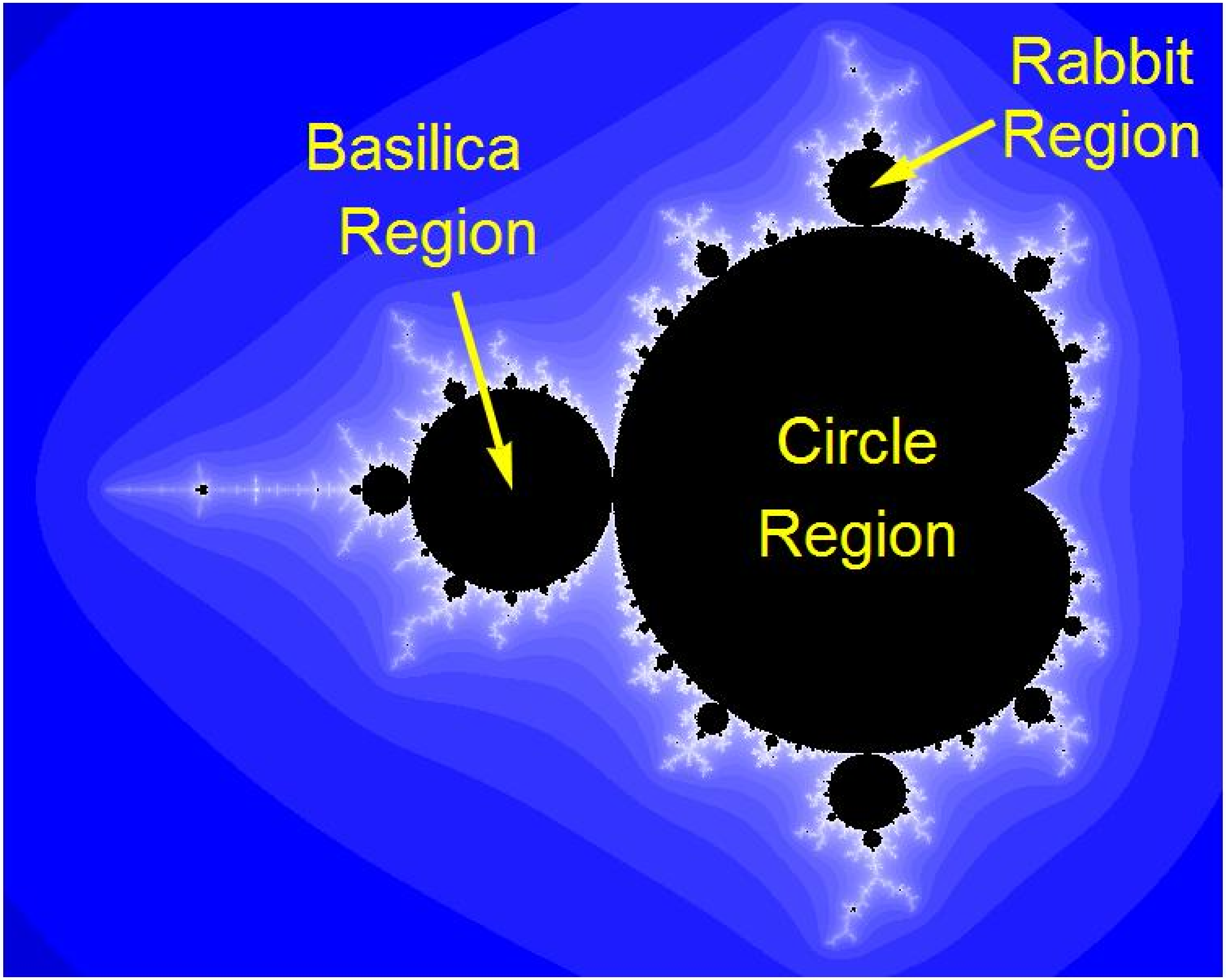
The Douady Rabbit

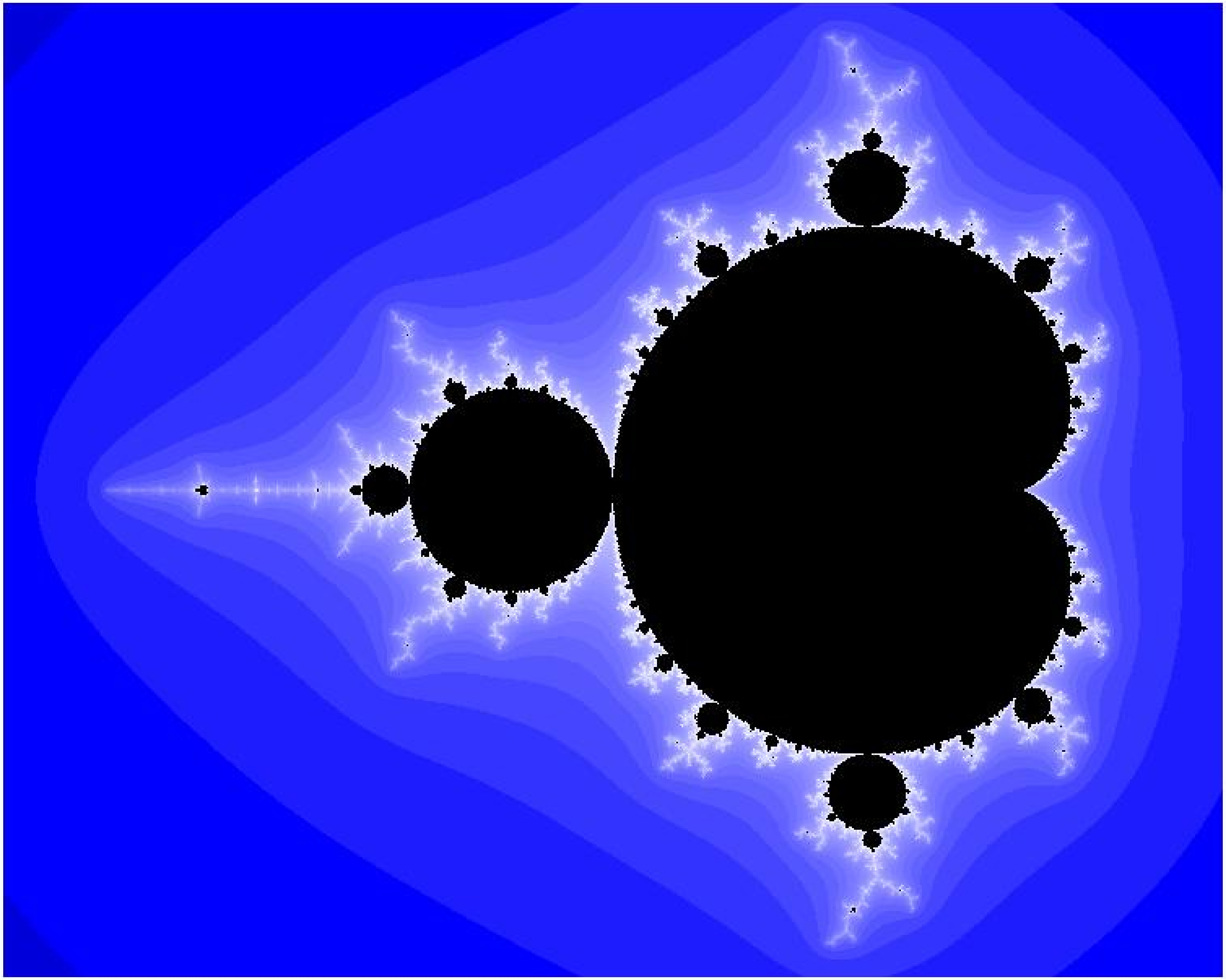


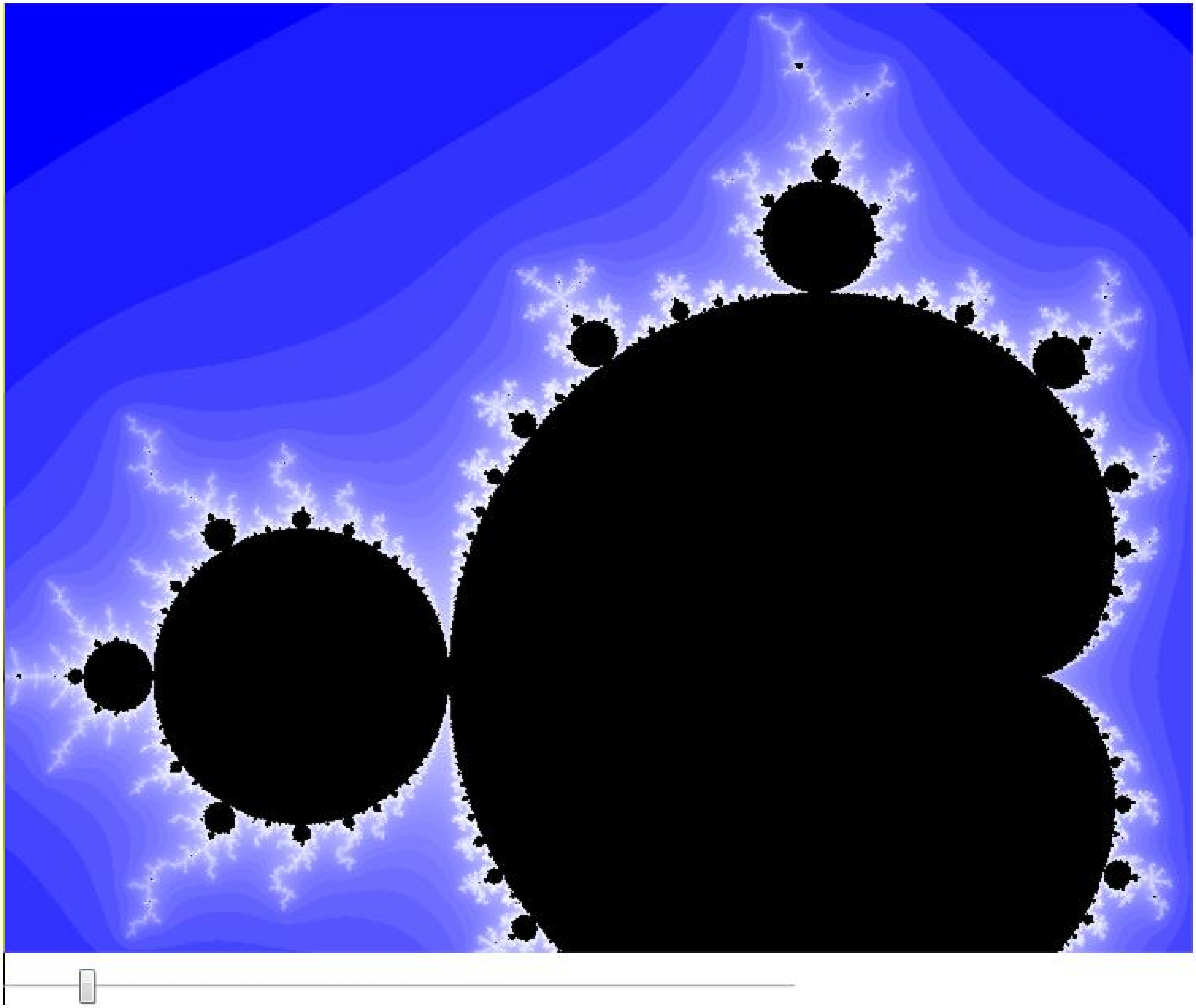
Theorem

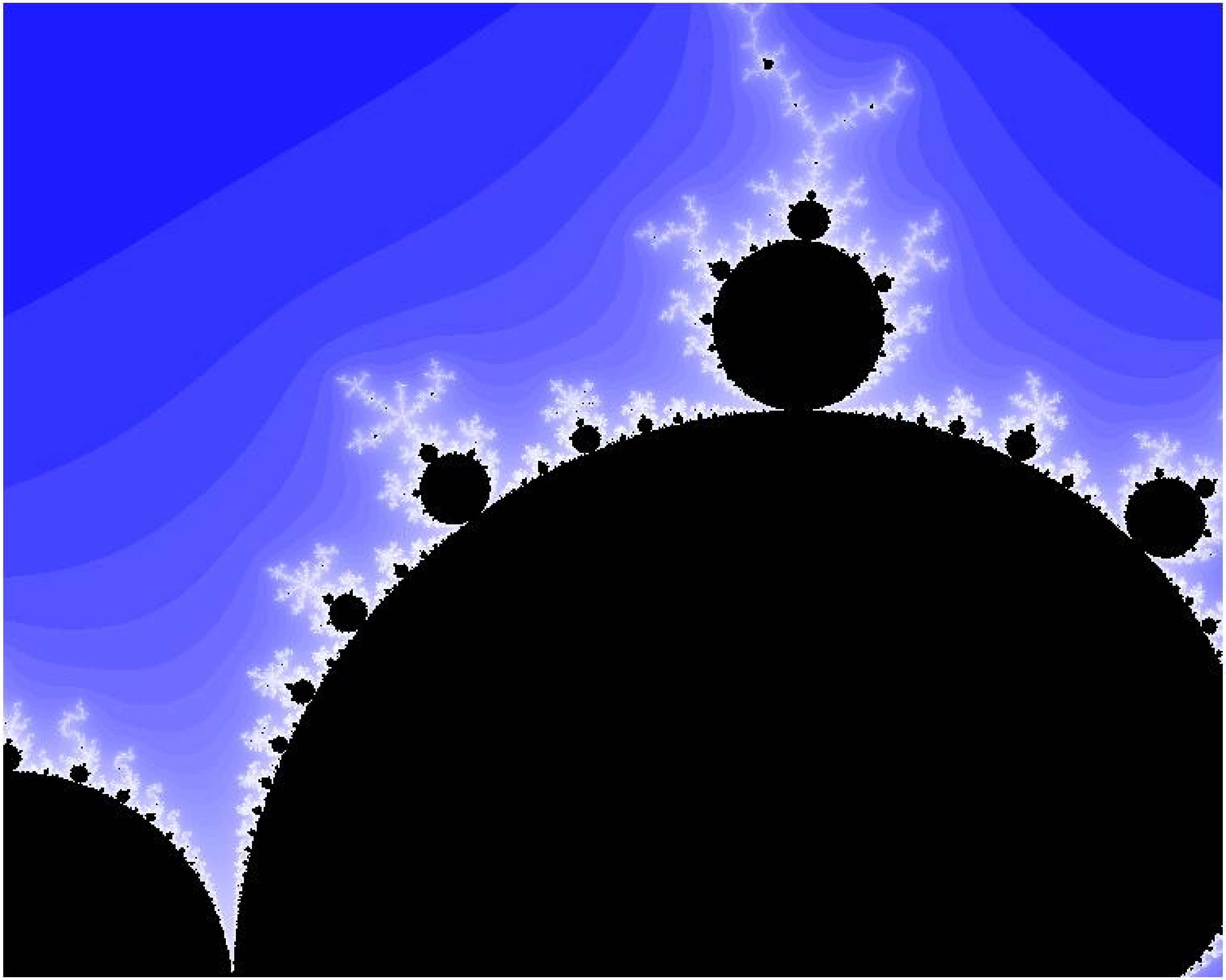
The rabbit Thompson group T_R has the following properties:

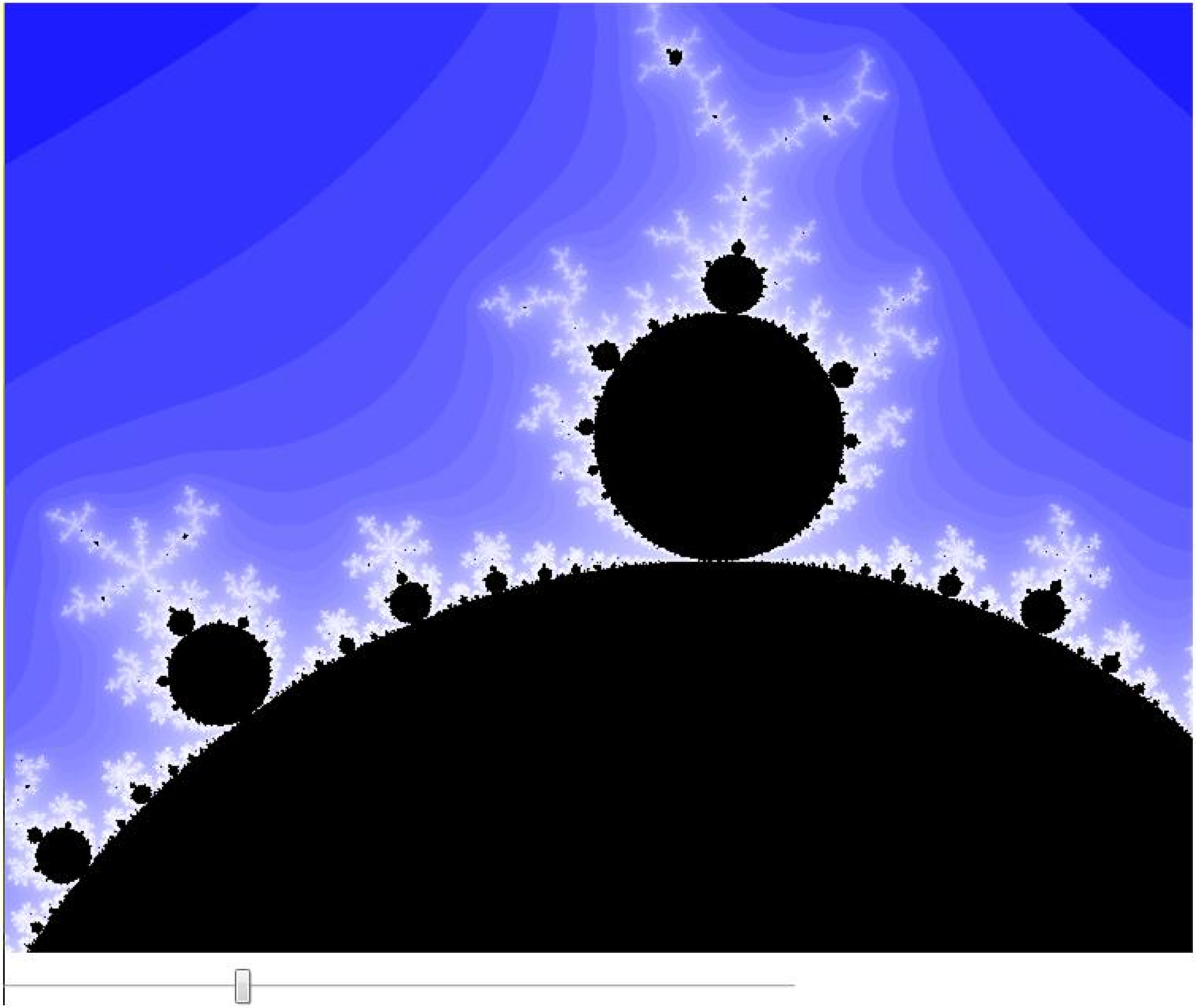
1. *T_R is generated by four elements.*
2. *$[T_R, T_R]$ has index three in T_R, and is simple.*

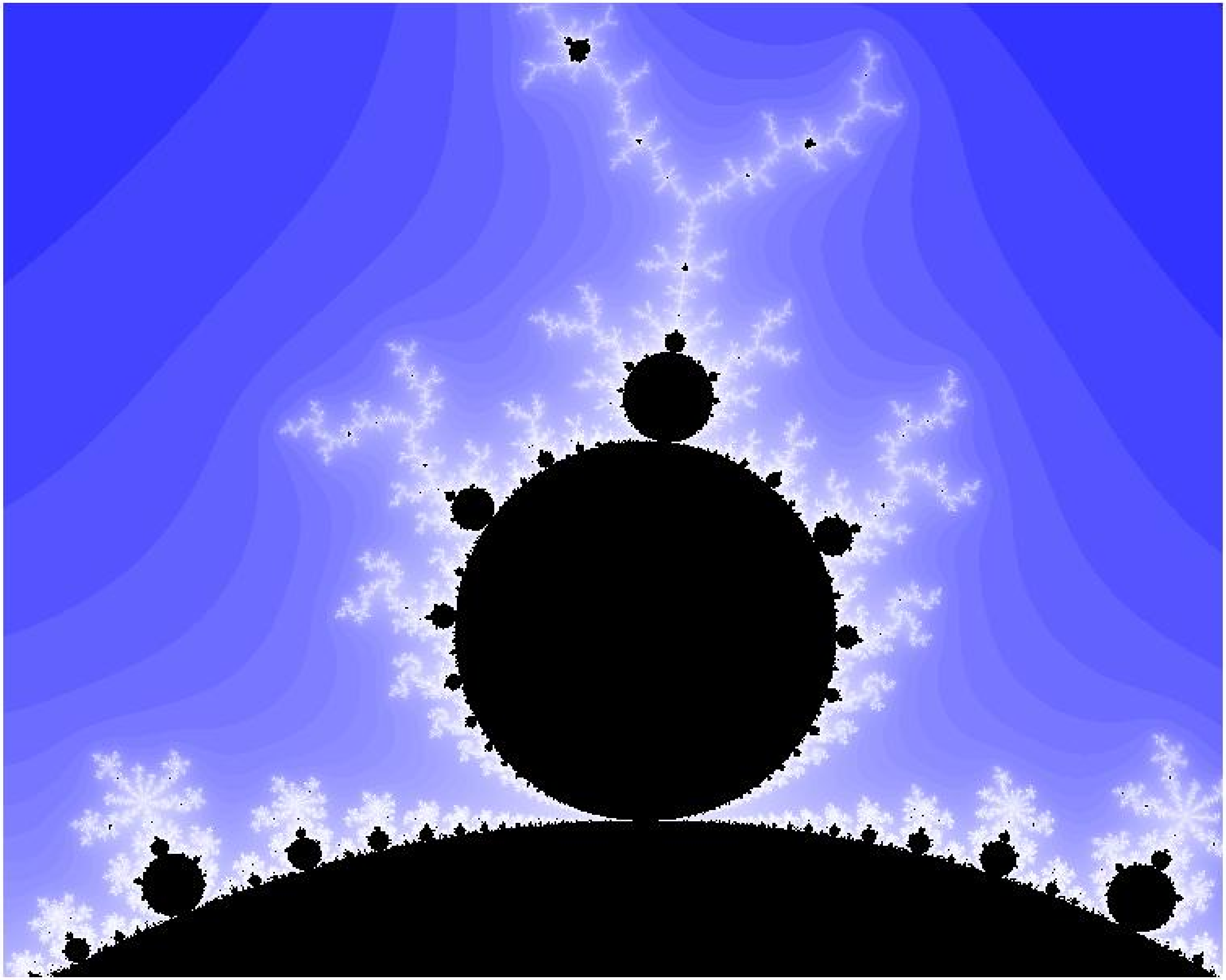

Basilica
Region
Rabbit
Region
Circle
Region

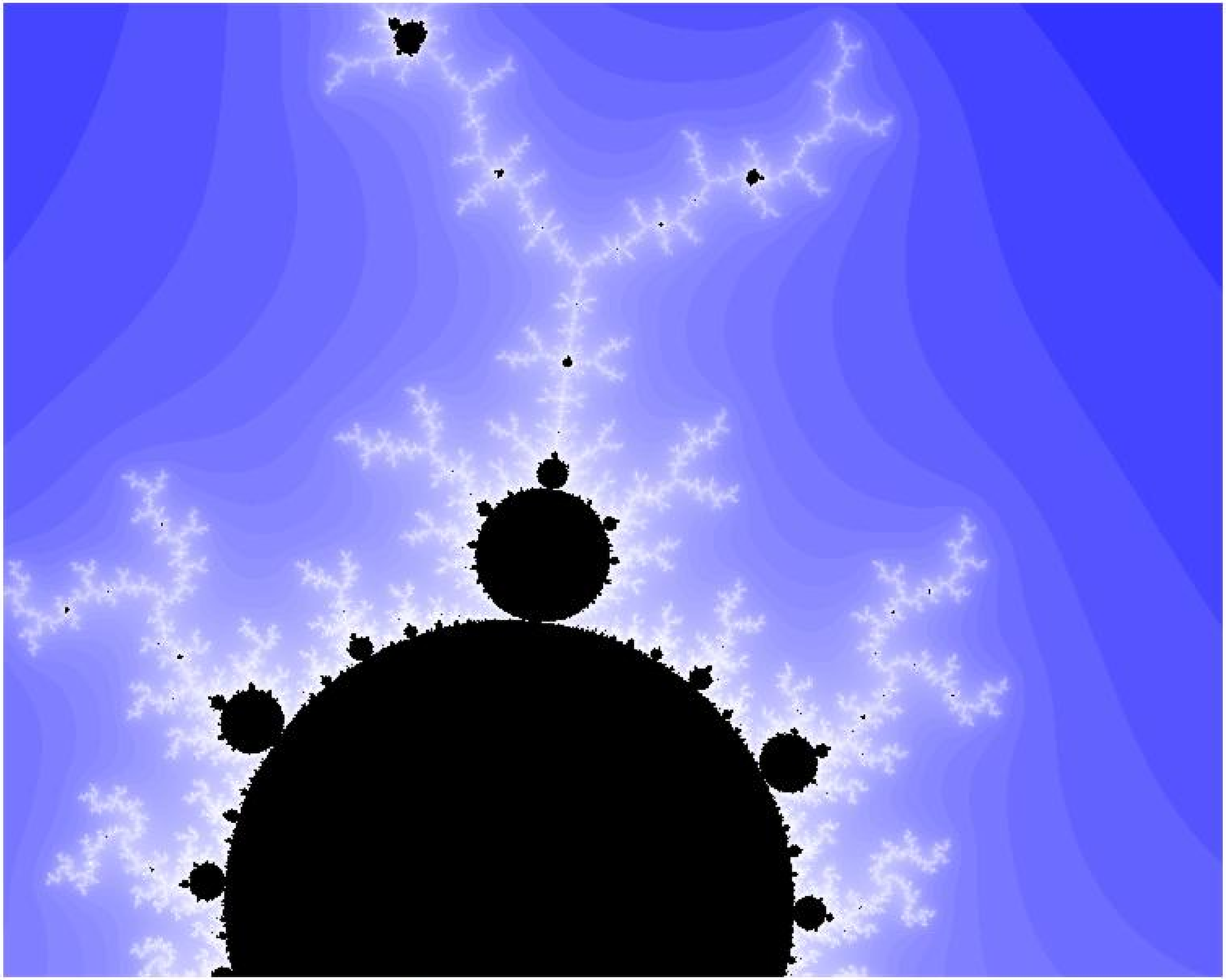


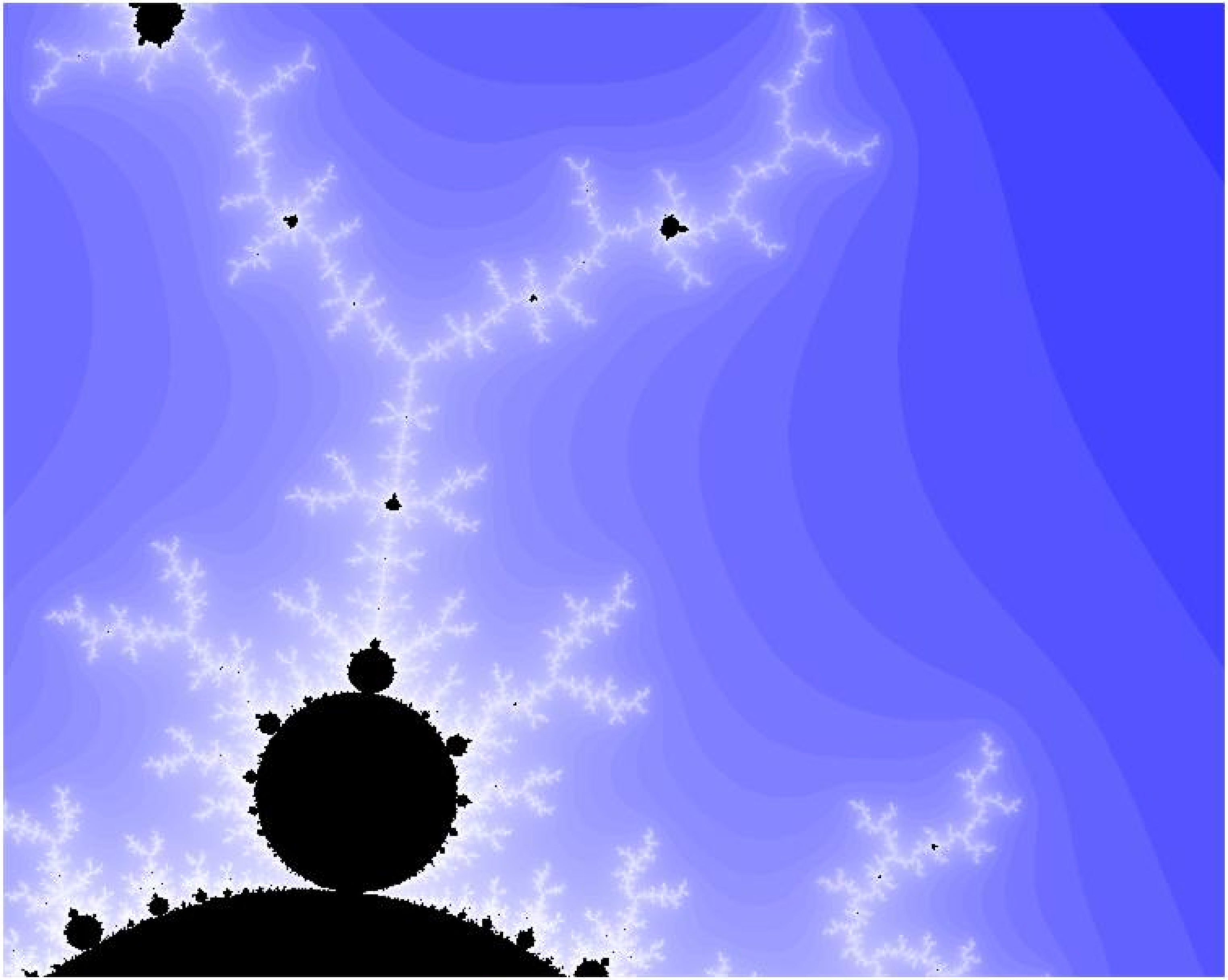


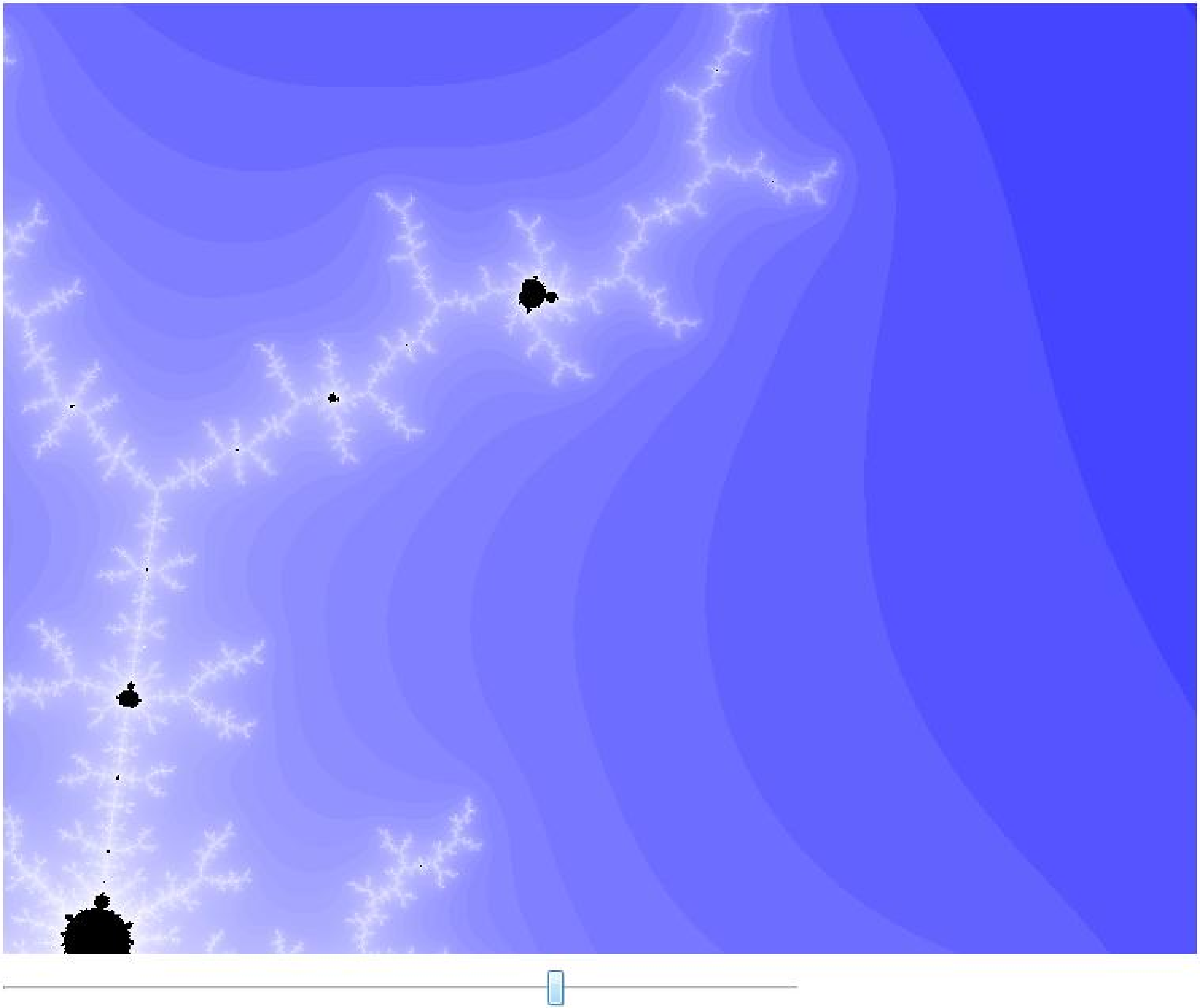


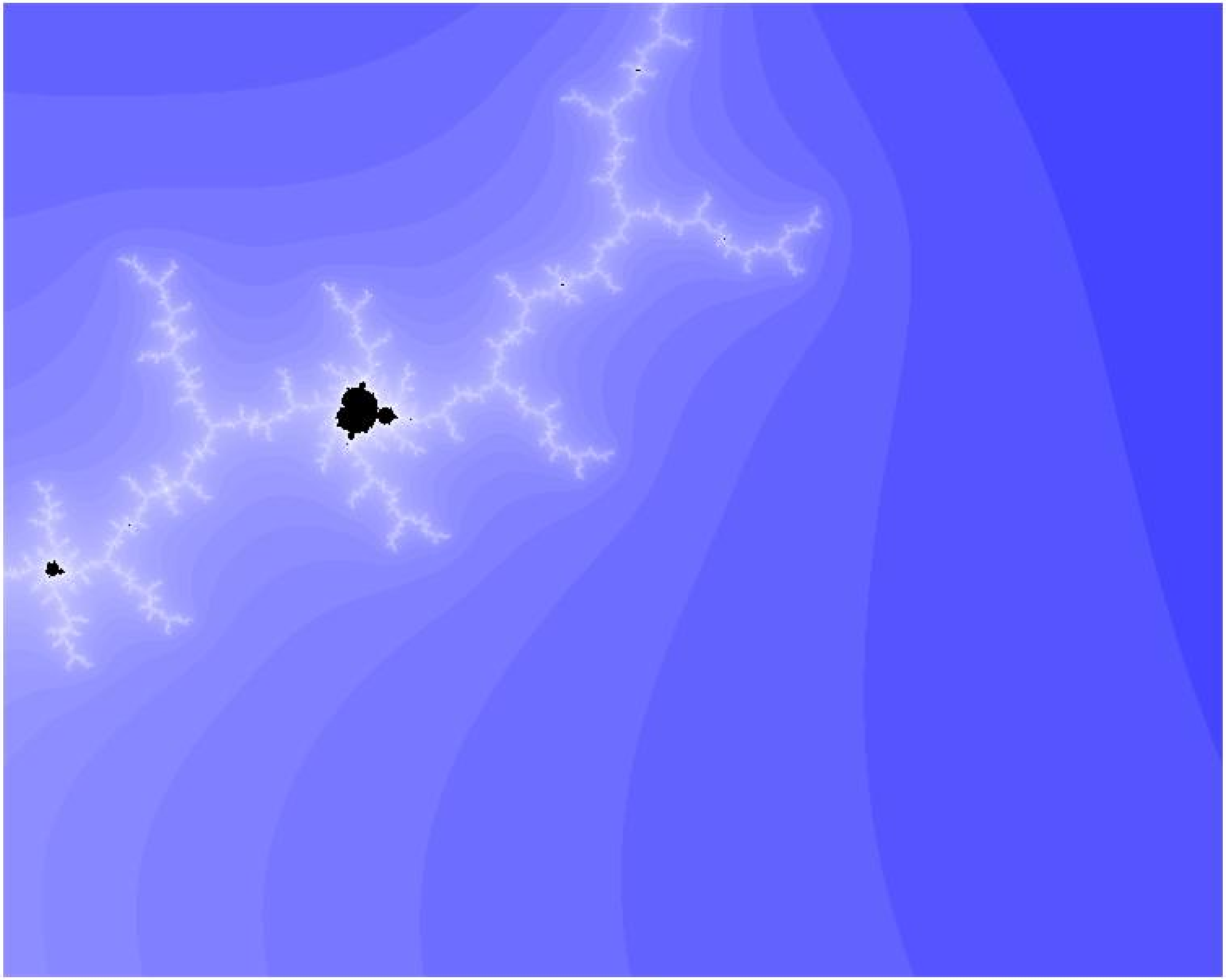


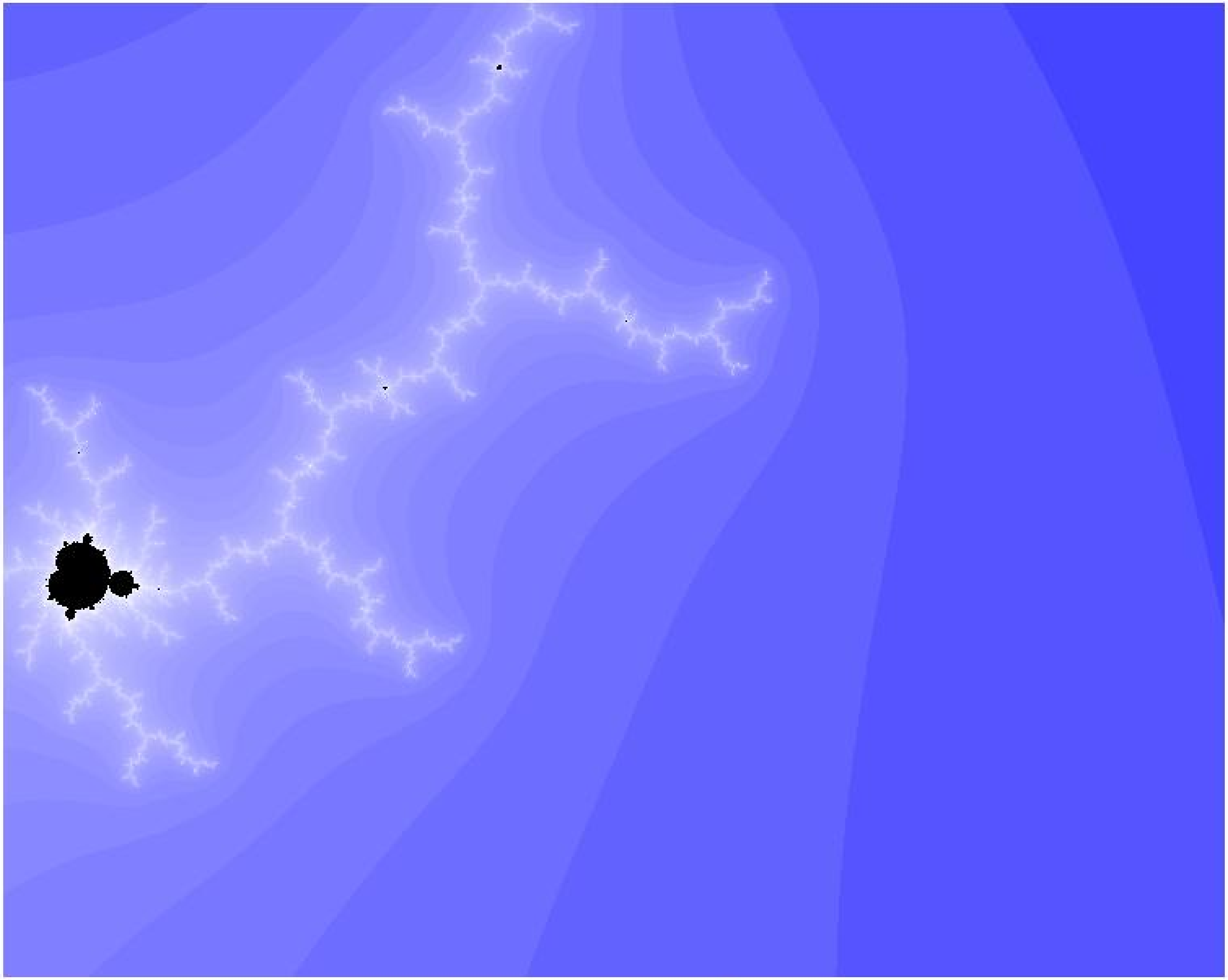


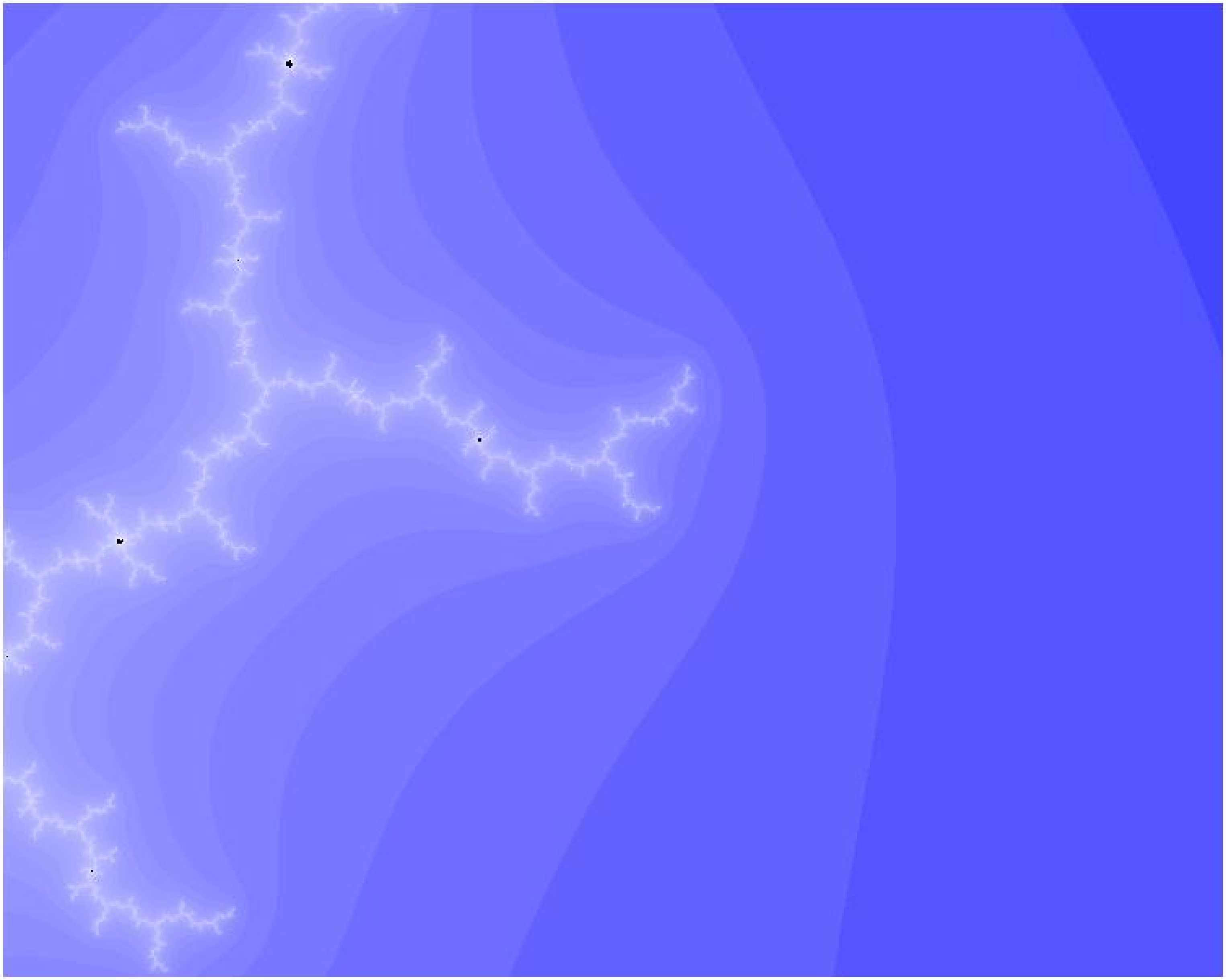


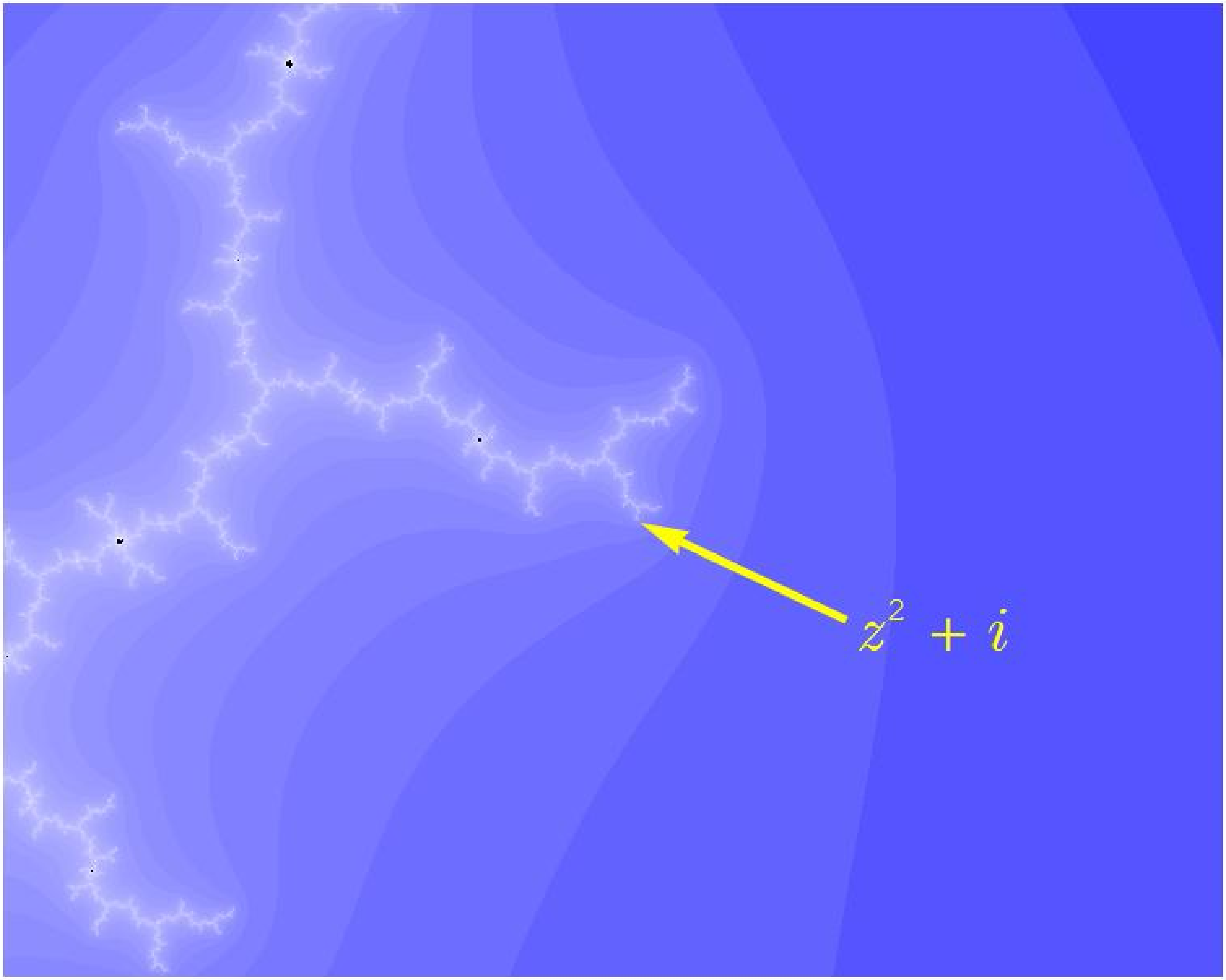




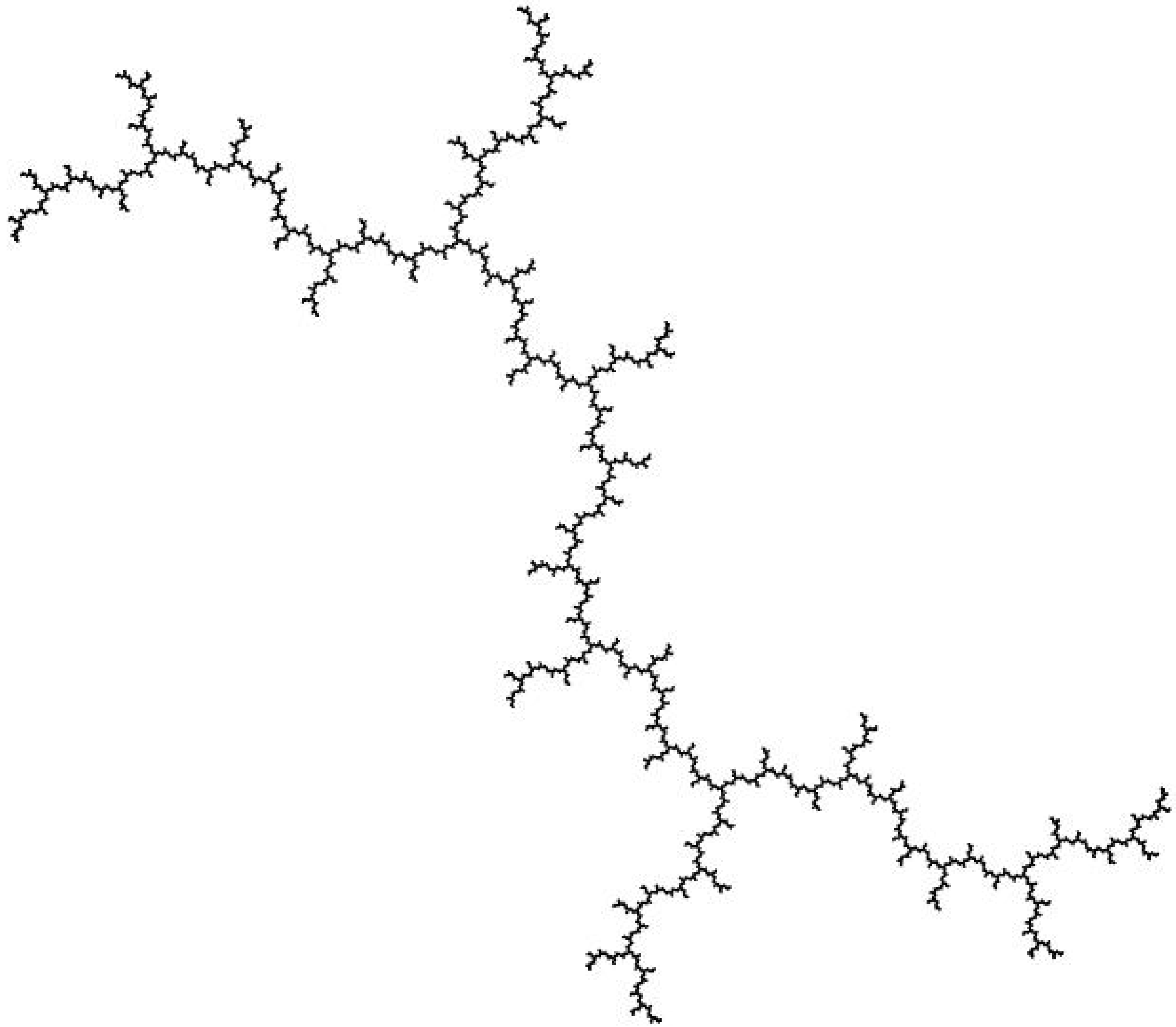




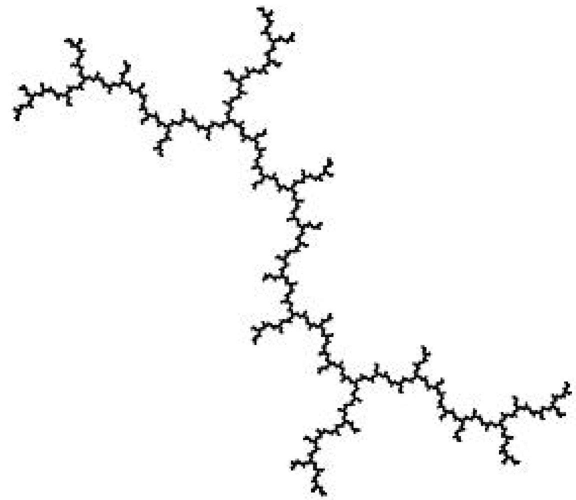



$z^2 + i$

A Dendrite Julia Set



A Dendrite Julia Set

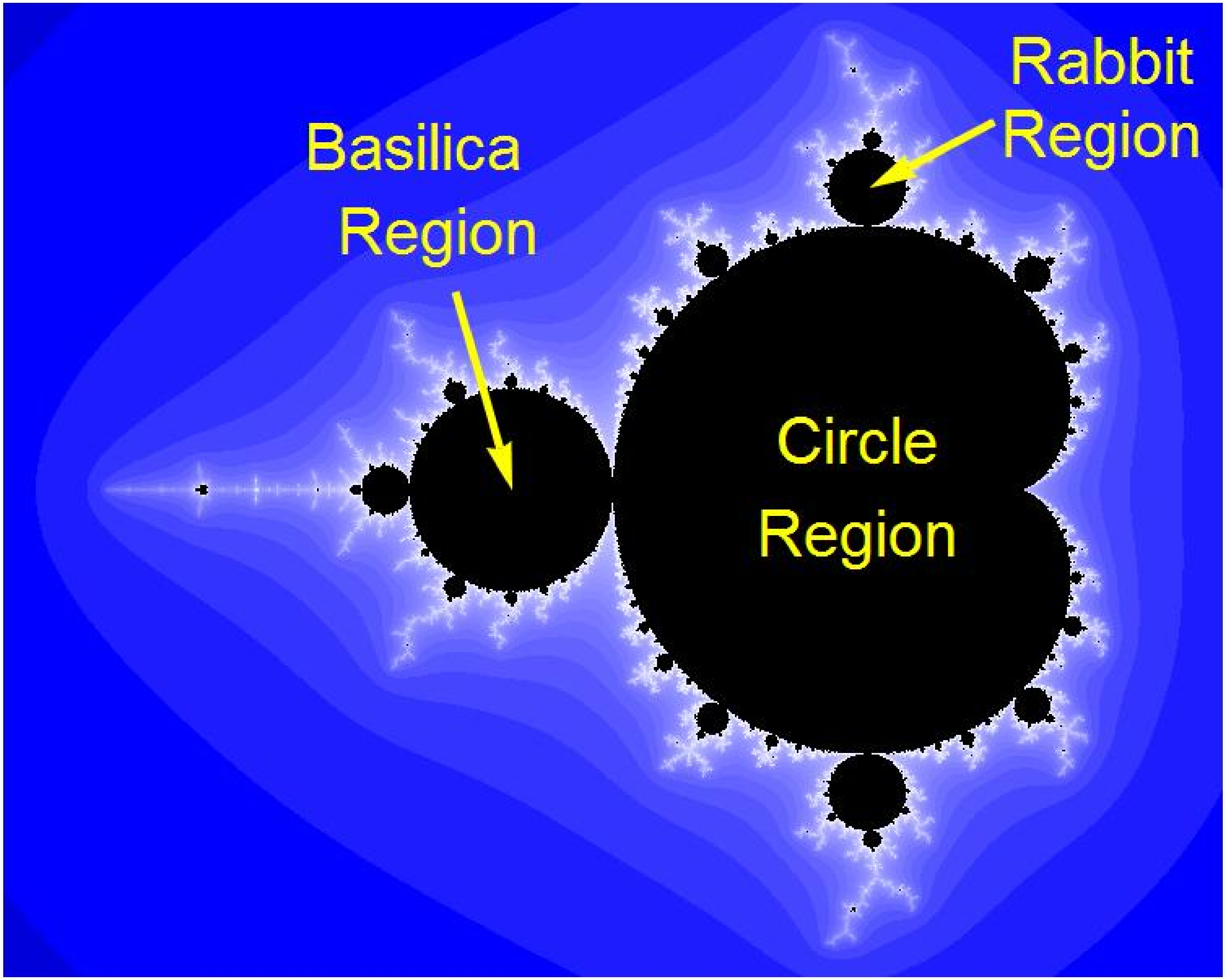


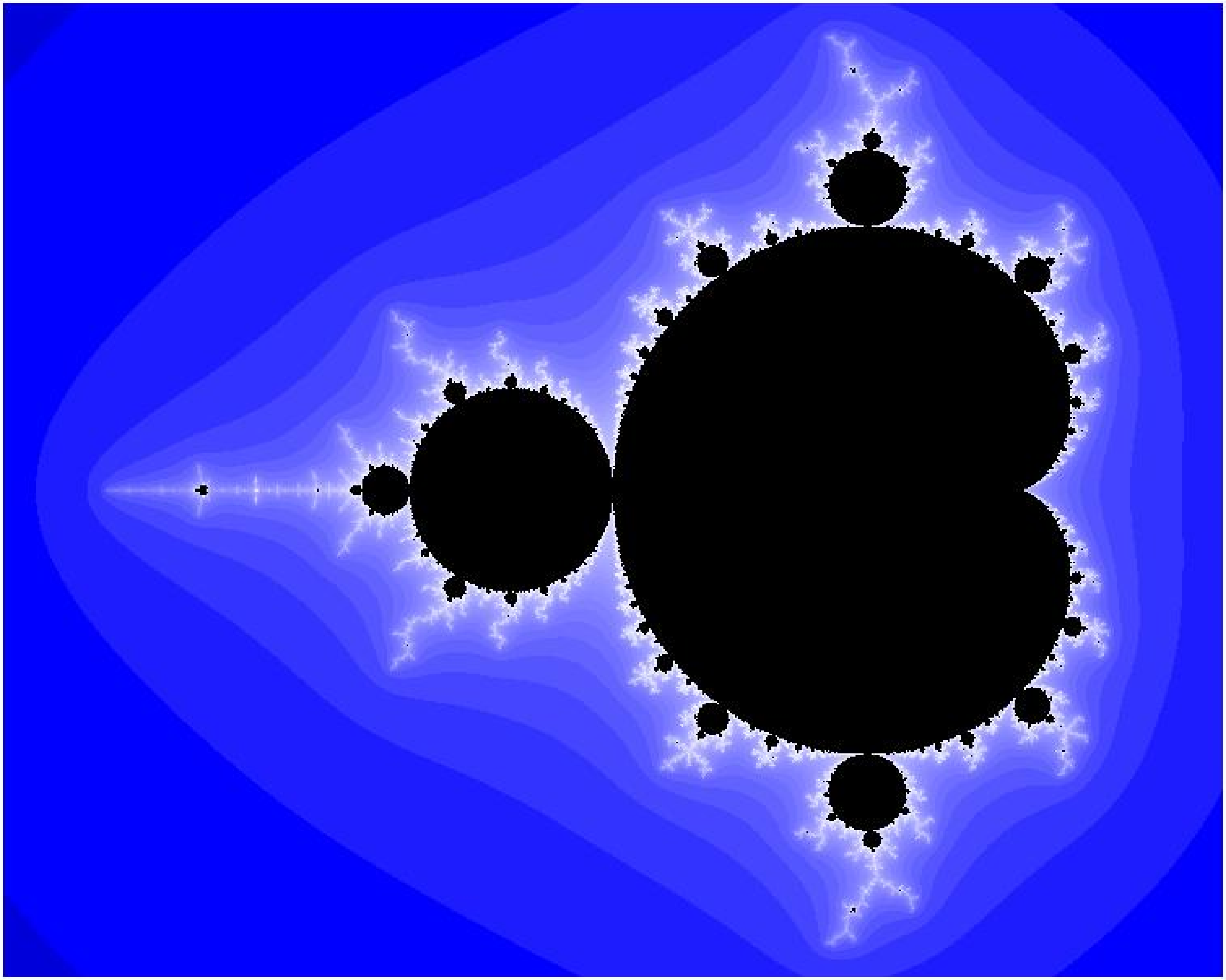
Theorem (Belk & Smith)

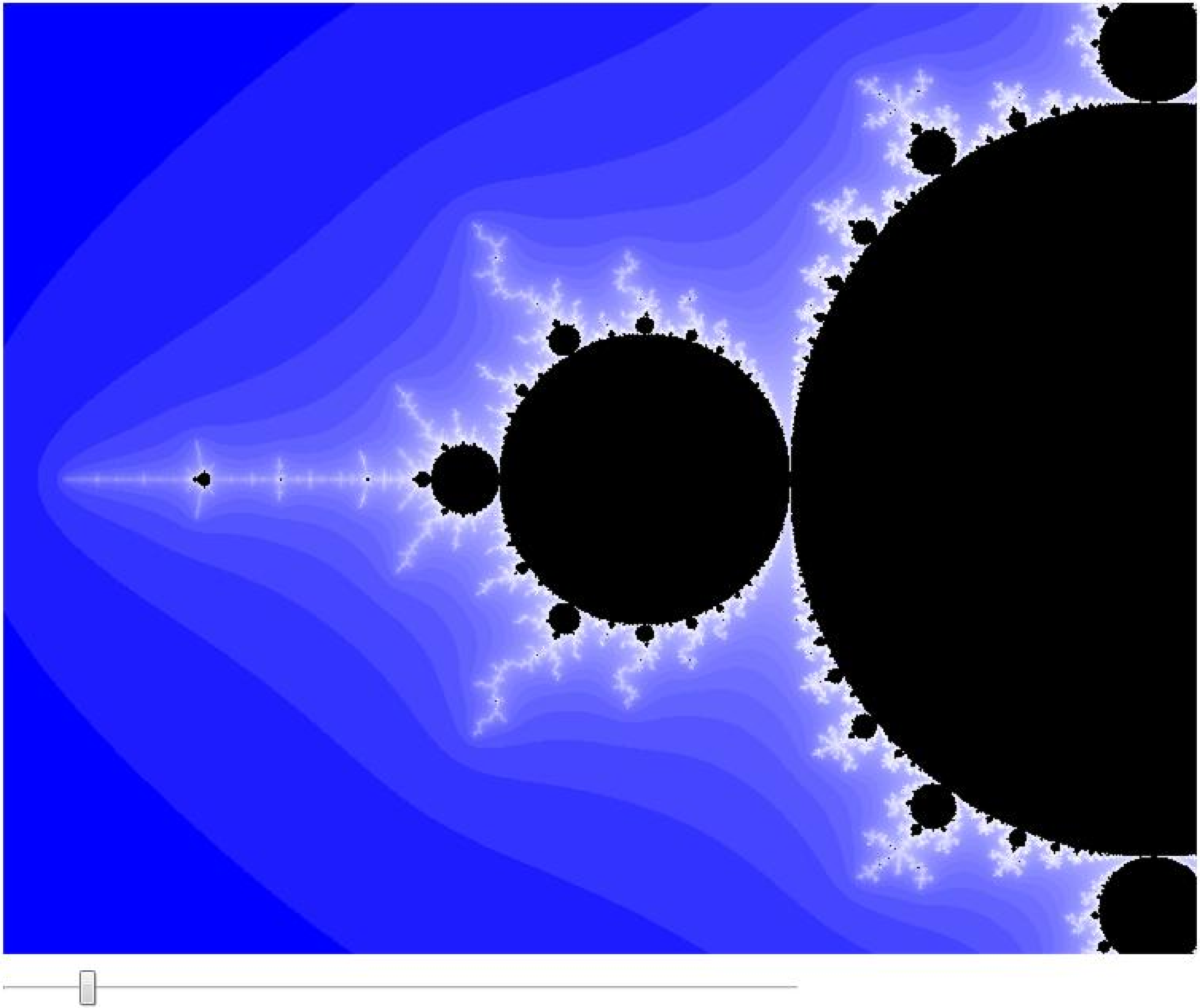
The dendrite Thompson group T_D has the following properties:

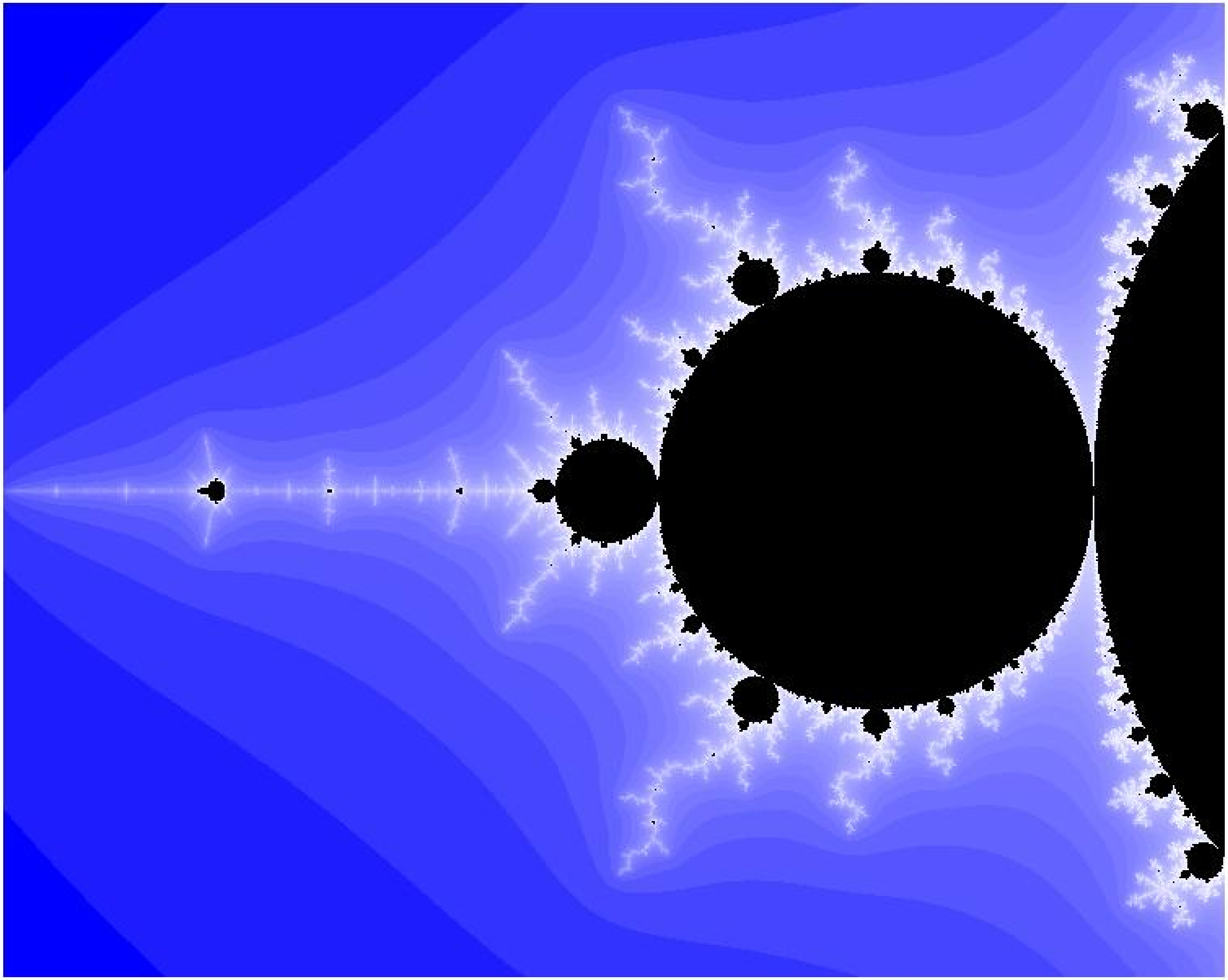
1. *T_D is generated by three elements.*
2. *$[T_D, T_D]$ has infinite index in T_D, with virtually cyclic quotient.*
3. *T_D contains Thompson's group F, but does not contain T.*

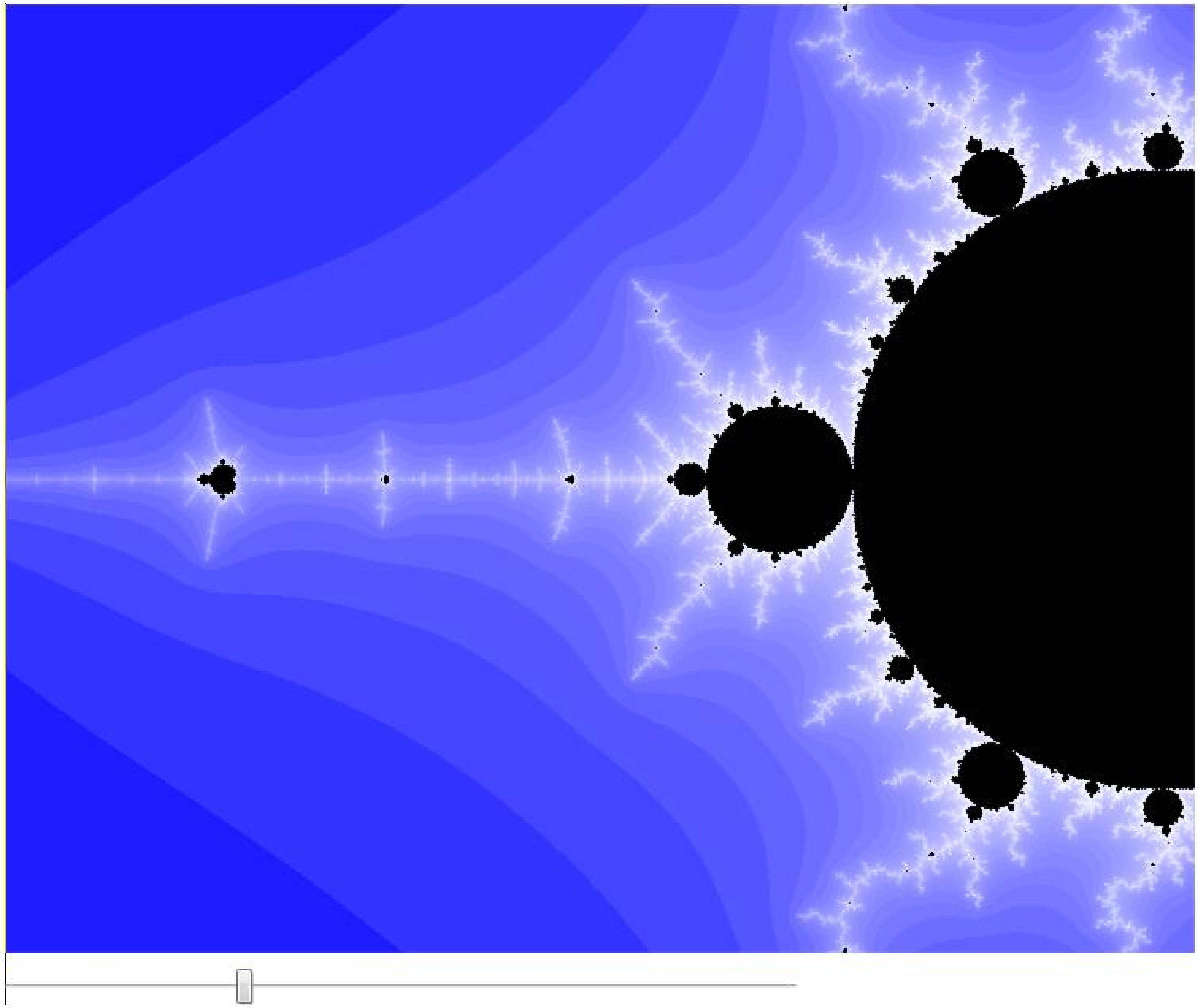
^

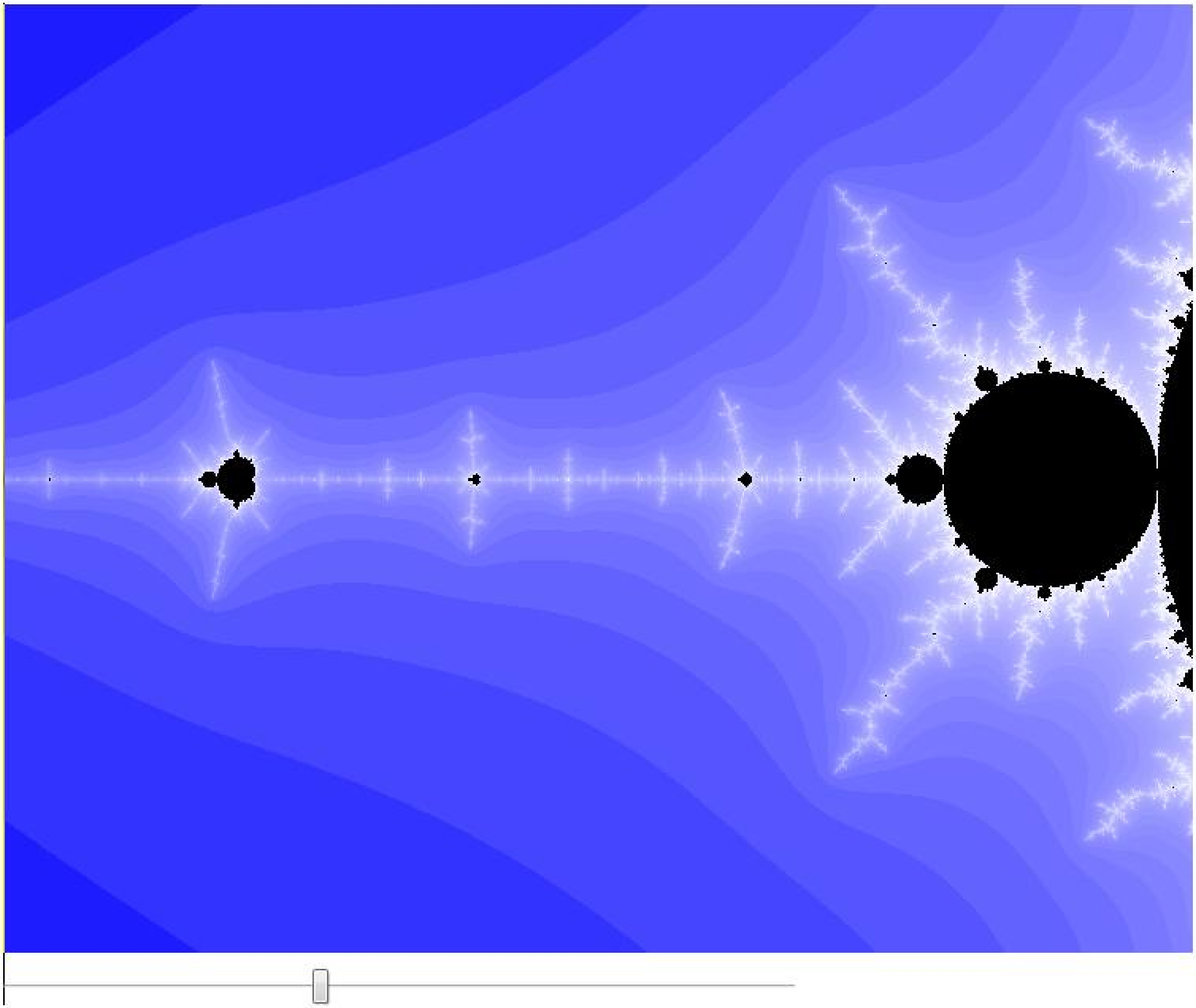

Rabbit
Region
Basilica
Region
Circle
Region

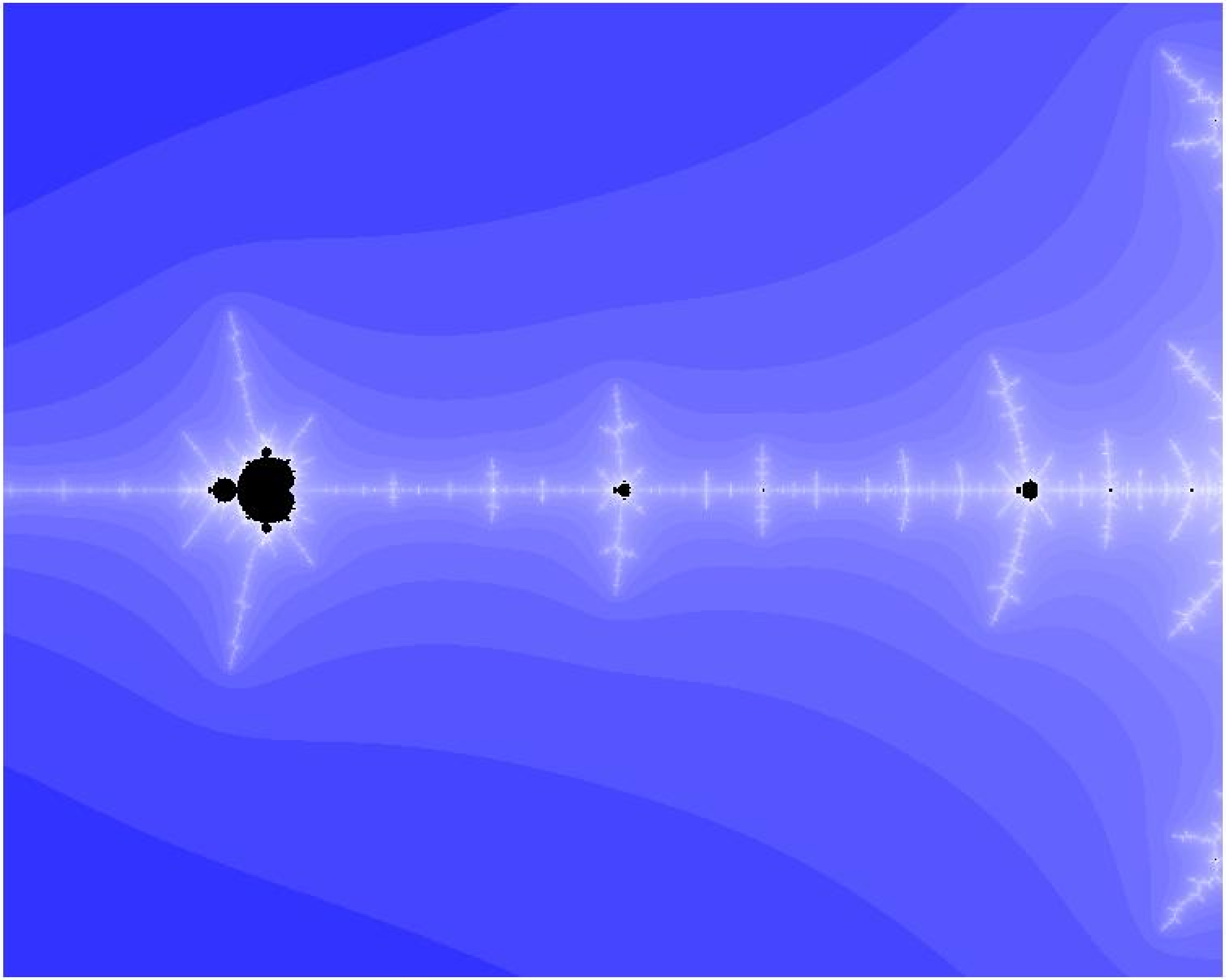


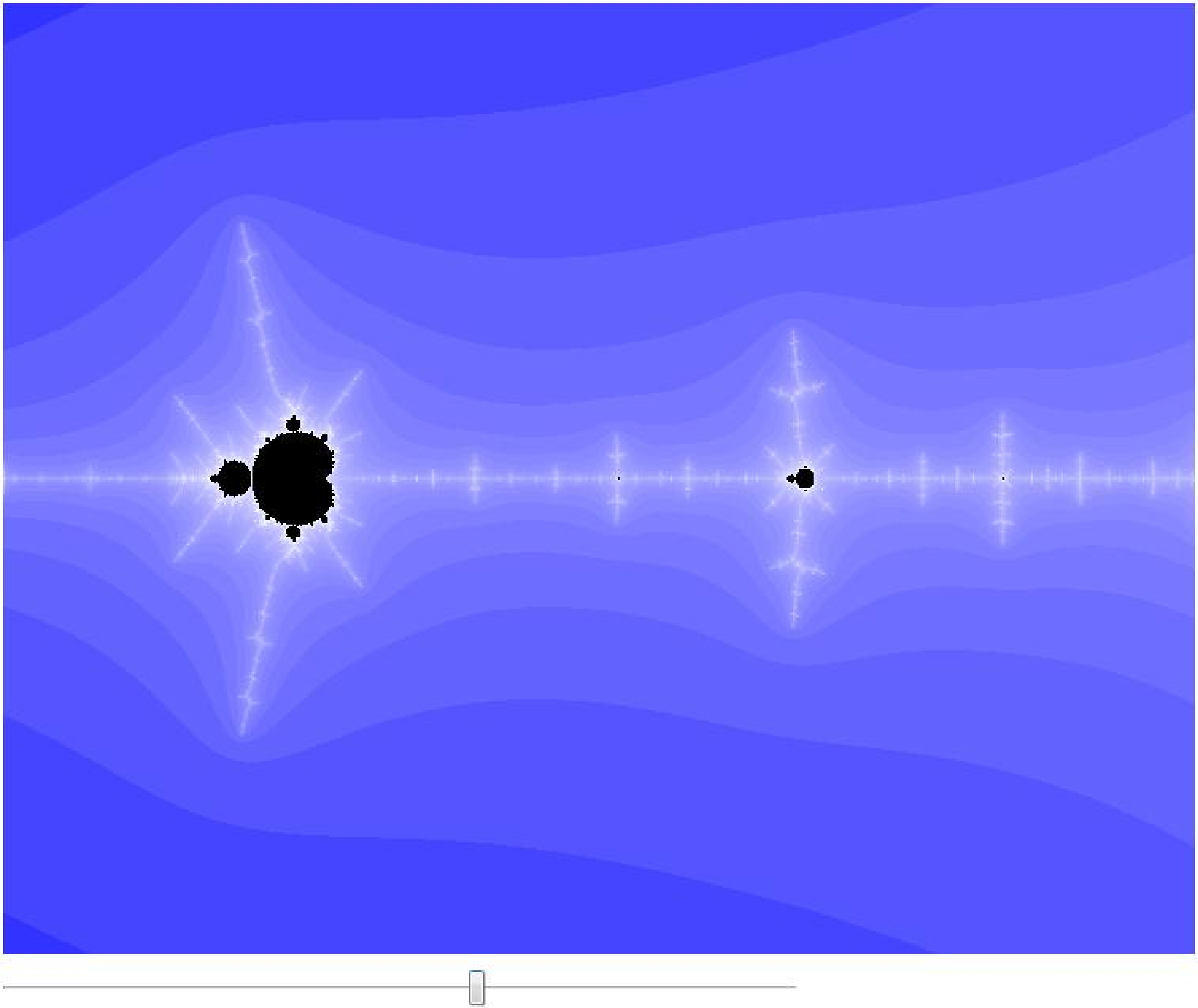


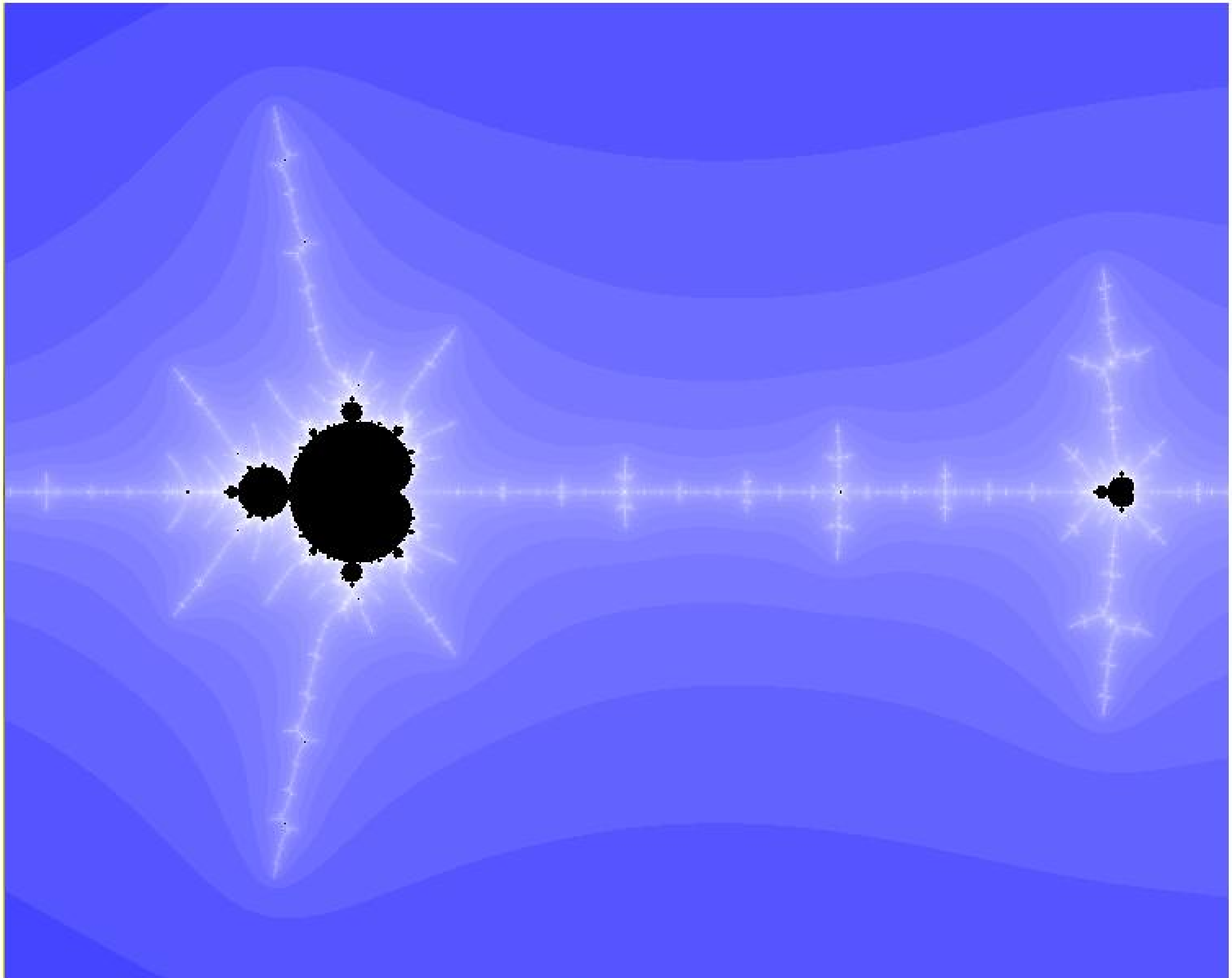


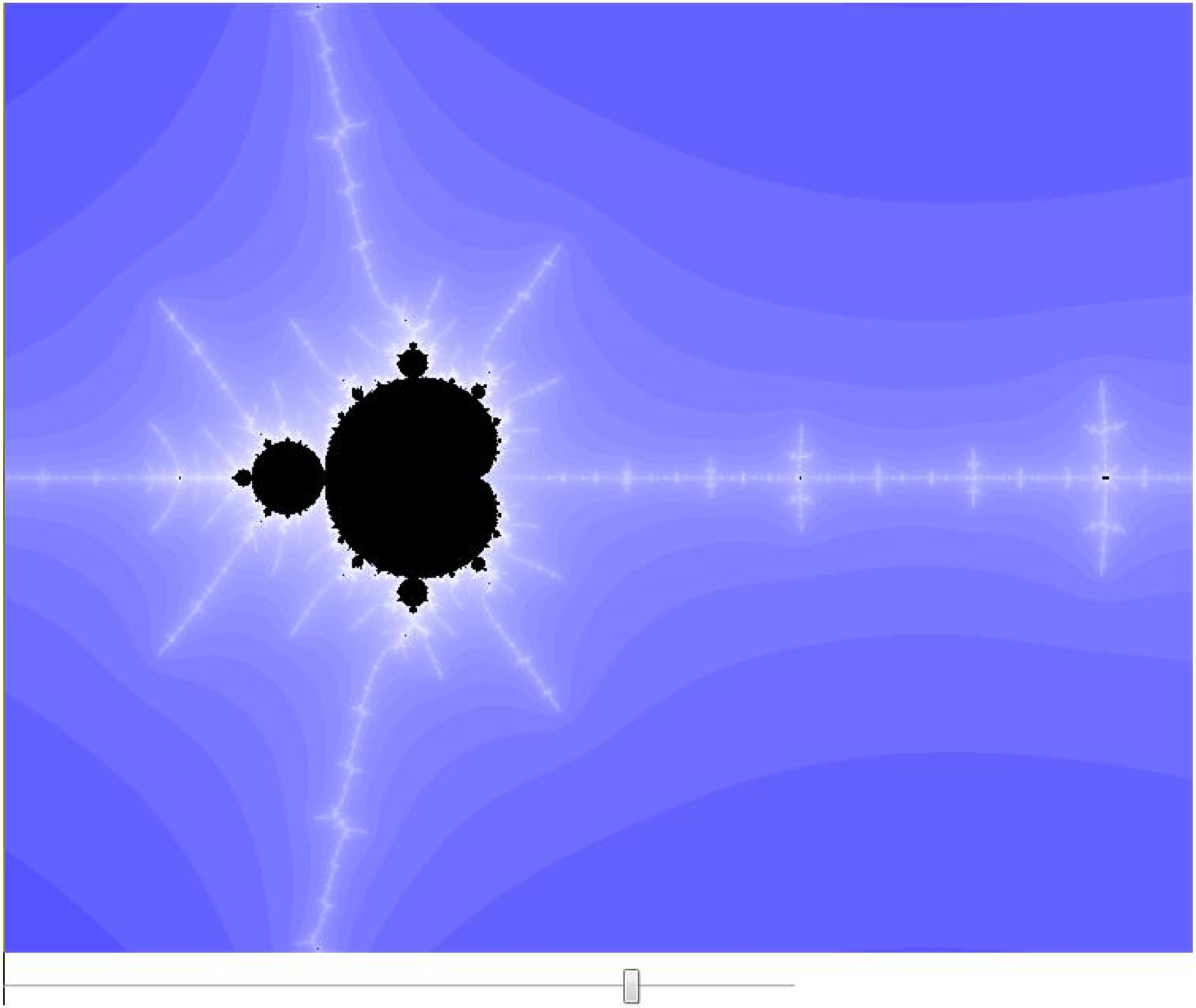


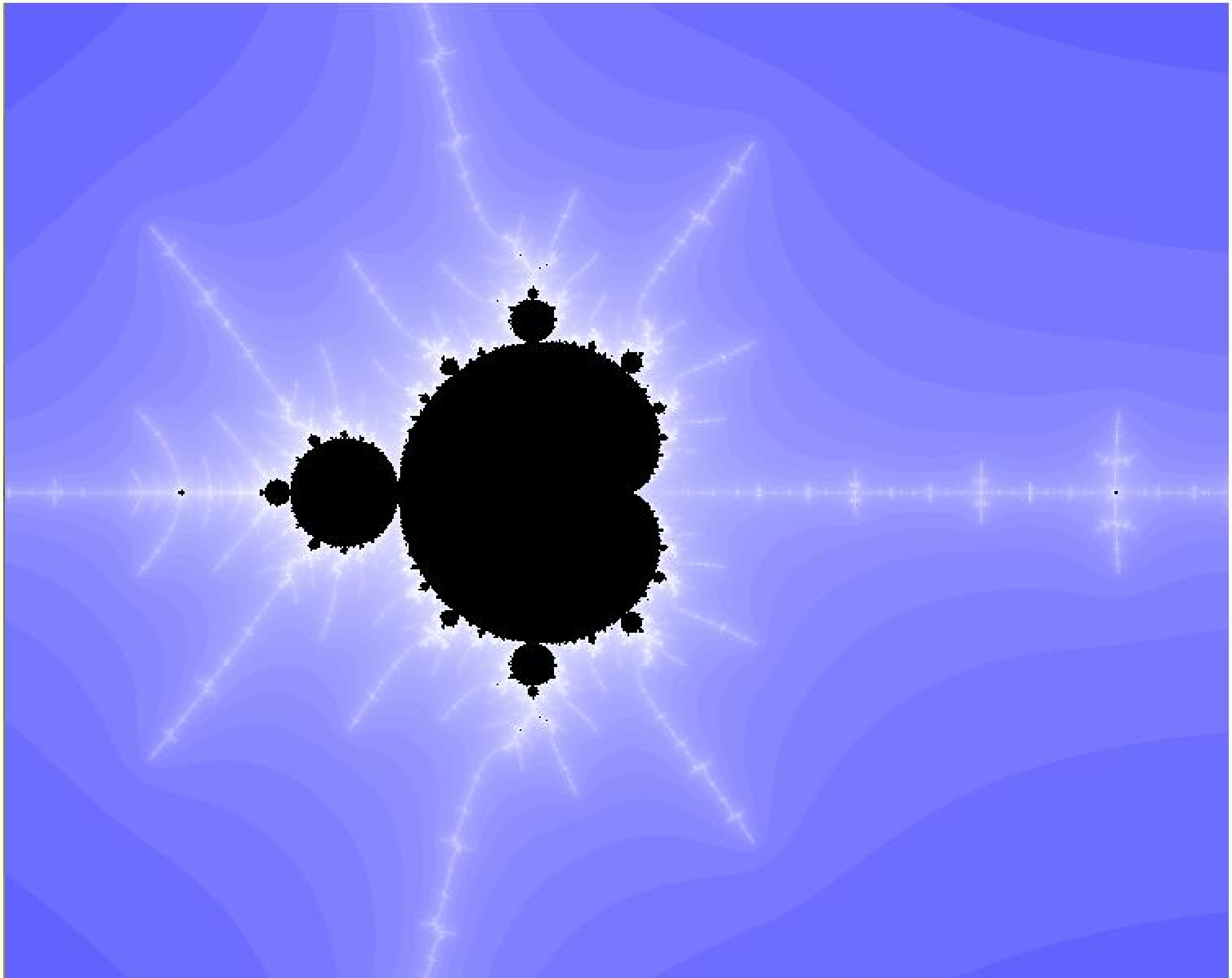


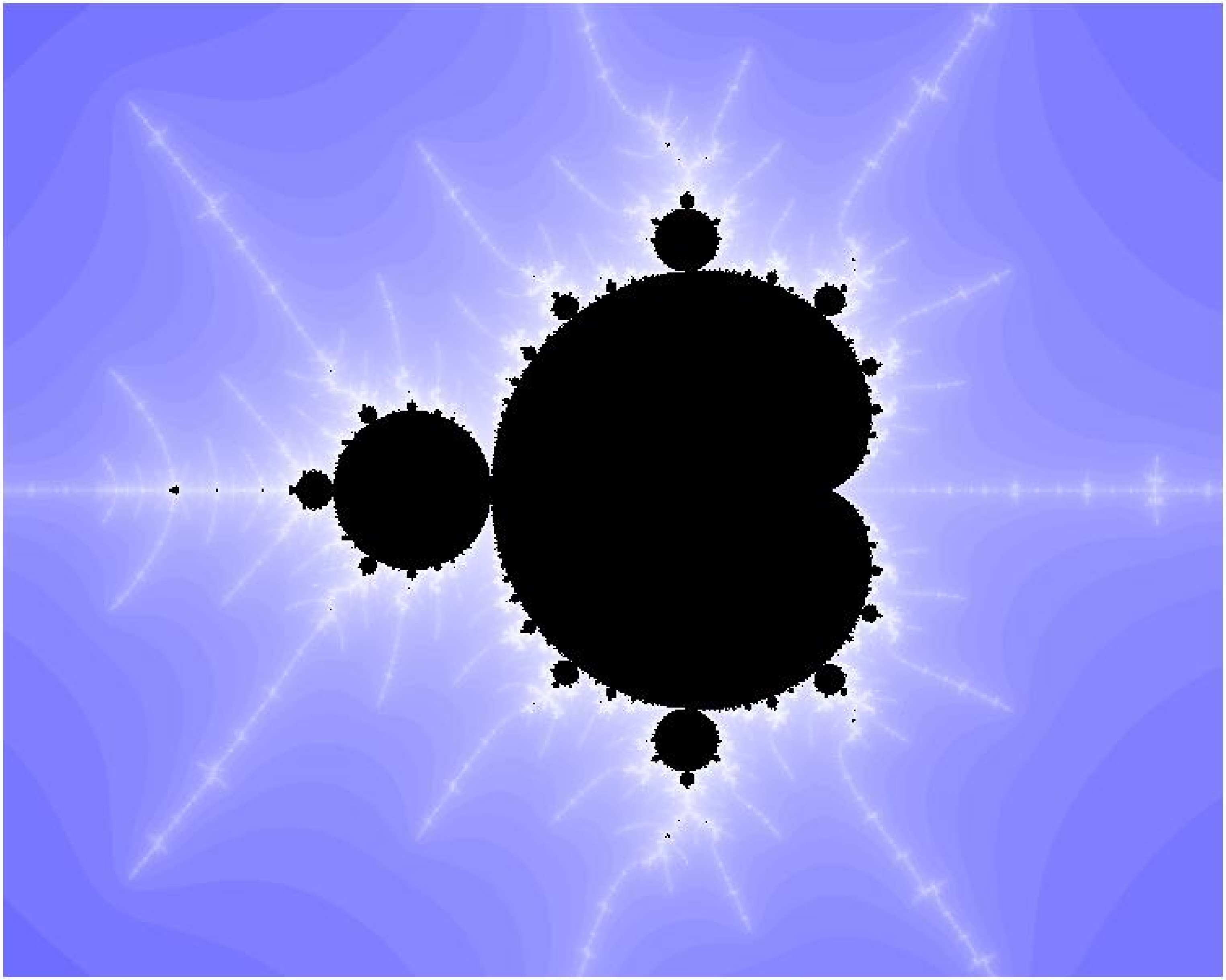




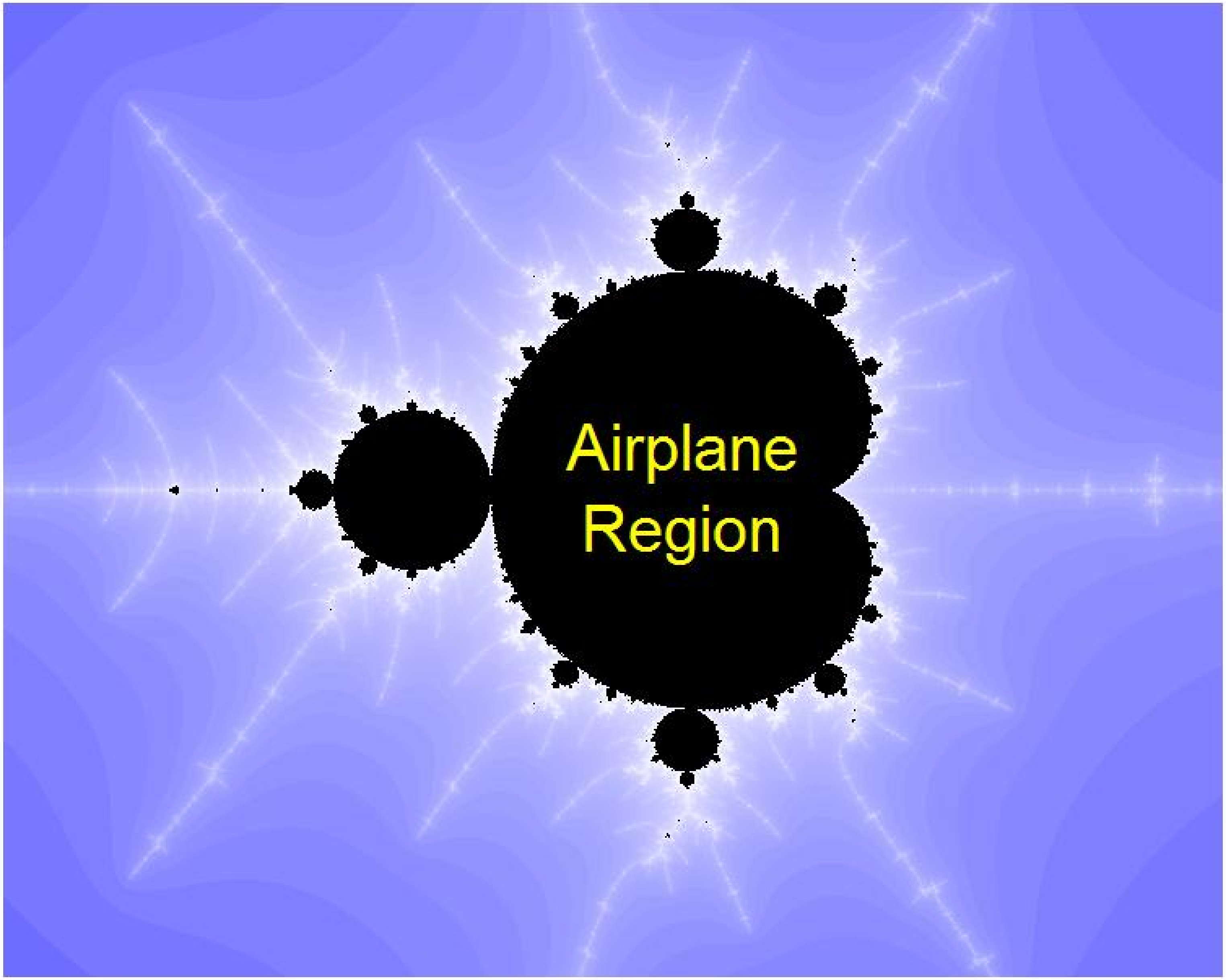




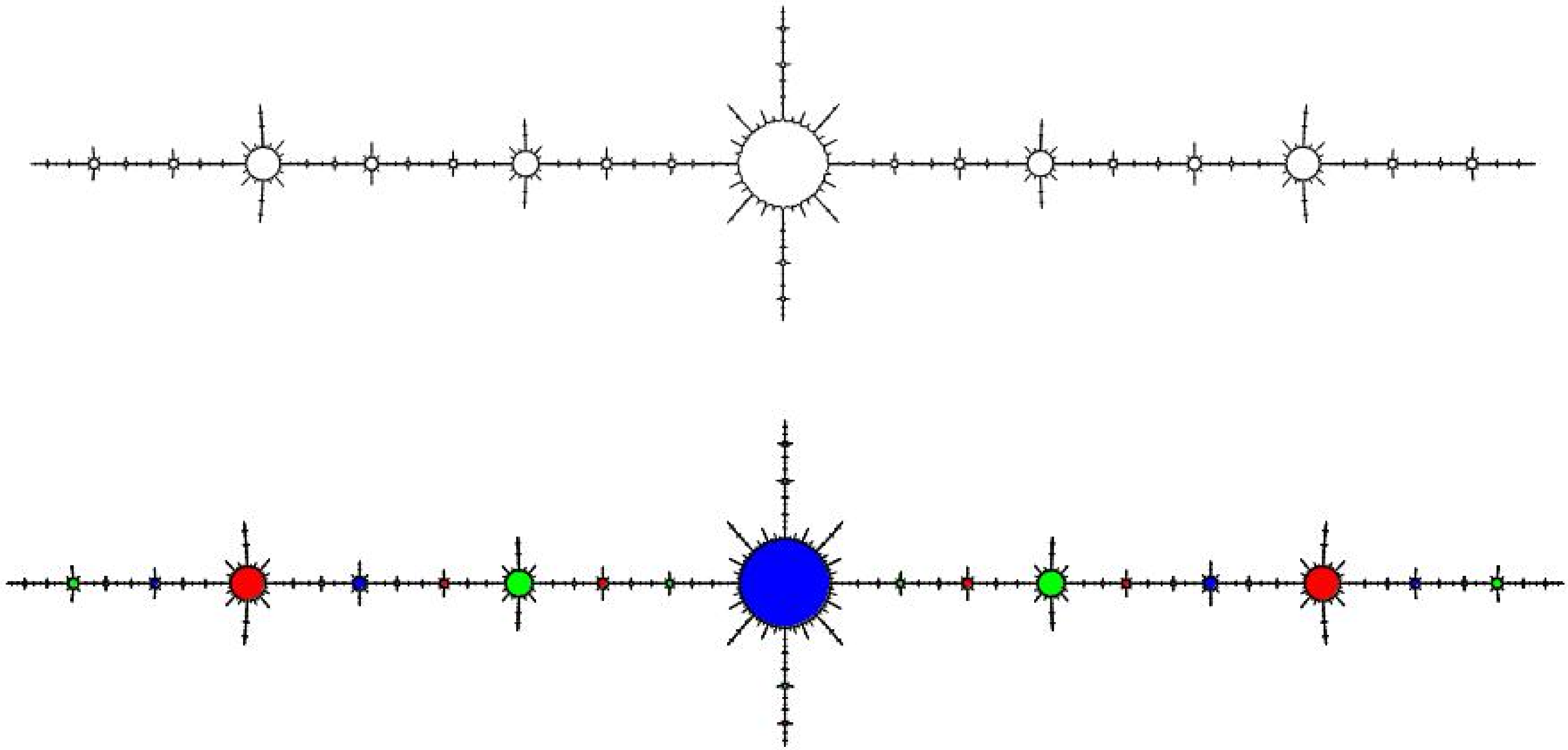




^

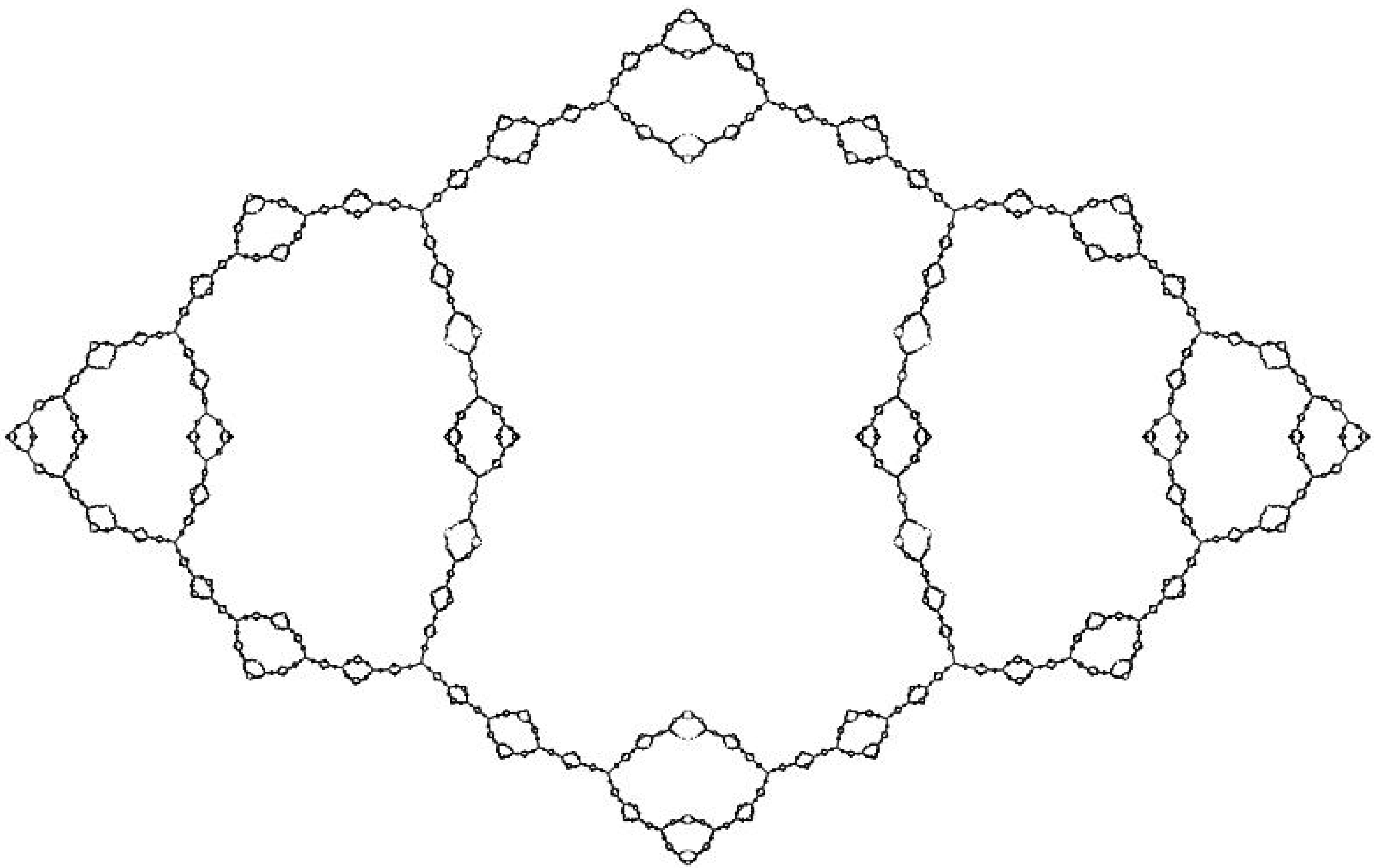

Airplane
Region

The Airplane



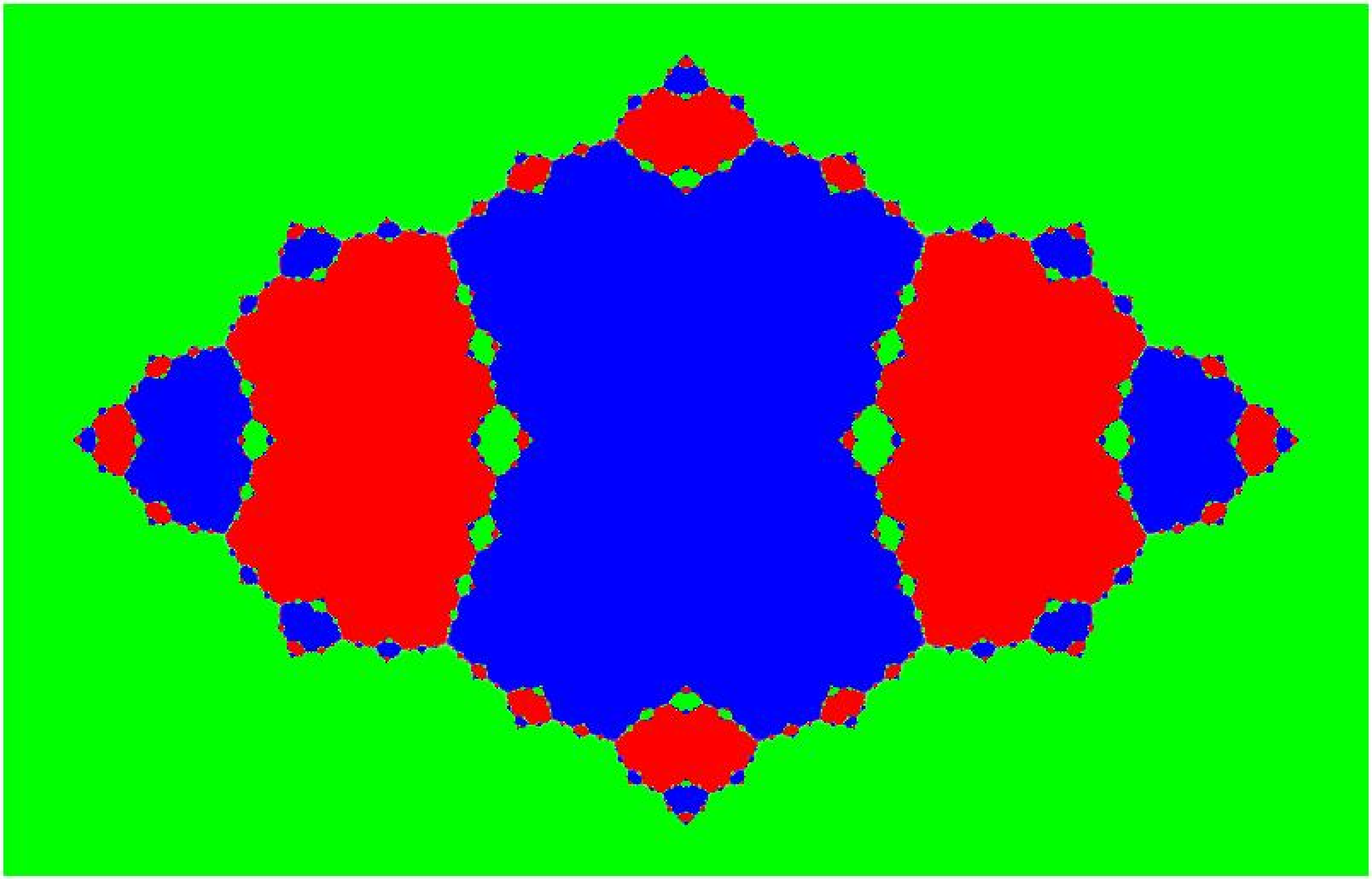
A Rational Julia Set

Here is the Julia set for $z \mapsto 1 - z^{-2}$.

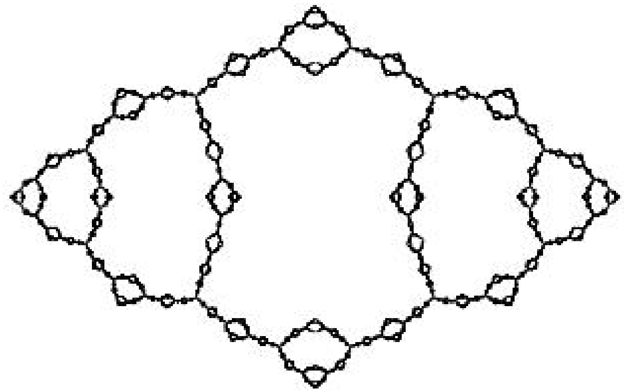


A Rational Julia Set

Here is the Julia set for $z \mapsto 1 - z^{-2}$.



A Rational Julia Set



Theorem (Belk & Weinrich-Burd)

The Thompson group T_{rat} for this Julia set has the following properties:

1. T_{rat} *is generated by four elements.*
2. $[T_{\mathrm{rat}}, T_{\mathrm{rat}}]$ *has index two, and is **not** simple.*
3. $T_{\mathrm{rat}} = H \rtimes S_3$, *where H is a simple normal subgroup.*

Note: T_{rat} is presumably ***not*** isomorphic to a subgroup of T.

The End