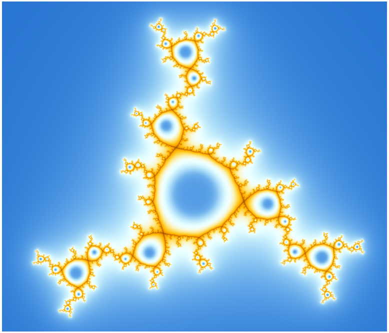
Quasisymmetries of Finitely Ramified Julia Sets



Jim Belk

University of St Andrews

Joint Work

Bradley Forrest

Stockton University

Quasiconformal Maps

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For a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$, let

$$\lfloor T \rfloor = \min_{v \neq 0} \frac{\|Tv\|}{\|v\|} \qquad \text{and} \qquad \lceil T \rceil = \max_{v \neq 0} \frac{\|Tv\|}{\|v\|}$$

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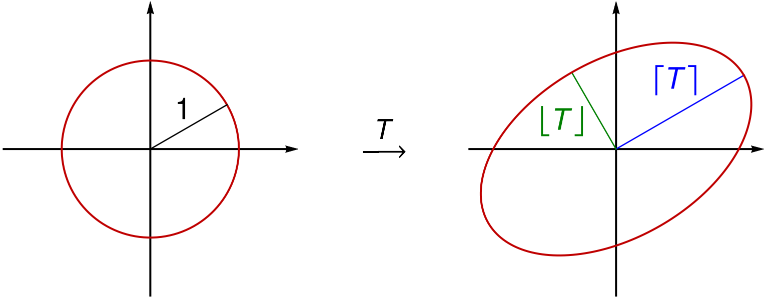

The ratio $\lceil T \rceil / \lfloor T \rfloor$ is a measure of ***eccentricity***.

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A diffeomorphism $f\colon U \longrightarrow U'$ between domains $\mathbb{R}^n$ is ***quasiconformal*** if there exists a $\lambda \geq 1$ so that

$$\frac{\lceil Df_p \rceil}{\lfloor Df_p \rfloor} \leq \lambda$$

for all $p \in U$.

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Note: If $\lambda = 1$ then f is ***conformal*** (or anticonformal).

More General Definition

More generally, let $f: U \longrightarrow U'$ be a homeomorphism.

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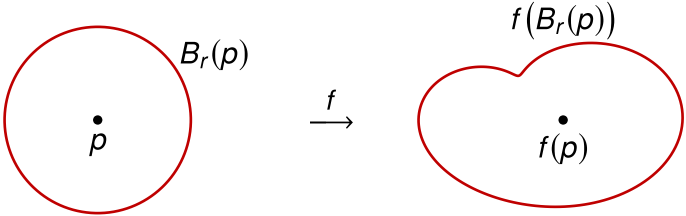

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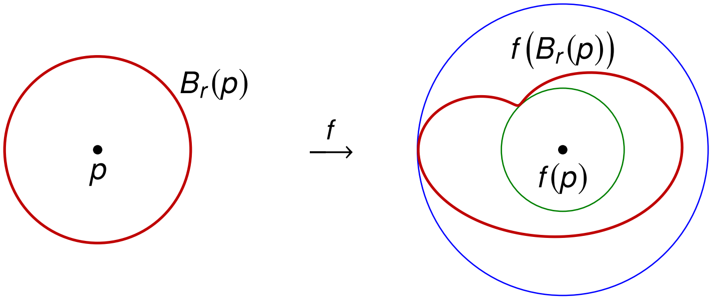

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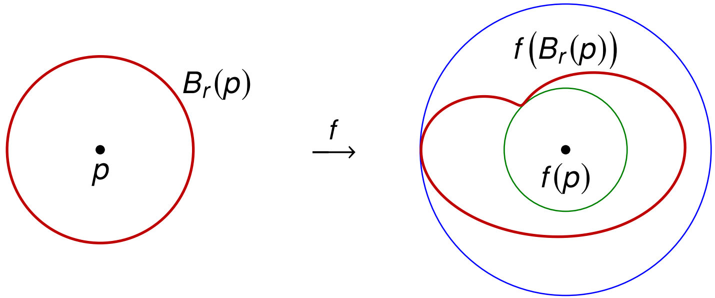


We say f is ***quasiconformal*** if there exists a $\lambda \geq 1$ so that

$$\limsup_{r \to 0^+} \frac{\text{outer radius of } f\big(B_r(p)\big)}{\text{inner radius of } f\big(B_r(p)\big)} \leq \lambda$$

for all $p \in U$.

Applications of Quasiconformal Maps

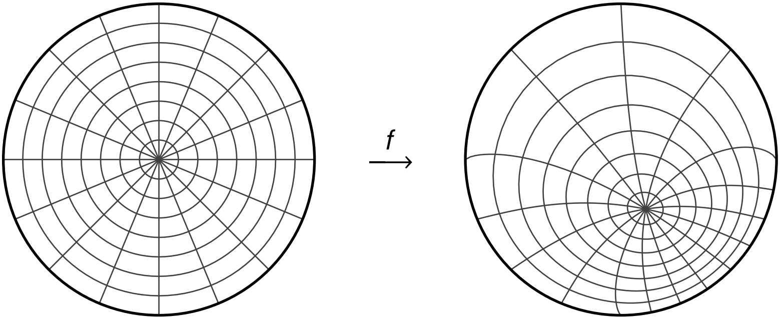
- ***Mostow rigidity:*** For $n \geq 3$, two compact hyperbolic n-manifolds are isometric if and only if their fundamental groups are isomorphic.
- ***No wandering domains:*** Every component of the Fatou set for a complex rational map is periodic or pre-periodic (Sullivan).
- ***Topological manifolds:*** Every n-manifold ($n \neq 4$) supports a "quasiconformal structure" (Sullivan). This allows a theory of characteristic classes for such manifolds (Sullivan, Connes, Teleman).
- ***Elliptic PDE's:*** Solution to Calderón's problem on electrical impedance tomography in two dimensions (Astala, Päivärinta).

Quasisymmetries

Quasiconformal Maps on a Disk

Let $f : D^2 \longrightarrow D^2$ be a homeomorphism which is quasiconformal on the interior.

What can the restriction of f to S^1 look like?



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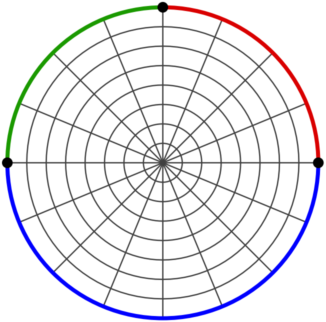
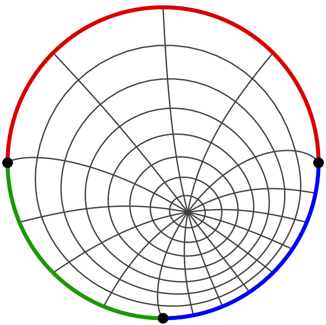


$\xrightarrow{f}$



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What can the restriction of f to S^1 look like?

Theorem (Beurling–Ahlfors 1956)

A homeomorphism $f: S^1 \longrightarrow S^1$ is a restriction of a quasiconformal map on D^2 iff there exists a homeomorphism $\eta: [0,\infty) \longrightarrow [0,\infty)$ so that

$$\frac{\|f(a) - f(b)\|}{\|f(a) - f(c)\|} \leq \eta\left(\frac{\|a - b\|}{\|a - c\|}\right)$$

for every triple a, b, c of distinct points in S^1.

These are the ***quasisymmetries*** of S^1.

General Definition

By a ***gauge*** we mean any homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$.

A homeomorphism $f\colon X \to Y$ between metric spaces is a ***quasisymmetry*** if there exists a gauge η so that

$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

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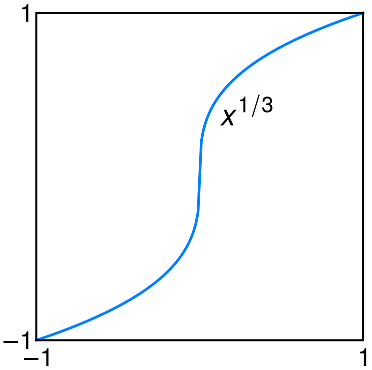
Example: If f is bilipschitz with

$$\frac{1}{K}\, d(x,x') \leq d\big(f(x), f(x')\big) \leq K\, d(x,x')$$

then f is quasisymmetric with gauge $\eta(t) = K^2 t$.

An Example

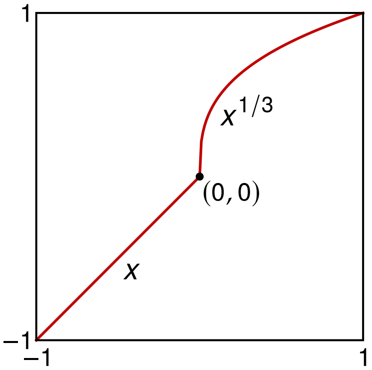


$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

The function $f(x) = x^{1/3}$ is a quasisymmetry of $[-1, 1]$, with gauge

$$\eta(t) = \begin{cases} 6t^{1/3} & \text{if } 0 \leq t \leq 1 \\ 6t & \text{if } t > 1. \end{cases}$$

A Non-Example



$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

This function is **not** a quasisymmetry of $[-1,1]$.

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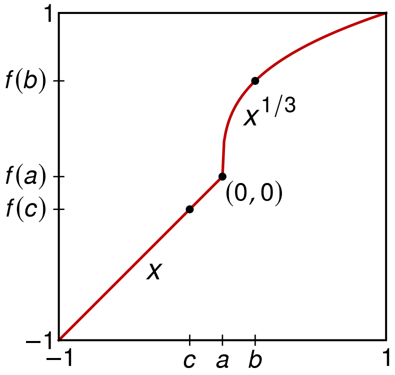


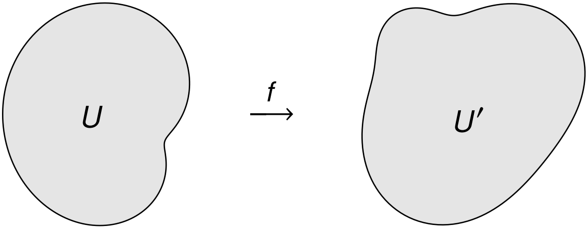
$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

For $a = 0$, $b = \varepsilon$, and $c = -\varepsilon$, we have

$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} = \frac{\varepsilon^{1/3}}{\varepsilon} = \frac{1}{\varepsilon^{2/3}} \qquad \text{and} \qquad \frac{d(a,b)}{d(a,c)} = 1.$$

Quasiconformal vs. Quasisymmetric

Let $f : U \to U'$ be a homeomorphism between domains in $\mathbb{R}^n$.



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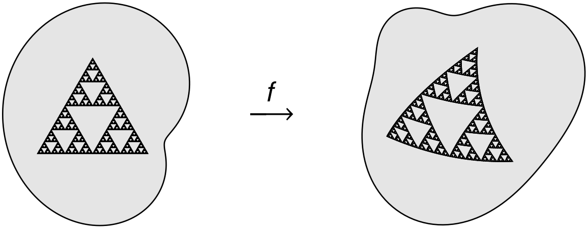

Theorem (Väisälä 1981)

If f is quasiconformal then f restricts to a quasisymmetry on every compact subset of U.

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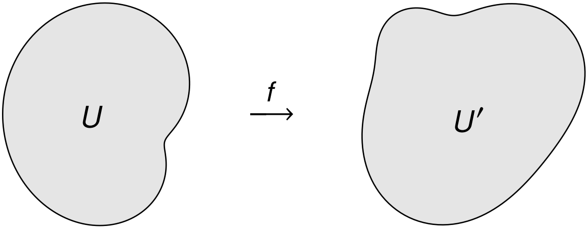

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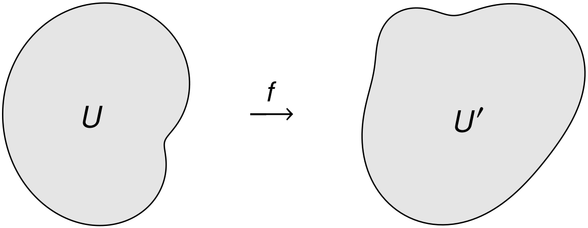

Theorem (Egg Yolk Principle, Väisälä 1981)

The following are equivalent:

1. *f is quasiconformal.*
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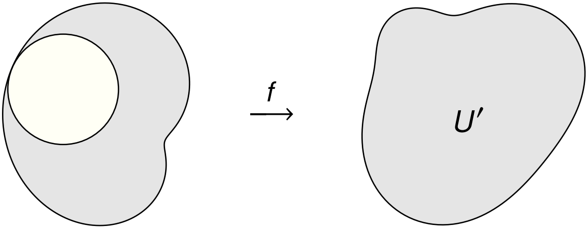

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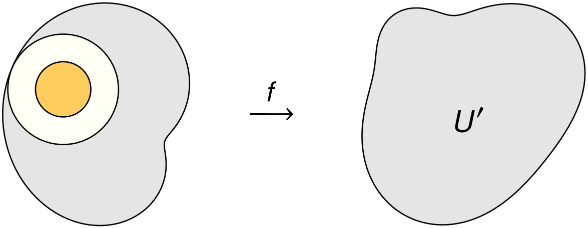

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Application: Hyperbolic Boundaries

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A homeomorphism of S^1 *is the restriction of a quasiconformal map on* D^2 *if and only if it is a quasisymmetry.*

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Cannon's Conjecture

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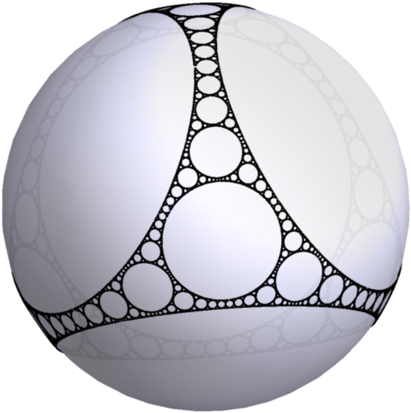

Quasisymmetries of Fractals

Quasisymmetry Groups

Observation

If X is a metric space, then the quasisymmetries $X \longrightarrow X$ form a group.

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Proof.

Suppose f and g are quasisymmetric with gauges η and ϑ.

$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right) \quad \text{and} \quad \frac{d\big(g(a), g(b)\big)}{d\big(g(a), g(c)\big)} \le \vartheta\left(\frac{d(a,b)}{d(a,c)}\right)$$

Then:

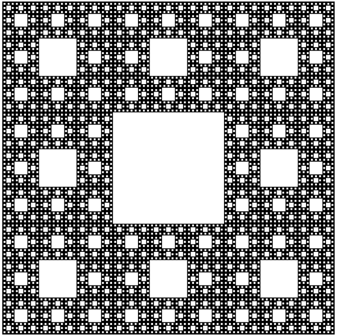
1. $f \circ g$ is quasisymmetric with gauge $\eta \circ \vartheta$, and
2. f^{-1} is quasisymmetric with gauge $t \mapsto 1 \big/ \eta^{-1}(1/t)$. $\square$

Sierpiński Carpets

Some metric spaces have surprisingly few quasisymmetries.

Theorem (Bonk–Merenkov 2013)

The quasisymmetry group of the square Sierpiński carpet is dihedral of order 8*.*



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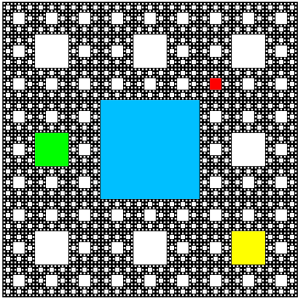
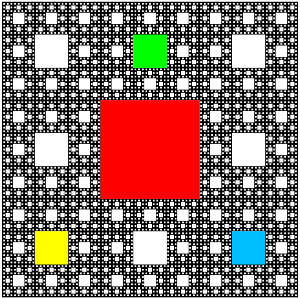
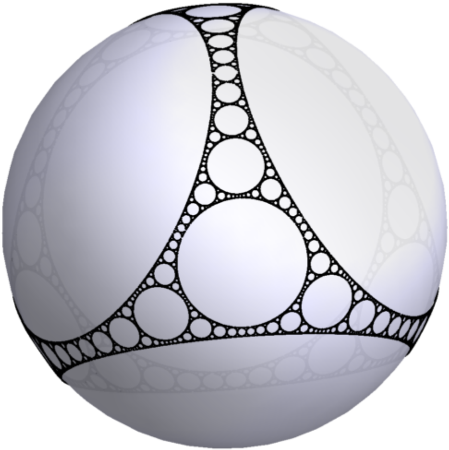


$\longrightarrow$



Sierpiński Carpets

Other Sierpiński carpets can have many quasisymmetries.



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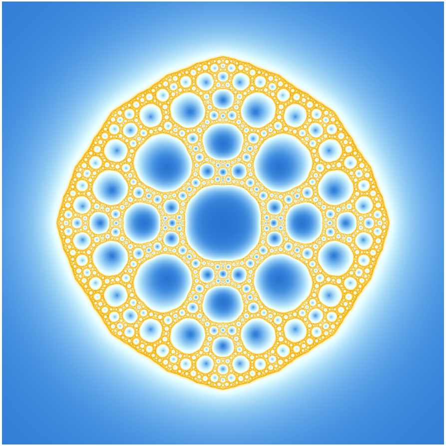

$$f(z) = z^2 - \frac{1}{16z^2}$$

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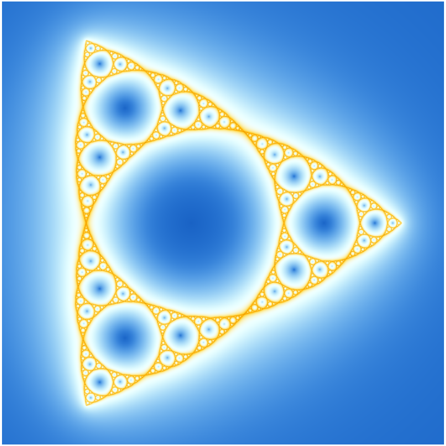
Theorem (Bonk–Lyubich–Merenkov 2016)

Let $f(z)$ be a postcritically finite rational function whose Julia set J_f is a Sierpiński carpet. Then J_f has only finitely many quasisymmetries.

Qiu, Yang, and Zeng (2019) extend this to a large family of semi-hyperbolic Sierpiński carpet Julia sets.

A Sierpiński Triangle

Some other Julia sets are also known to have only finitely many quasisymmetries.

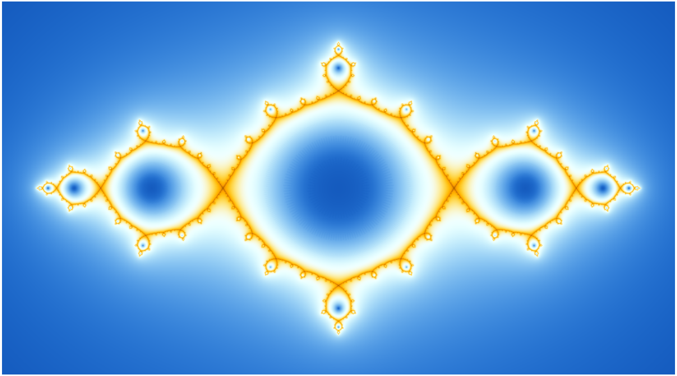


$$f(z) = z^2 - \frac{16}{27z}$$

(Ushiki 1991,
Kameyama 2000)

The Basilica

The ***basilica*** is the Julia set for $f(z) = z^2 - 1$



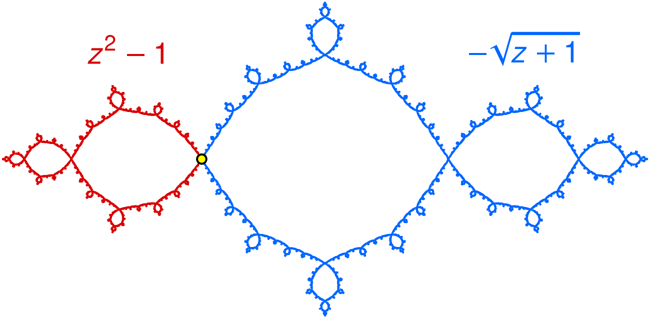
Theorem (Lyubich–Merenkov 2018)

The basilica has infinitely many quasisymmetries.

Quasisymmetries of the Basilica

Definition (B–Forrest 2015)

The ***basilica Thompson group*** consists of all piecewise-conformal homeomorphisms of the basilica whose breakpoints are cut points.



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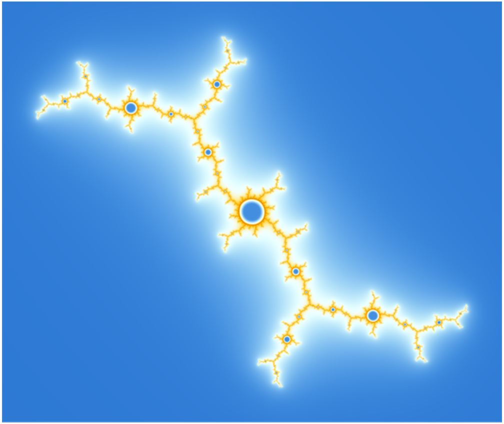
Theorem (Lyubich–Merenkov 2018)

Elements of the basilica Thompson group are quasisymmetries.

Other Julia Sets

Can we extend this to other polynomial Julia sets?



Julia set for $f(z) = z^2 - 0.157 + 1.032i$

Finitely Ramified Fractals

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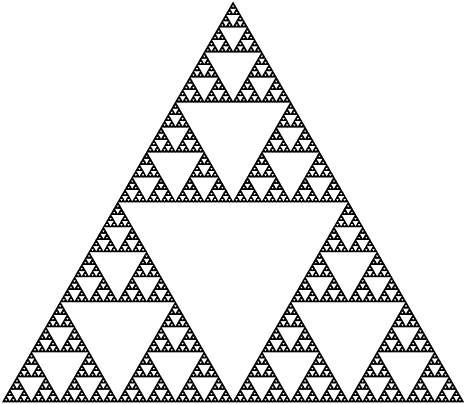
Roughly speaking, a fractal is ***finitely ramified*** if it is made from pieces (called ***cells***) that have finitely many boundary points.



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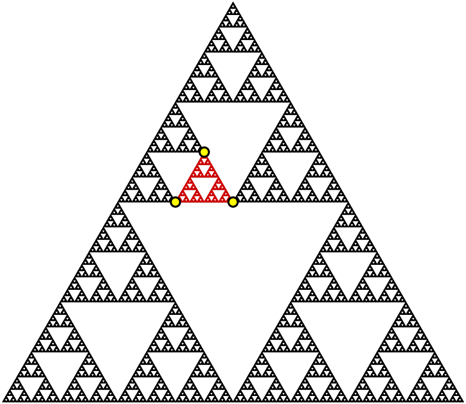

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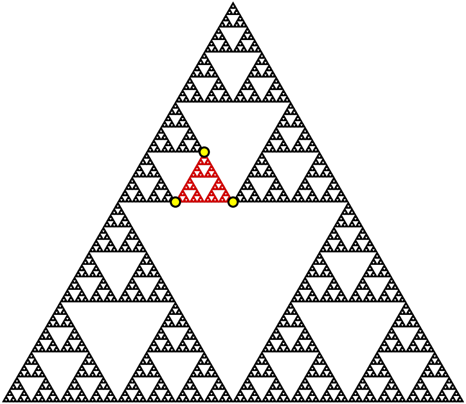

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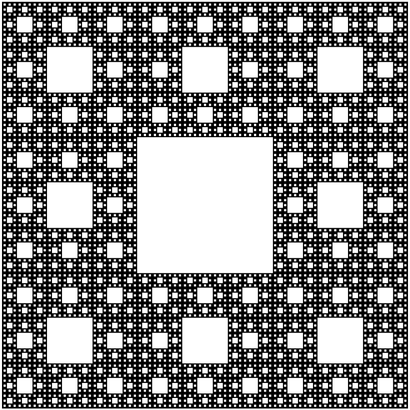

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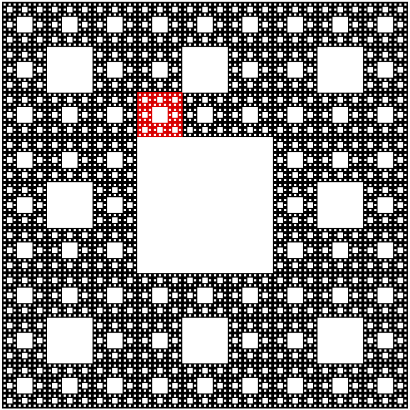

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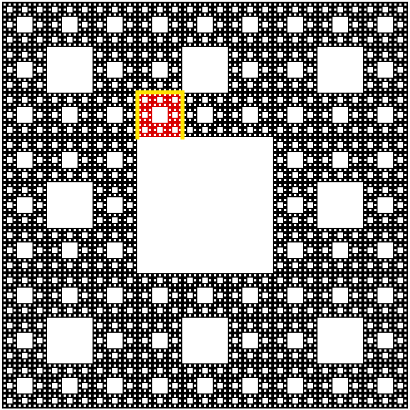

Not finitely ramified

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Definition (Teplyaev 2008)

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For each $n \geq 0$, fix a finite collection of subsets of X (the ***n-cells***).

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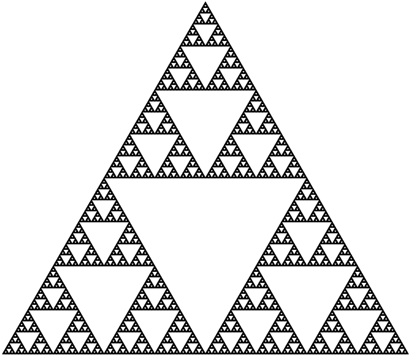

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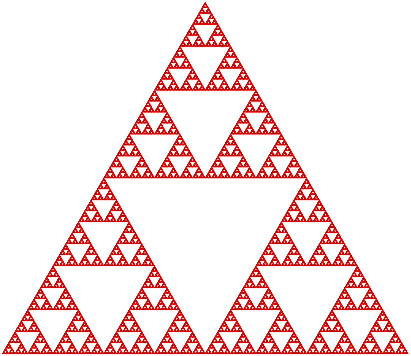

One 0-cell

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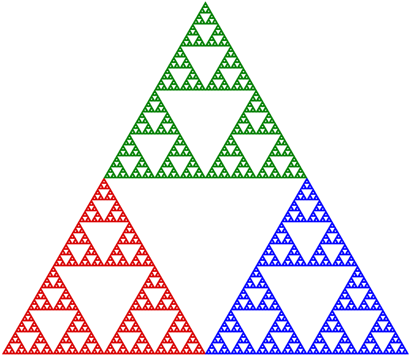

Three 1-cells

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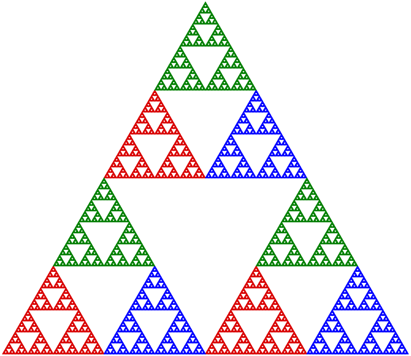

Nine 2-cells

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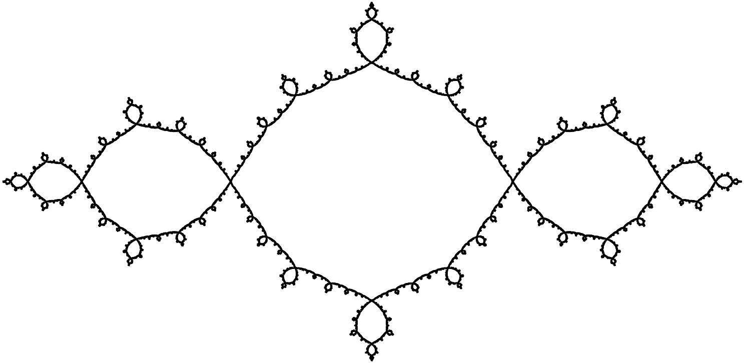
4. Every n-cell is a union of $(n+1)$-cells.
5. If $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ with each E_n an n-cell, then $\bigcap_{n=0} E_n$ is a single point.

Example: The Basilica

The basilica Julia set can be viewed as a finitely ramified fractal.



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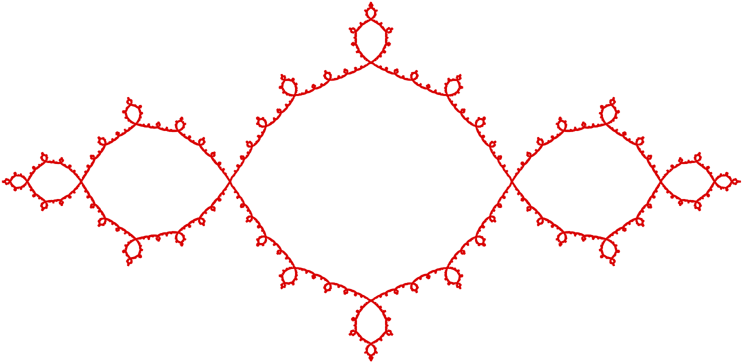

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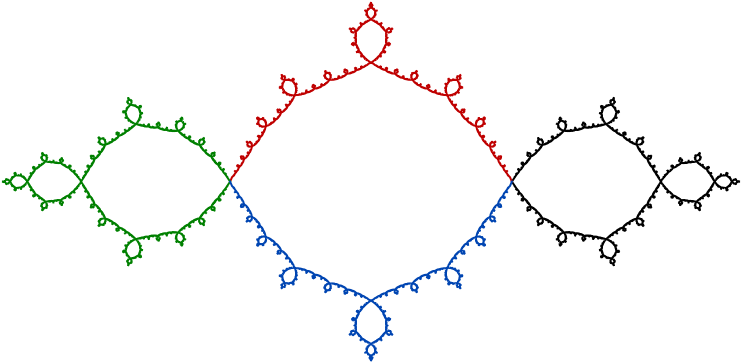

Four 1-cells

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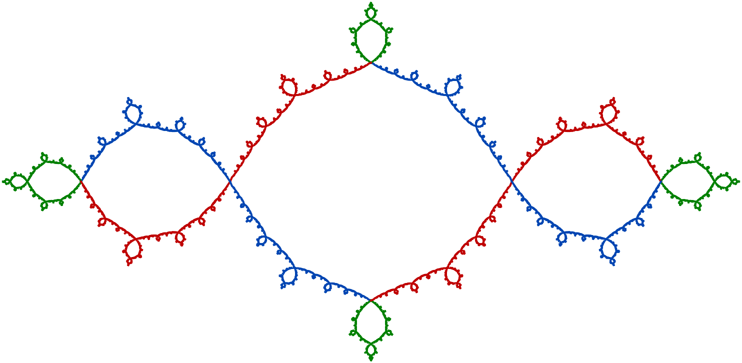

Twelve 2-cells

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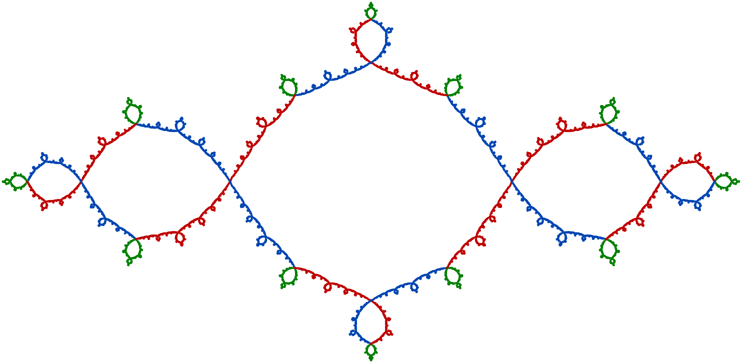
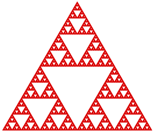


Thirty-six 3-cells

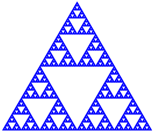
Cellular Maps

Let X be a finitely ramified fractal, and let E, E' be cells in X.



E (n-cell)

$\longrightarrow$



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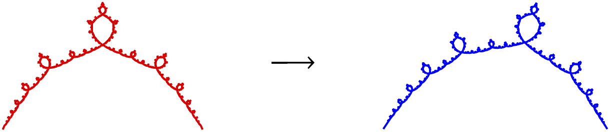

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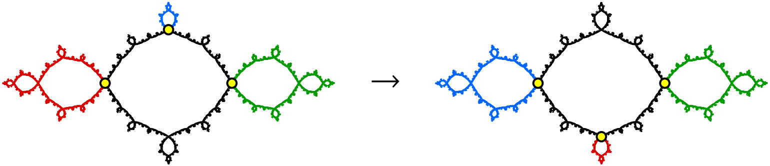
Piecewise-Cellular Maps

A homeomorphism $f\colon X \to X$ is ***piecewise-cellular*** if there exist subdivisions

$$\{E_1, \ldots, E_n\} \qquad \text{and} \qquad \{E'_1, \ldots, E'_n\}$$

of X into cells so that f maps each E_i to E'_i by a cellular map.



Note: The piecewise-cellular homeomorphisms of X form a group.

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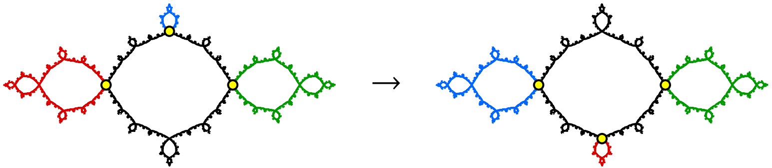

Question: When are piecewise-cellular homeomorphisms quasisymmetries?

Main Results

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A metric on a finitely ramified fractal X is ***quasiregular*** if:

1. It satisfies the exponential decay condition.
2. It has bounded neighbor ratios, and
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Theorem (B–Forrest 2021)

If the metric on X is quasiregular then any piecewise-cellular homeomorphism of X is a quasisymmetry.

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Exponential Decay Condition:

There exist constants $0 < r < R < 1$ and $C \geq 1$ so that

$$\frac{r^k}{C} \leq \frac{\operatorname{diam}(E')}{\operatorname{diam}(E)} \leq CR^k$$

for any n-cell E and any $(n+k)$-cell E' contained in E.

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Bounded Neighbor Ratios:

There exists a constant $\lambda \geq 1$ so that

$$\frac{1}{\lambda} \leq \frac{\mathrm{diam}(E')}{\mathrm{diam}(E)} \leq \lambda$$

for any two n-cells E and E' that intersect.

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$$d(E, E') \geq \delta \,\mathrm{diam}(E)$$

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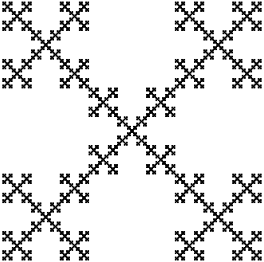
The ***Vicsek fractal*** is the following subset of $\mathbb{R}^2$.



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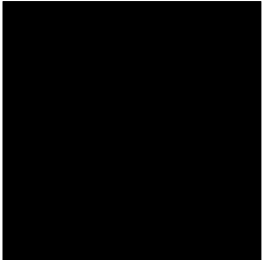

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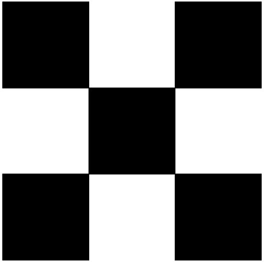

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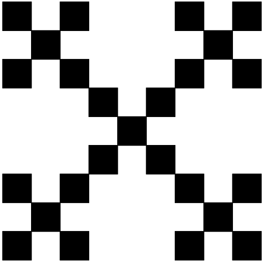

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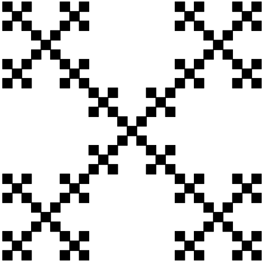

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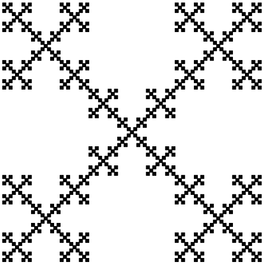

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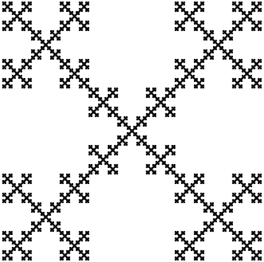

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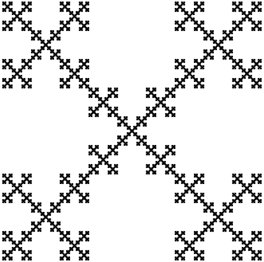

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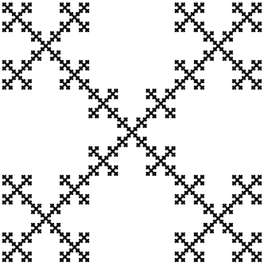

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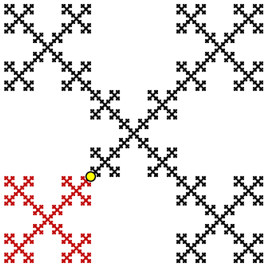

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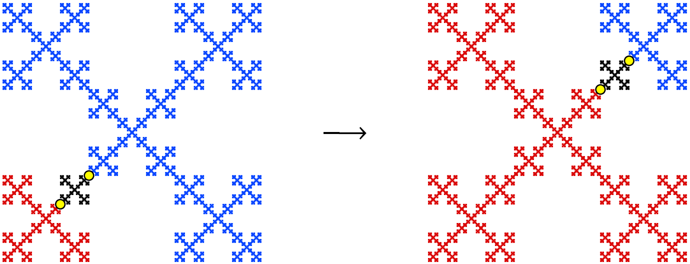

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Theorem (B–Forrest 2021)

Let X be a finitely ramified fractal.

1. *Any two quasiregular metrics on X are quasi-equivalent.*
2. *If d and d' are quasi-equivalent, then d is quasiregular iff d' is quasiregular.*

So any finitely ramified fractal that admits a quasiregular metric has a natural *topological* notion of quasisymmetry.

Finitely Ramified Julia Sets

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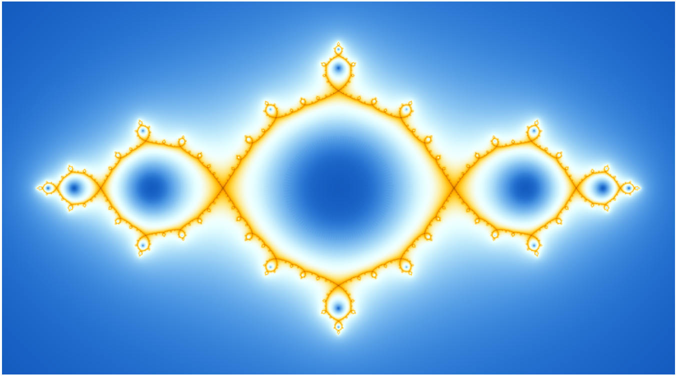
Julia sets for polynomials tend to be finitely ramified.



Julia set for $f(z) = z^2 - 1$

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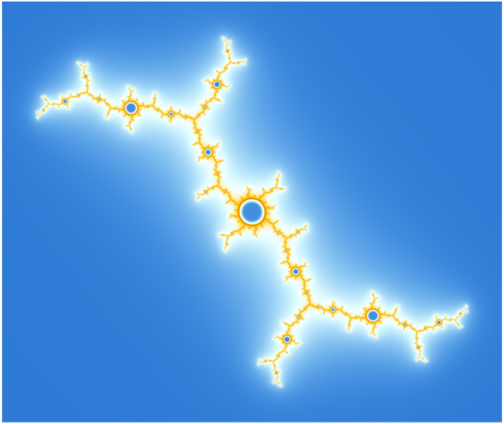

Julia set for $f(z) = z^2 - 0.157 + 1.032i$

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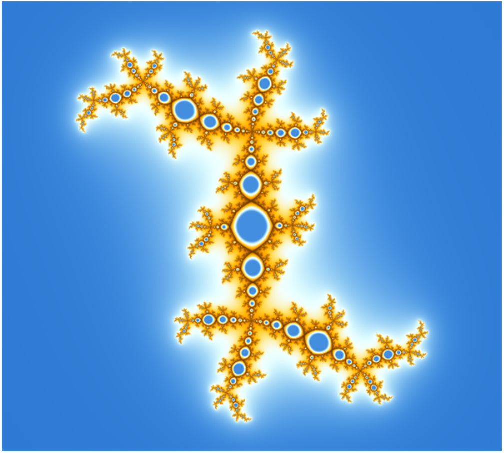

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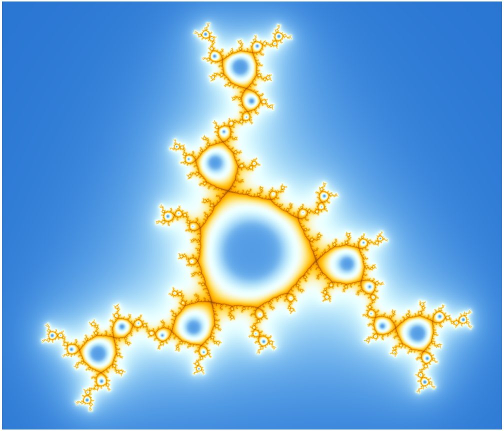

Julia set for $f(z) = z^3 - 0.21 + 1.09i$

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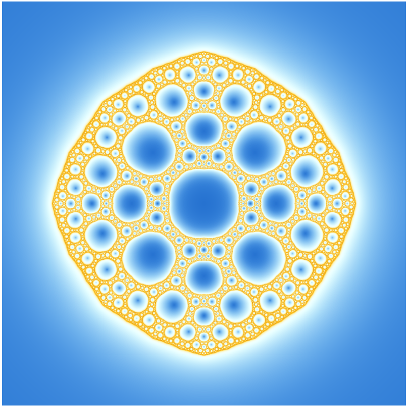

Julia set for $f(z) = z^2 - \dfrac{1}{16z^2}$

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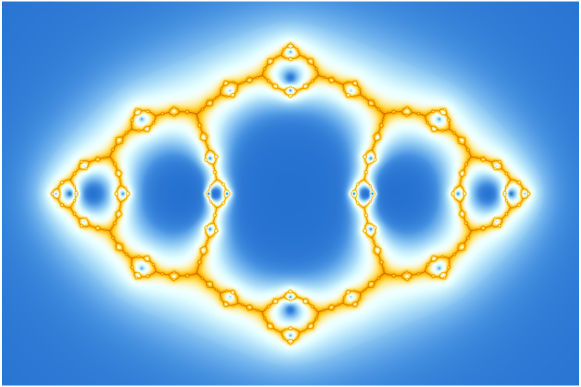

Julia set for $f(z) = \dfrac{1}{z^2} - 1$

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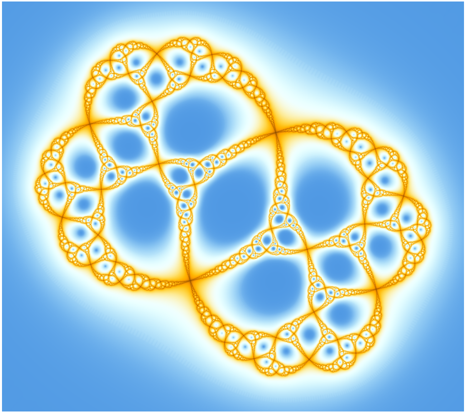

Julia set for $f(z) = \dfrac{e^{2\pi i/3}z^2 - 1}{z^2 - 1}$

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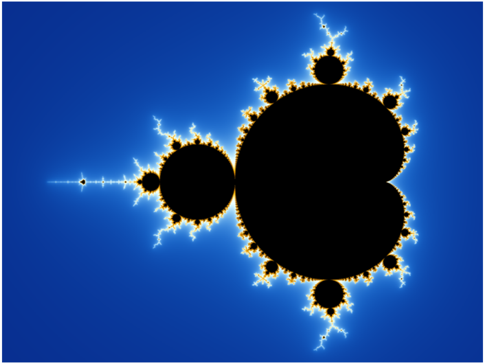

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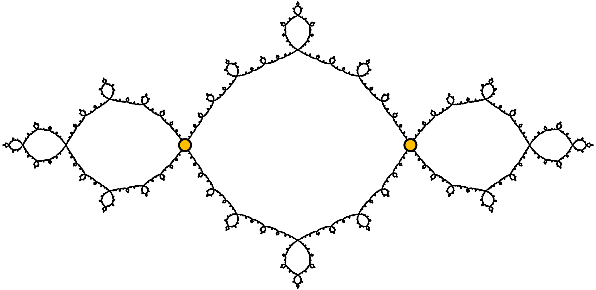

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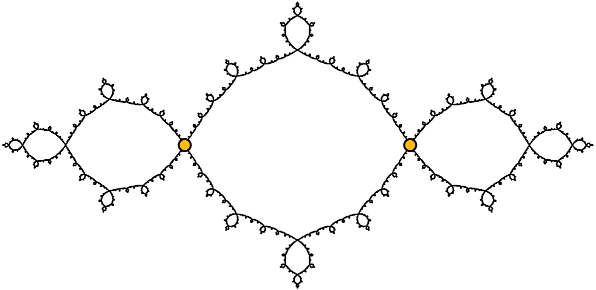

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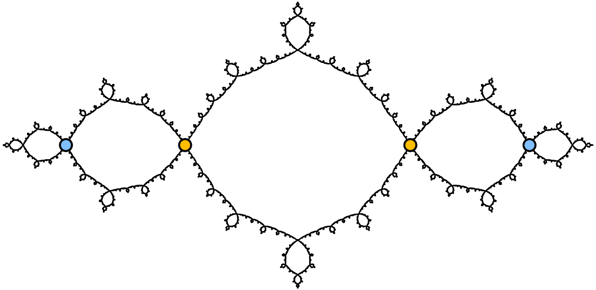

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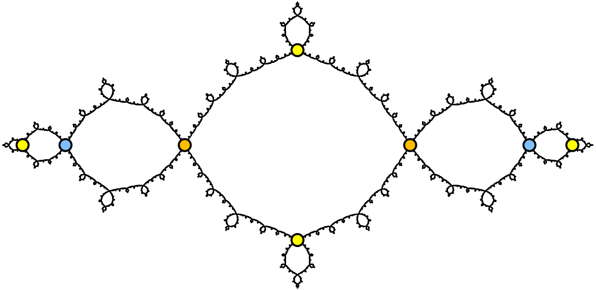

$f^{-2}(S)$

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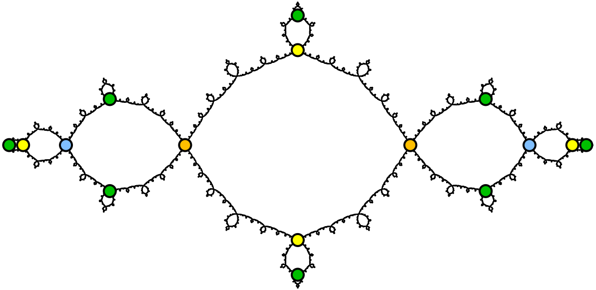

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For such a set, the iterated preimages $f^{-n}(S)$ cut J_f into cells.

Main Results

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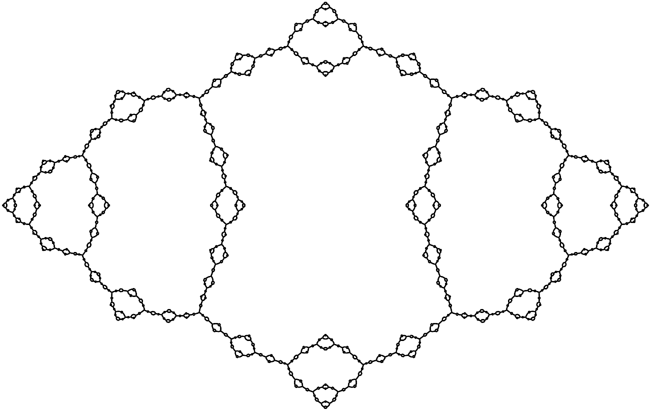
Theorem (B–Forrest 2021)

If J_f has a simple cut set, then the restriction of the Euclidean metric to J_f is quasiregular with respect to the resulting cell structure.

Example: $z^{-2} - 1$

Consider the Julia set for $f(z) = \dfrac{1}{z^2} - 1$.



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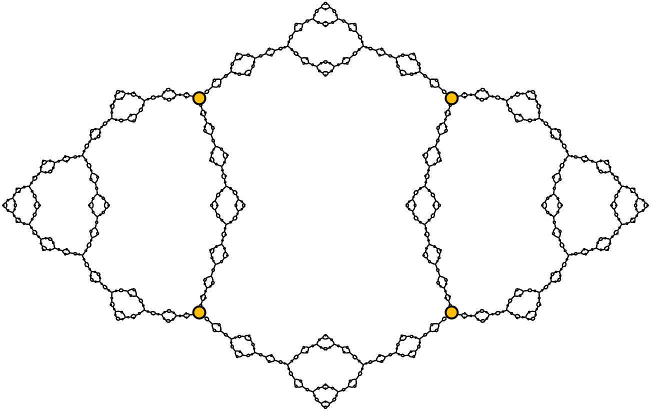

Six 1-cells

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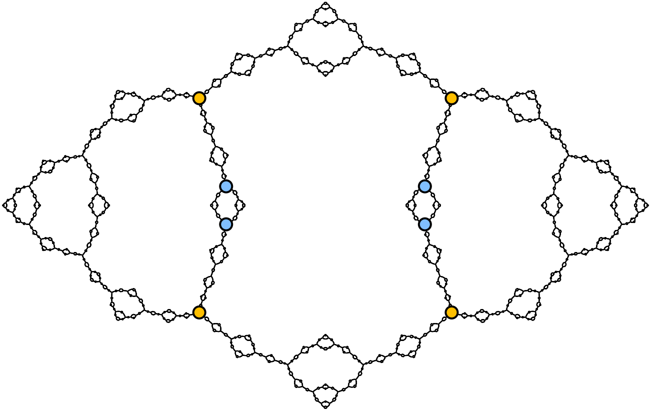
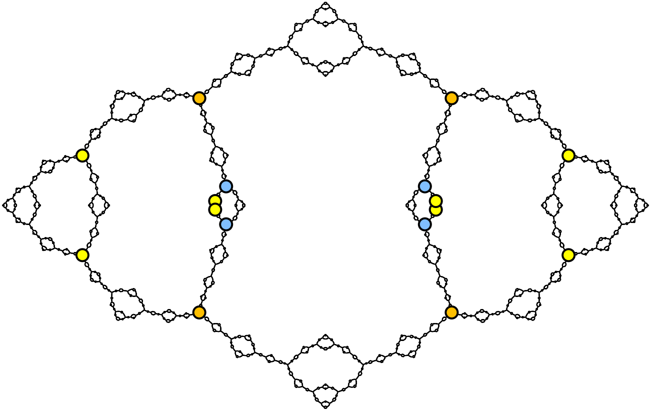


Twelve 2-cells

Example: $z^{-2} - 1$

Consider the Julia set for $f(z) = \dfrac{1}{z^2} - 1$.



Twenty-four 3-cells

Example: $z^{-2} - 1$

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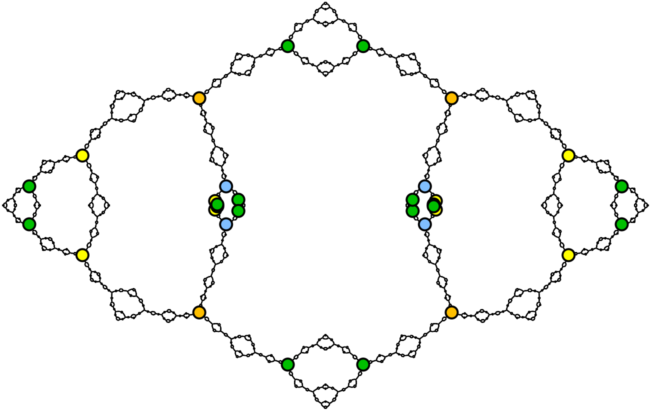

Forty-eight 4-cells

Example: $z^{-2} - 1$

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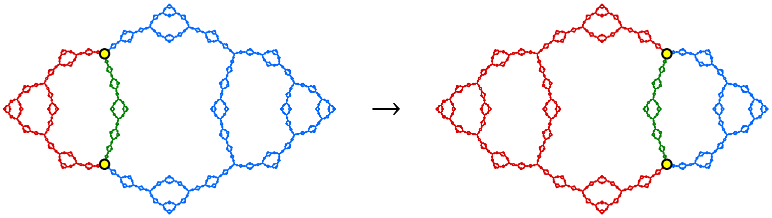
Consider the Julia set for $f(z) = \dfrac{1}{z^2} - 1$.

Any piecewise-cellular homeomorphism of J_f is a quasisymmetry.



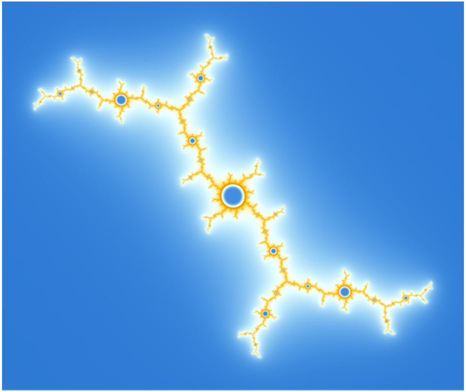
It follows that J_f has infinitely many quasisymmetries.

Polynomials

Main Results

Theorem (B–Forrest 2021)

If $f(z)$ is a hyperbolic polynomial, then J_f has a simple cut set.



Julia set for $f(z) = z^2 - 0.157 + 1.032i$

Main Results

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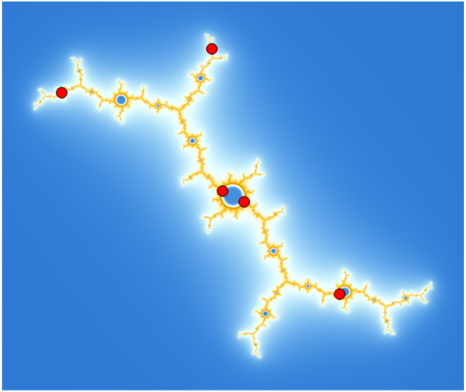

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Sketch of Proof.

Let $U_1, \ldots, U_n$ be the bounded components of $\mathbb{C} \setminus J_f$ that contain the critical points.

If U_i contains a critical point of local degree d_i, then choose d_i pre-periodic points on ∂U_i that have the same image under f.

The union of all of these points and their forward orbits is a simple cut set. $\square$

Main Results

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Thus:

- Every such Julia set has a finitely ramified cell structure, and
- The restriction of the Euclidean metric is quasiregular, so
- Piecewise-cellular homeomorphisms are quasisymmetries.

Main Results

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Theorem (B–Forrest 2021)

If f is a hyperbolic quadratic polynomial, then J_f has infinitely many quasisymmetries.

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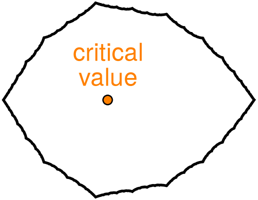

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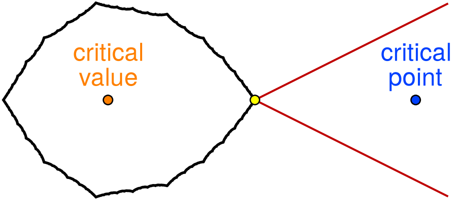

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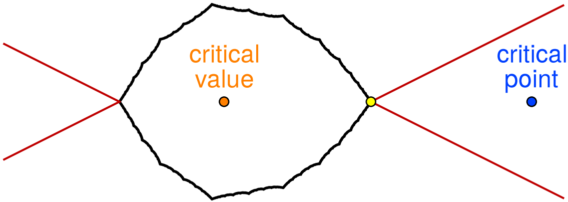

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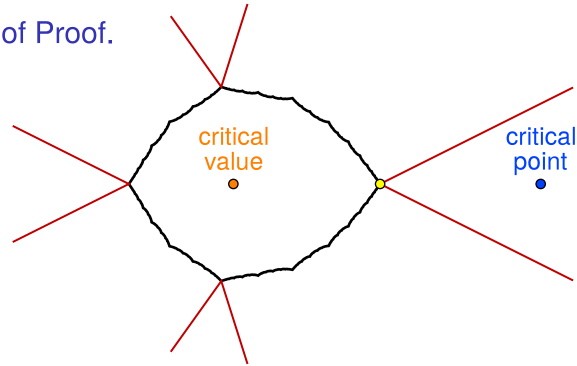

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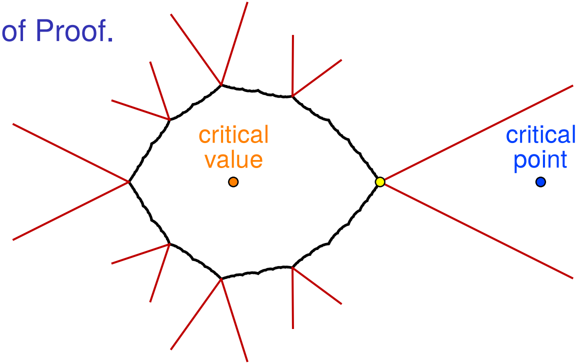

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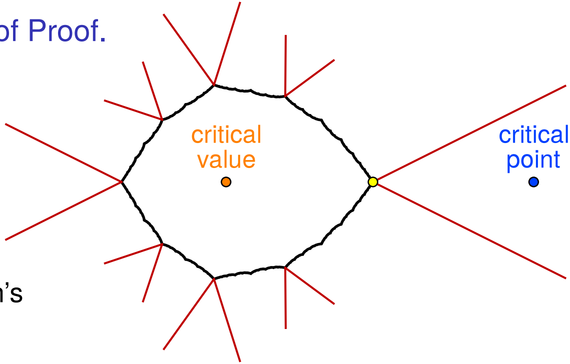

Thompson’s
group F

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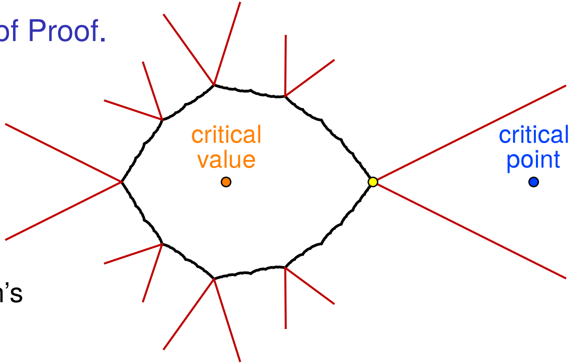
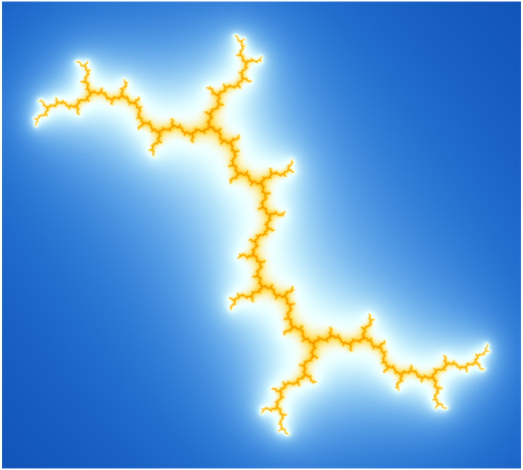


Thompson's group F

□

Open Questions

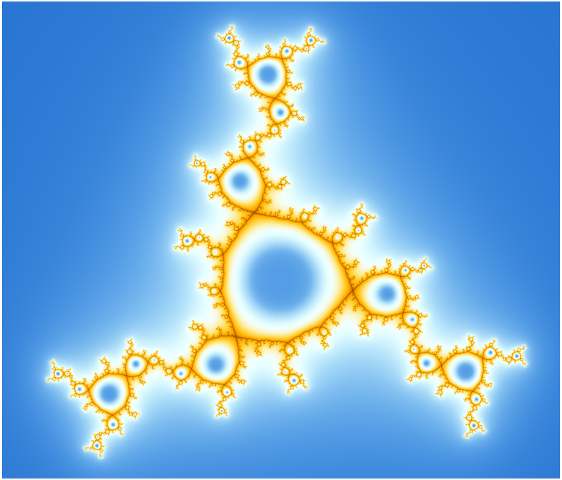
Can this theory be extended to the subhyperbolic case?



Julia set for $f(z) = z^2 + i$

Open Questions

What about hyperbolic cubic polynomials?



Julia set for $f(z) = z^3 - 0.21 + 1.09i$

The End