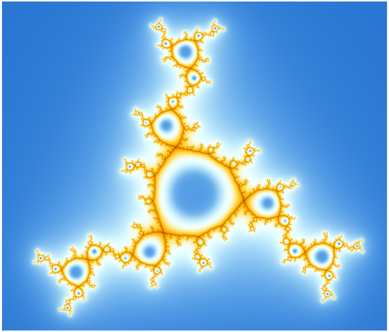
Quasisymmetry Groups of Finitely Ramified Fractals



Jim Belk

Cornell University

Joint Work

Bradley Forrest

Stockton University

Quasiconformal Geometry

Quasiconformal Maps

For a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$, let

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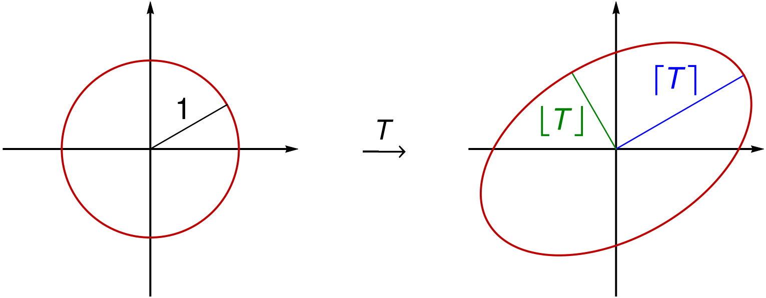

The ratio $\lceil T \rceil / \lfloor T \rfloor$ is a measure of ***eccentricity***.

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A diffeomorphism $f \colon U \longrightarrow U'$ between open subsets of $\mathbb{R}^n$ is ***quasiconformal*** if there exists a $\lambda \geq 1$ so that

$$\frac{\lceil Df_p \rceil}{\lfloor Df_p \rfloor} \leq \lambda$$

for all $p \in U$.

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Note: If $\lambda = 1$ then f is ***conformal*** (or anticonformal).

Applications of Quasiconformal Maps

- ***Teichmüller theory:*** Metric on the Teichmüller space of a hyperbolic surface. Leads to the Nielsen–Thurston classification of mapping classes (Bers).

- ***Mostow rigidity:*** For $n \geq 3$, if X and Y are closed hyperbolic n-manifolds and $\pi_1(X) \cong \pi_1(Y)$ then X and Y are isometric.

- ***Groups quasi-isometric to $\mathbb{H}^n$:*** Any f.g. group which is quasi-isometric to $\mathbb{H}^n$ has a geometric action on $\mathbb{H}^n$ (Tukia, Cannon, Cooper, Gromov).

- ***Further Applications:*** Complex dynamics, characteristic classes, elliptic P.D.E.'s

Quasisymmetries

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Definition

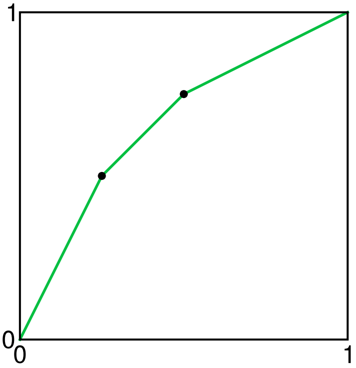
A homeomorphism $f\colon X \longrightarrow Y$ between metric spaces is a ***quasisymmetry*** if

$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

for some homeomorphism $\eta\colon [0,\infty) \longrightarrow [0,\infty)$ and every triple a, b, c of distinct points in X.

Note: The quasisymmetries $X \longrightarrow X$ form a group.

Examples



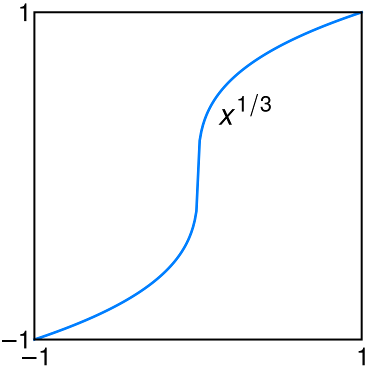
$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

If f is bilipschitz with

$$\frac{1}{K}\,d(x,x') \le d\big(f(x),f(x')\big) \le K\,d(x,x')$$

then f is quasisymmetric with $\eta(t) = K^2 t$.

Examples

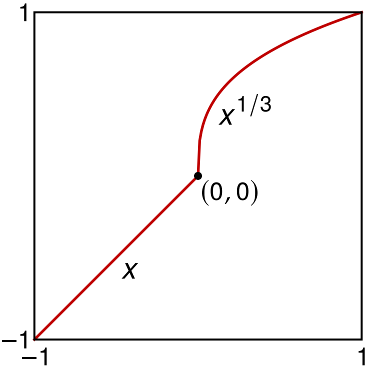


$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

The function $f(x) = x^{1/3}$ is a quasisymmetry of $[-1,1]$, with

$$\eta(t) = \begin{cases} 6t^{1/3} & \text{if } 0 \le t \le 1 \\ 6t & \text{if } t > 1. \end{cases}$$

A Non-Example



$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

This function is **not** a quasisymmetry of $[-1,1]$.

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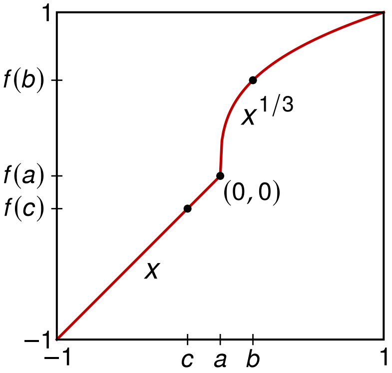


$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

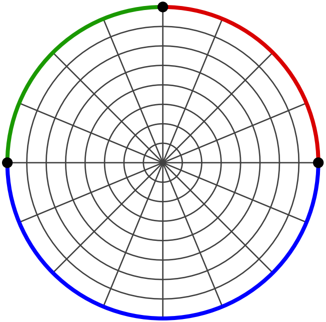
For $a = 0$, $b = \varepsilon$, and $c = -\varepsilon$, we have

$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} = \frac{\varepsilon^{1/3}}{\varepsilon} = \frac{1}{\varepsilon^{2/3}} \qquad \text{and} \qquad \frac{d(a,b)}{d(a,c)} = 1.$$

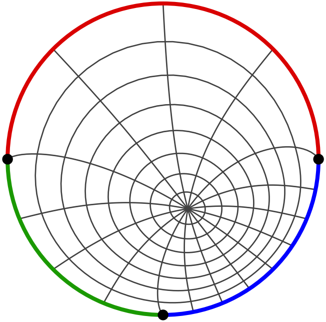
Quasiconformal vs. Quasisymmetric

Theorem (Beurling–Ahlfors 1956)

A homeomorphism of S^1 is the restriction of a quasiconformal map on D^2 if and only if it is a quasisymmetry.

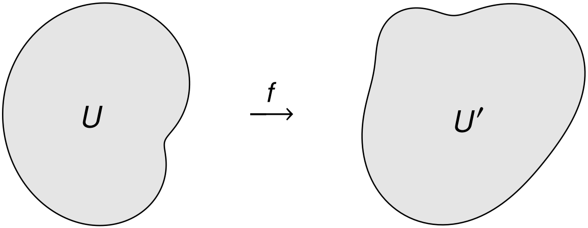


$\longrightarrow$



Quasiconformal vs. Quasisymmetric

Let f be a homeomorphism between open subsets of $\mathbb{R}^n$ ($n \geq 2$).



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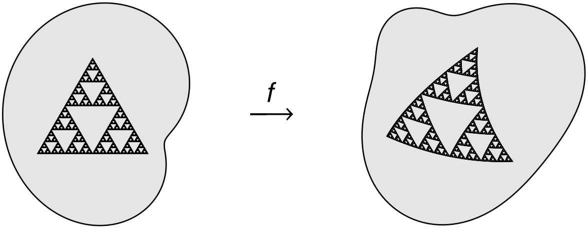

Theorem (Väisälä 1981)

If f is quasiconformal then f restricts to a quasisymmetry on every compact subset of its domain.

Relation to Hyperbolic Groups

Hyperbolic Groups

A group is ***hyperbolic*** if its Cayley graph satisfies Gromov's thin triangles condition.

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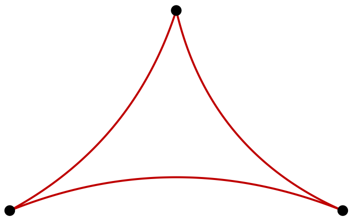

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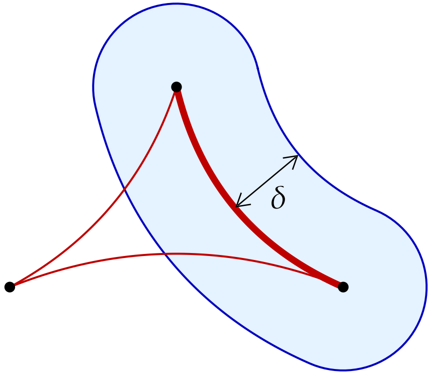

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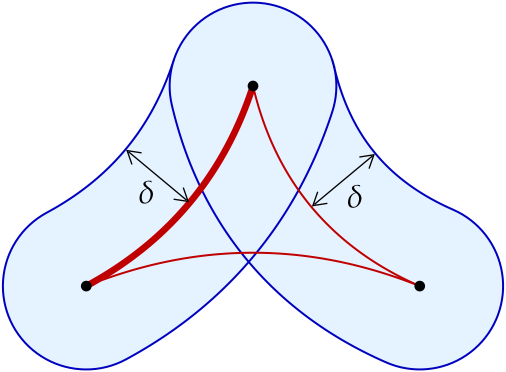

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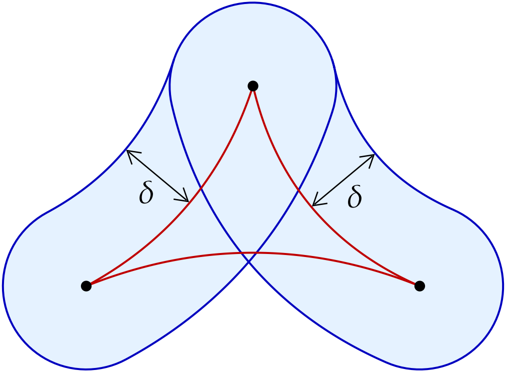

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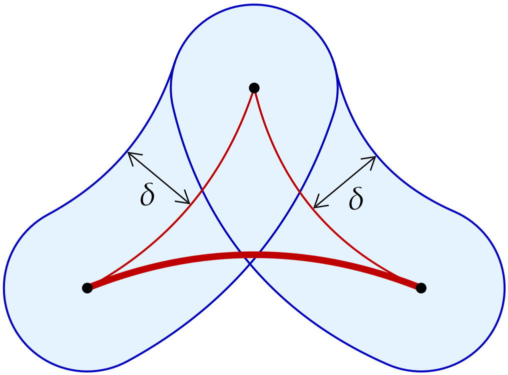

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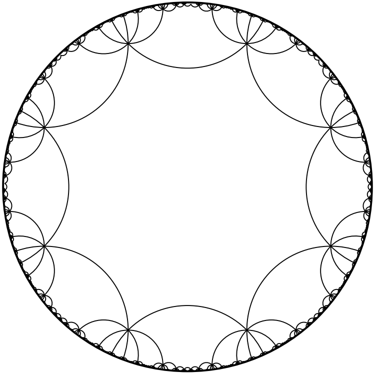

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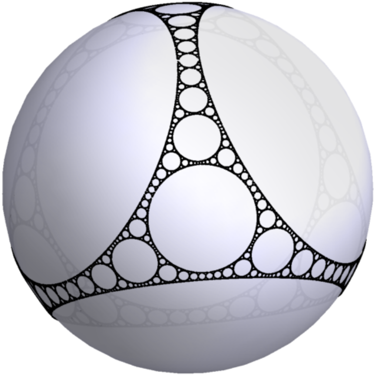

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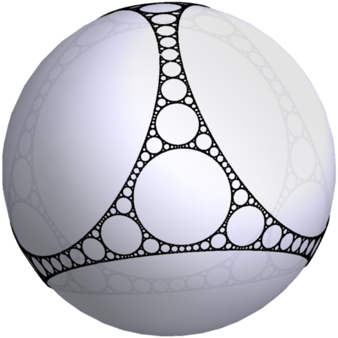
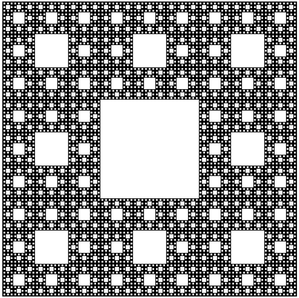


$\partial_\infty G$

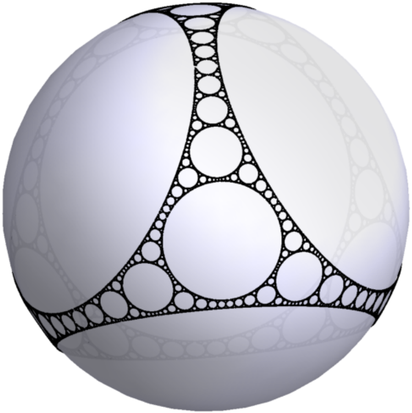


Sierpiński carpet

Quasi-Isometries

Theorem (Bonk–Schramm 2000)

Any quasi-isometry $G \longrightarrow H$ between hyperbolic groups induces a quasisymmetry $\partial_\infty G \longrightarrow \partial_\infty H$.



Cannon's Conjecture

Let G be a hyperbolic group.

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If there exists a homeomorphism $\partial_\infty G \longrightarrow S^2$, then G acts geometrically on $\mathbb{H}^3$.

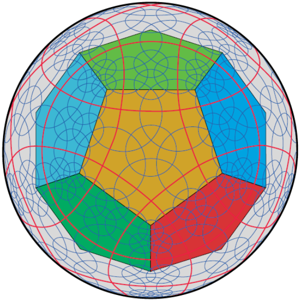

figure from Cannon, Floyd, and Parry 2001

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Theorem (Sullivan–Tukia 1986)

*If there exists a **quasisymmetry** $\partial_\infty G \to S^2$ then G acts geometrically on $\mathbb{H}^3$.*

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Theorem (Sullivan–Tukia 1986)

If there exists a ***quasisymmetry*** *$\partial_\infty G \to S^2$ then G acts geometrically on $\mathbb{H}^3$.*

Conjecture (Kapovich–Kleiner)

If $\partial_\infty G$ is homeomorphic to the Sierpiński carpet, then G acts geometrically on a convex subset of $\mathbb{H}^3$ with totally geodesic boundary.

By the Way

Theorem (Dahmani–Guirardel–Przytycki 2011)

The boundary of a "random" hyperbolic group is homeomorphic to the Menger sponge.

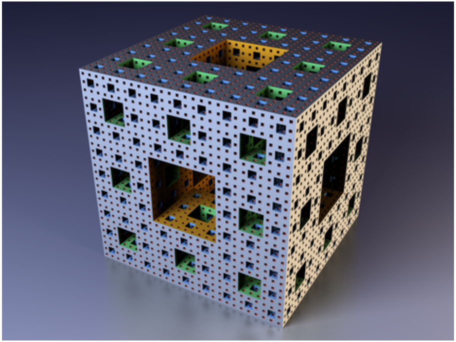
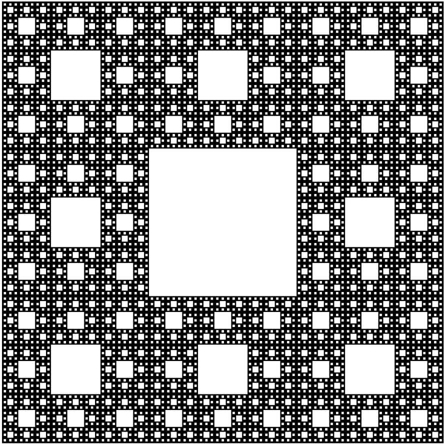


figure by Niabot from Wikimedia Commons

Quasisymmetries of Fractals

Quasisymmetries of Fractals

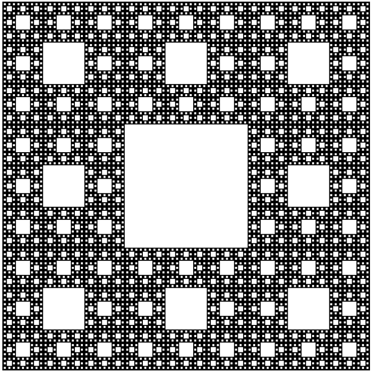
We want to understand the quasisymmetry groups of fractal spaces such as the Sierpiński carpet.



Quasisymmetries of Fractals

Theorem (Bonk–Merenkov 2013)

The quasisymmetry group of the square Sierpiński carpet is dihedral of order 8*.*



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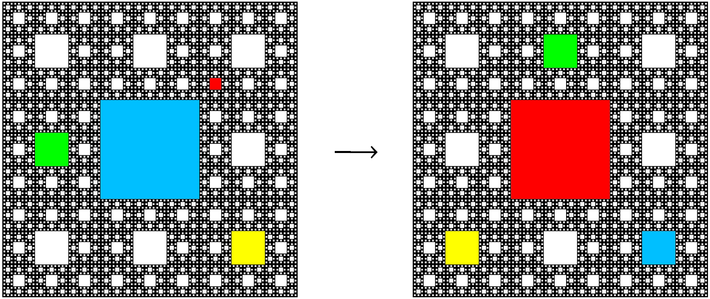
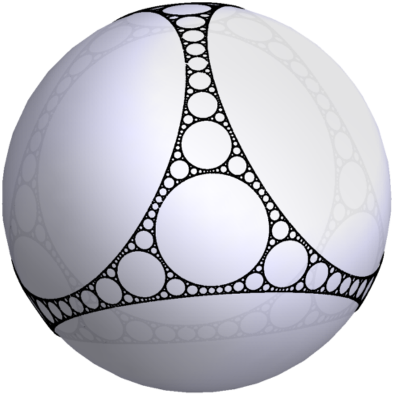


The full homeomorphism group is very large.

Quasisymmetries of Fractals

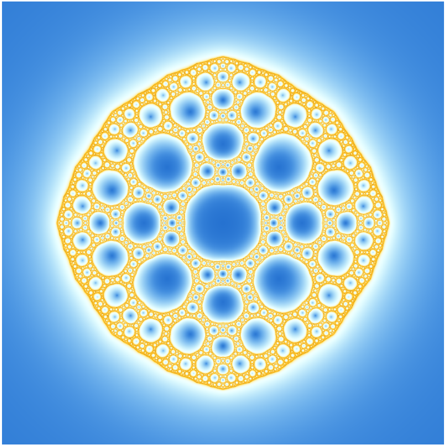
Other Sierpiński carpets can have many quasisymmetries.



So the quasisymmetry group depends on the metric.

Julia Sets

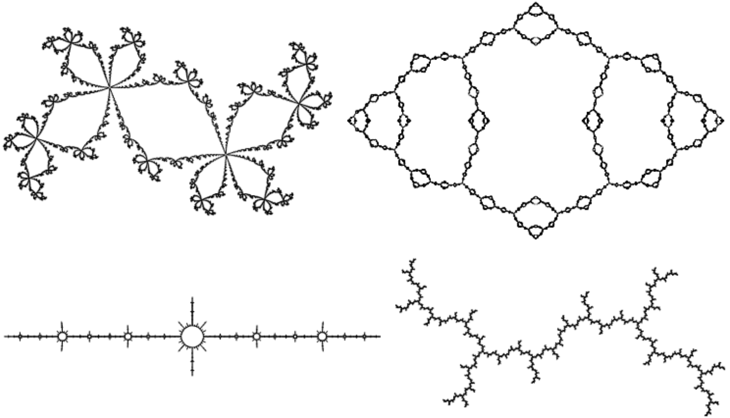
Sierpiński carpets also arise as Julia sets for certain rational functions (Milnor–Lei 1993).



$$f(z) = z^2 - \frac{1}{16z^2}$$

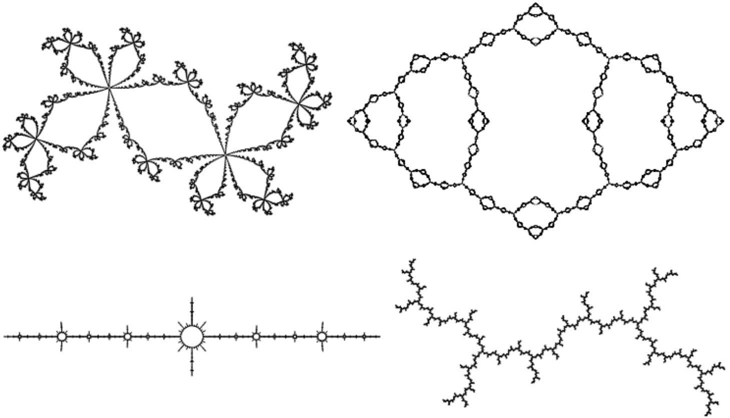
Julia Sets

Every rational function on the Riemann sphere has an associated ***Julia set***.



Julia Sets

The Julia set is the closure of the set of repelling periodic points.



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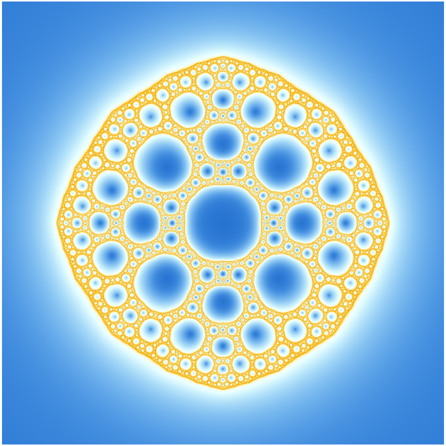

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Julia Sets

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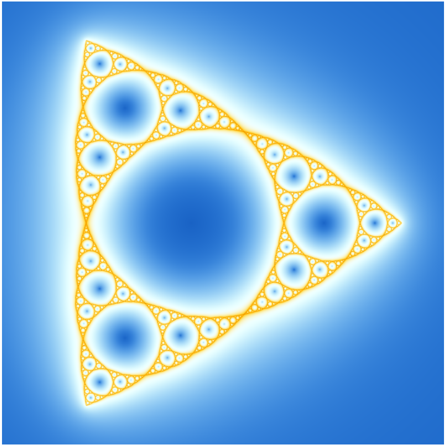
Theorem (Bonk–Lyubich–Merenkov 2016)

Let $f(z)$ be a rational function whose Julia set J_f is a Sierpiński carpet. If f is postcritically finite, then the quasisymmetry group of J_f is finite.

Qiu, Yang, and Zeng (2019) extend this to a large family of semi-hyperbolic Sierpiński carpet Julia sets.

Julia Sets

Some other Julia sets are also known to have finite quasisymmetry group.

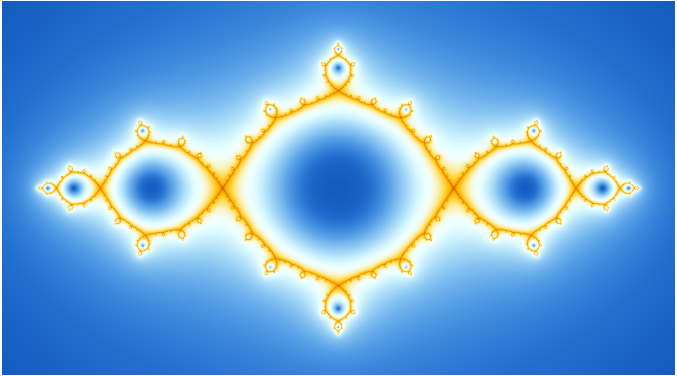


$$f(z) = z^2 - \frac{16}{27z}$$

(Ushiki 1991,
Kameyama 2000)

The Basilica

The ***basilica*** is the Julia set for $f(z) = z^2 - 1$



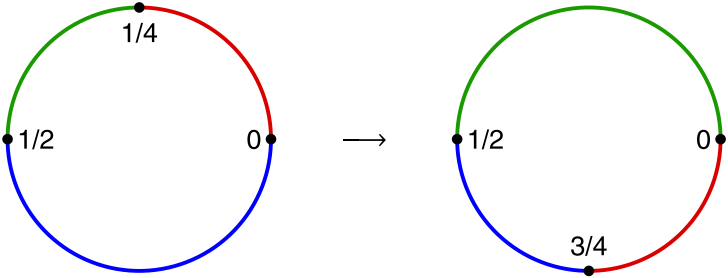
Theorem (Lyubich–Merenkov 2018)

The quasisymmetry group of the basilica is infinite.

Quasisymmetries of the Basilica

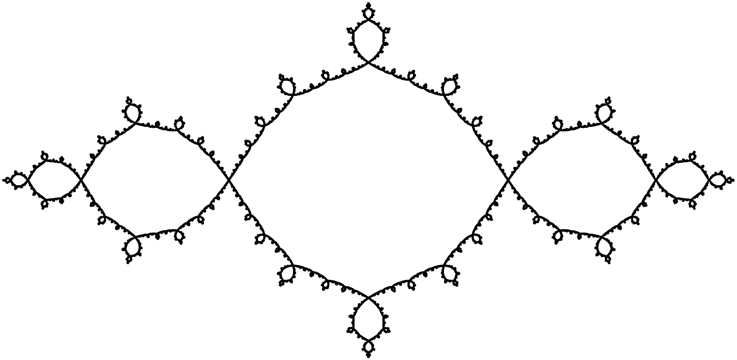
Thompson's group T is the group of all piecewise-linear homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ that satisfy the following conditions:

1. All slopes have the form 2^n for some $n \in \mathbb{Z}$.
2. Each breakpoint is a dyadic rational, as is the image of 0.



Quasisymmetries of the Basilica

In 2015, Bradley Forrest and I proved that Thompson's group T acts on the basilica in a natural way.



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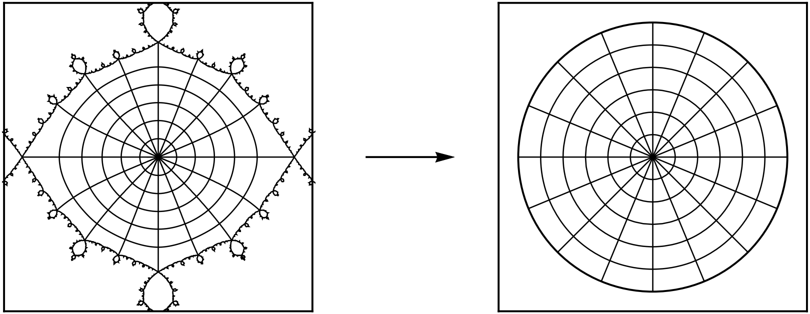

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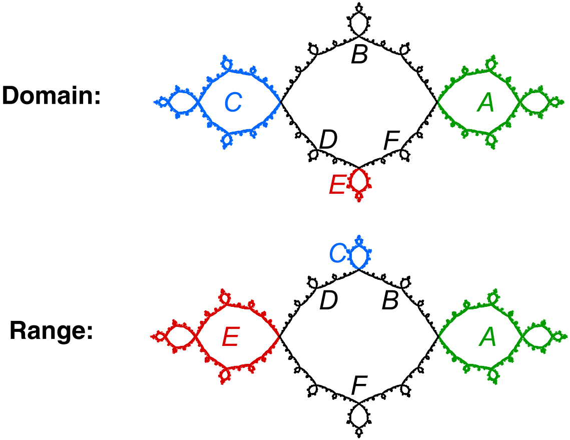

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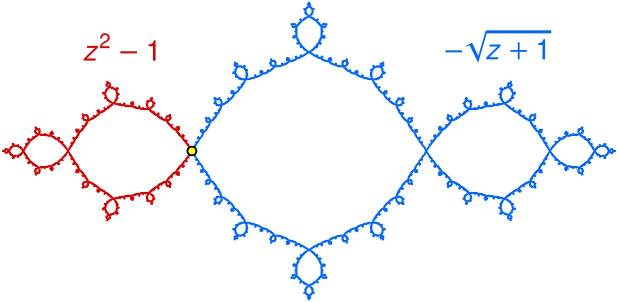
This T is contained in a larger group of piecewise-conformal homeomorphisms that we called the ***basilica Thompson group***.



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Theorem (B–Forrest 2015)

The basilica Thompson group is finitely generated, co-embeddable with T, and has an index-two subgroup which is simple.

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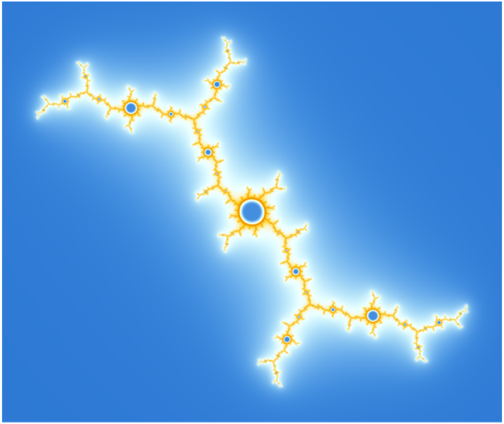
Theorem (Lyubich–Merenkov 2018)

All elements of the basilica Thompson group are quasisymmetries.

Other Julia Sets

Can we extend this to other polynomial Julia sets?



Julia set for $f(z) = z^2 - 0.157 + 1.032i$

Finitely Ramified Fractals

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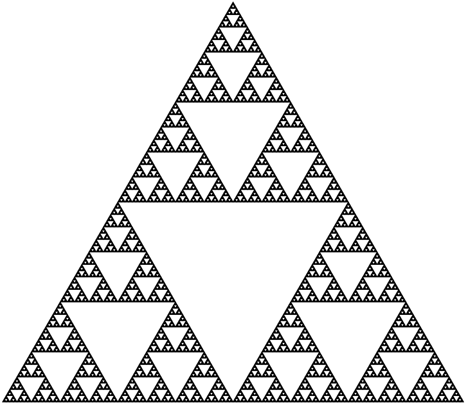
Roughly speaking, a fractal is ***finitely ramified*** if it is made from pieces (called ***cells***) that have finitely many boundary points.



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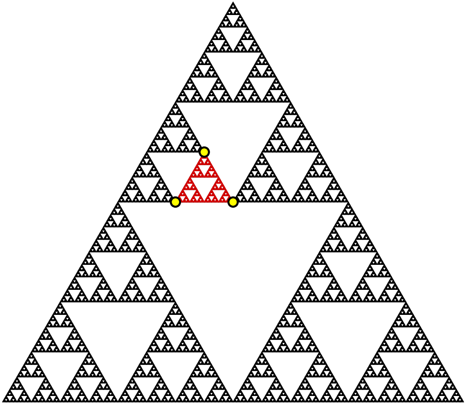

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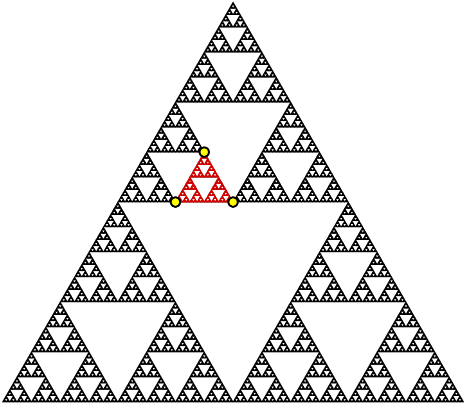
Roughly speaking, a fractal is ***finitely ramified*** if it is made from pieces (called ***cells***) that have finitely many boundary points.



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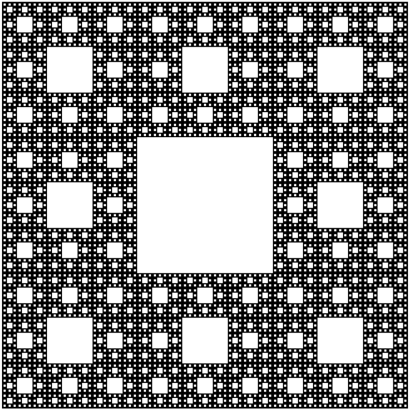

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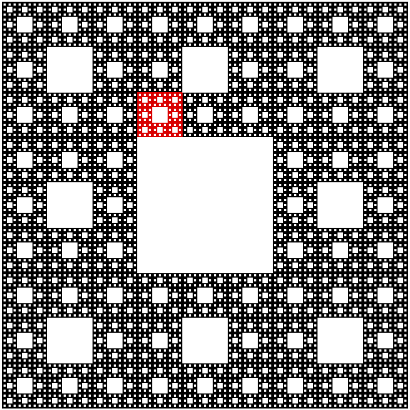

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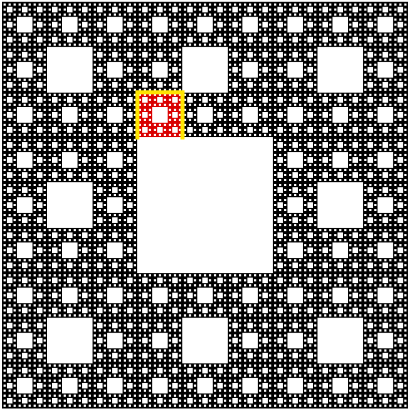

Not finitely ramified

General Definition

Definition (Teplyaev 2008)

Let X be a compact, connected metrizable space.

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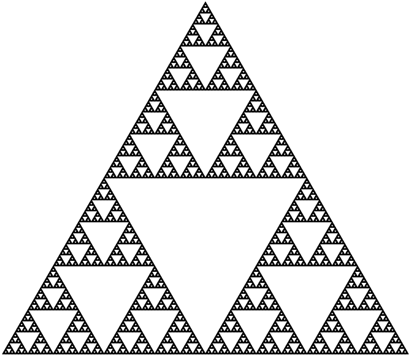

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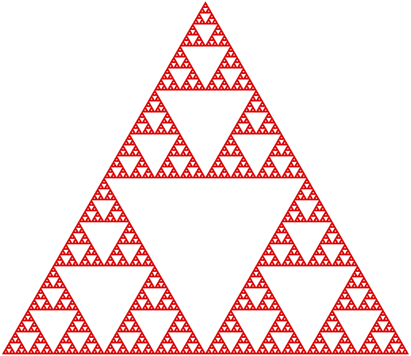

One 0-cell

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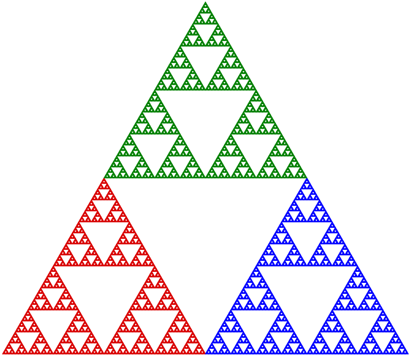

Three 1-cells

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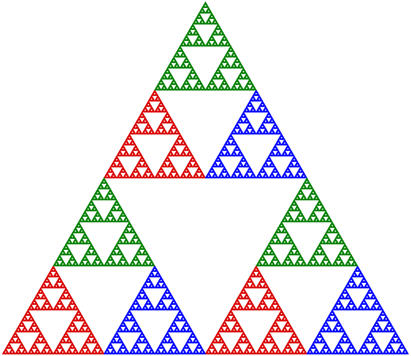

Nine 2-cells

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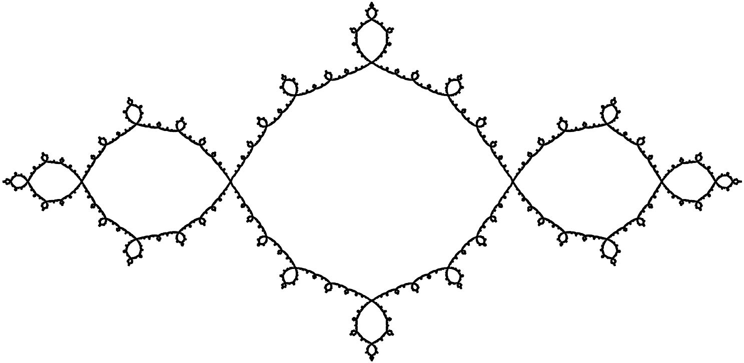
4. If $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ with each E_n an n-cell, then $\bigcap_{n=0} E_n$ is a single point.

Example: The Basilica

The basilica Julia set can be viewed as a finitely ramified fractal.



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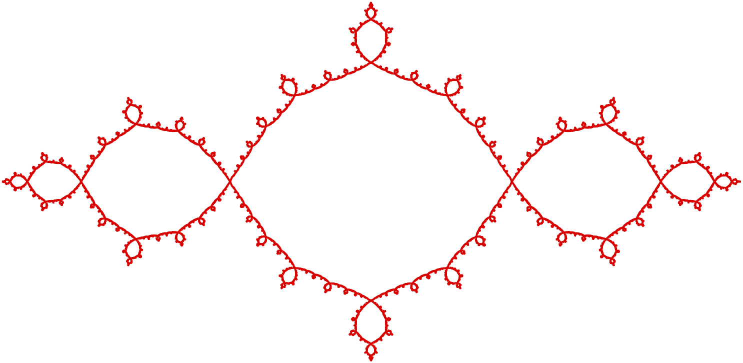

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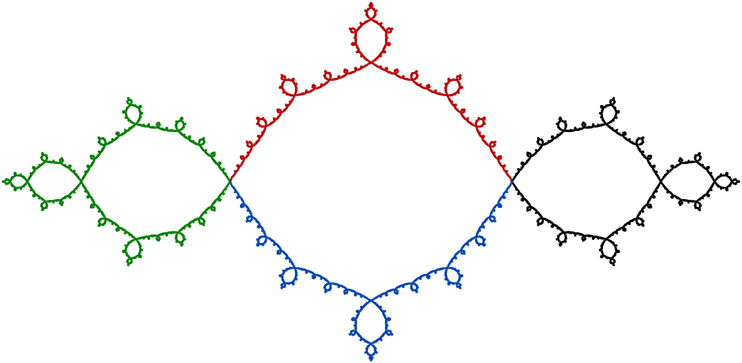

Four 1-cells

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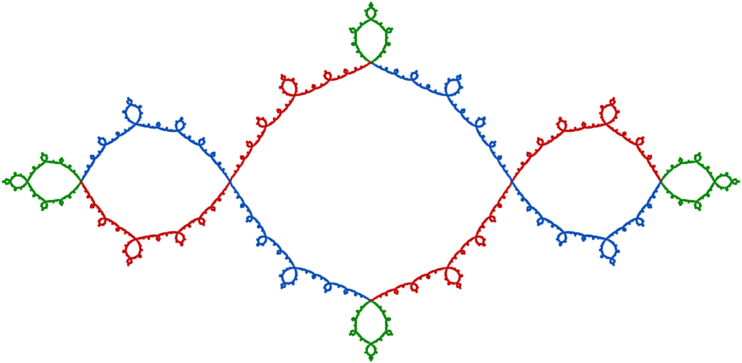

Twelve 2-cells

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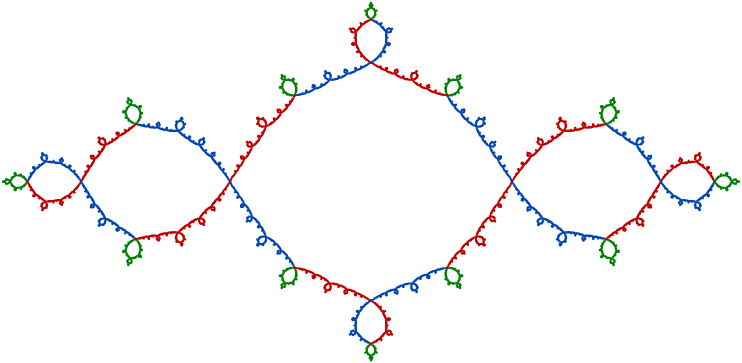
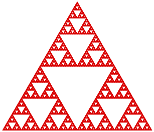


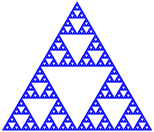
Thirty-six 3-cells

Cellular Maps

Let X be a finitely ramified fractal, and let E, E' be cells in X.



$\longrightarrow$



E (n-cell) E' (n'-cell)

A homeomorphism $E \longrightarrow E'$ is ***cellular*** if it maps $(n+k)$-cells in E to $(n'+k)$-cells in E' for all $k \geq 0$.

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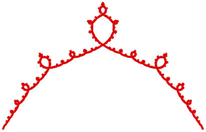

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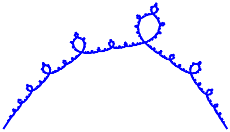

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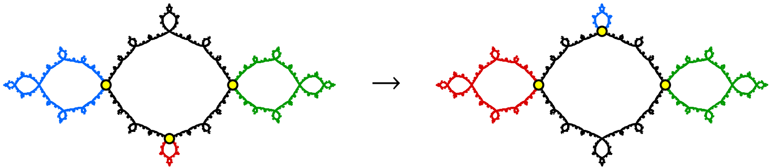
Piecewise-Cellular Maps

A homeomorphism $f : X \to X$ is ***piecewise-cellular*** if there exist subdivisions

$$\{E_1, \ldots, E_n\} \qquad \text{and} \qquad \{E'_1, \ldots, E'_n\}$$

of X into cells so that f maps each E_i to E'_i by a cellular map.



Note: The piecewise-cellular homeomorphisms of X form a group.

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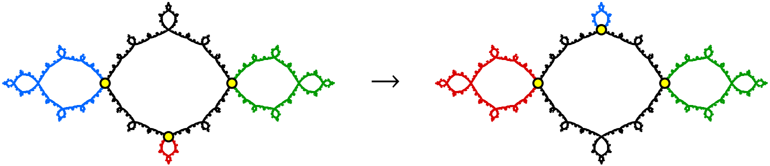

Question: When are piecewise-cellular homeomorphisms quasisymmetries?

Main Results

Main Results

A metric on a finitely ramified fractal X is ***quasiregular*** if:

1. It satisfies the exponential decay condition.
2. It has bounded neighbor ratios, and
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Theorem (B–Forrest 2021)

Any two quasiregular metrics on X are quasisymmetrically equivalent.

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Exponential Decay Condition:

There exist constants $0 < r < R < 1$ and $C \geq 1$ so that

$$\frac{r^k}{C} \leq \frac{\mathrm{diam}(E')}{\mathrm{diam}(E)} \leq CR^k$$

for any n-cell E and any $(n+k)$-cell E' contained in E.

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Bounded Neighbor Ratios:

There exists a constant $\lambda \geq 1$ so that

$$\frac{1}{\lambda} \leq \frac{\mathrm{diam}(E')}{\mathrm{diam}(E)} \leq \lambda$$

for any two n-cells E and E' that intersect.

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There exists a constant $\delta > 0$ so that

$$d(E, E') \geq \delta \, \mathrm{diam}(E)$$

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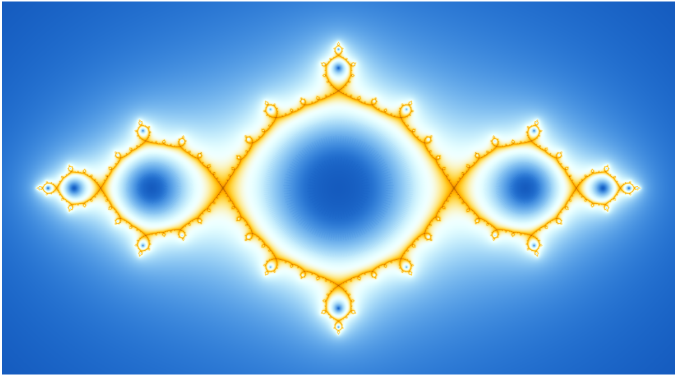
Julia sets for polynomials tend to be finitely ramified.



Julia set for $f(z) = z^2 - 1$

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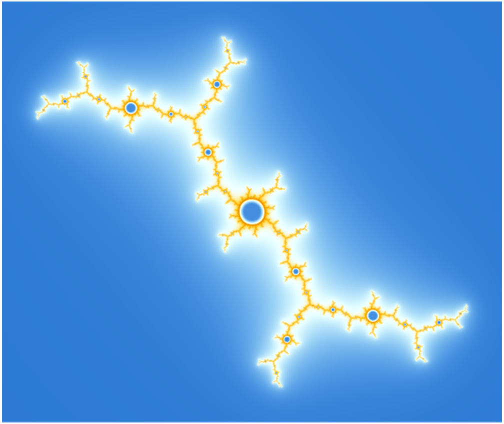

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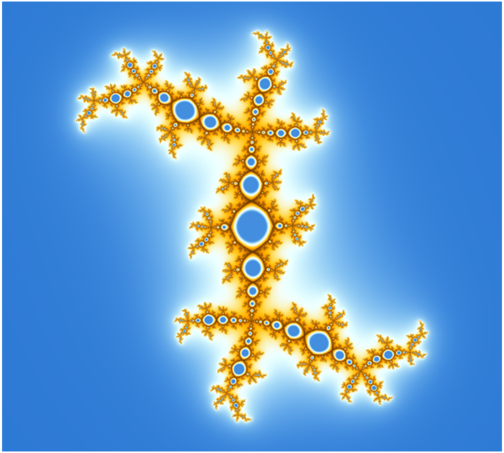

Julia set for $f(z) = z^2 + 0.32 + 0.56i$

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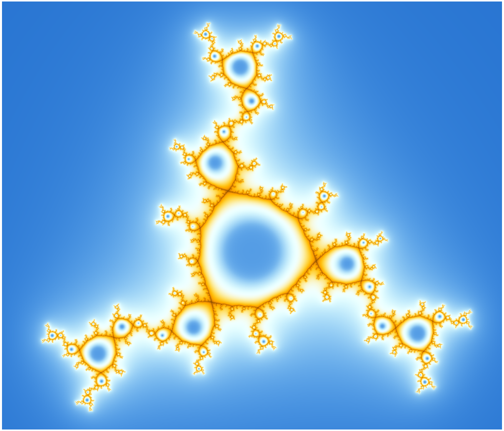

Julia set for $f(z) = z^3 - 0.21 + 1.09i$

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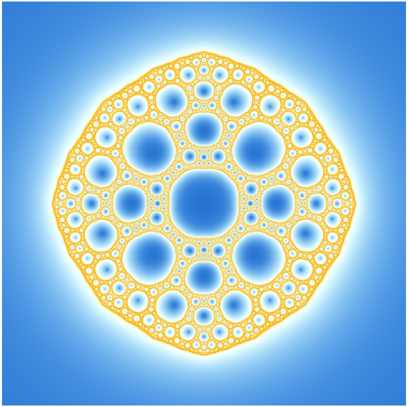

Julia set for $f(z) = z^2 - \dfrac{1}{16z^2}$

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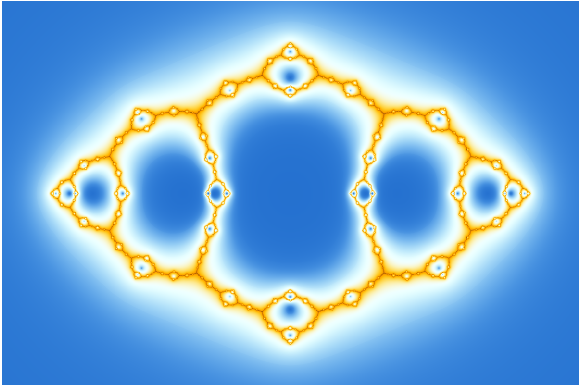

Julia set for $f(z) = \dfrac{1}{z^2} - 1$

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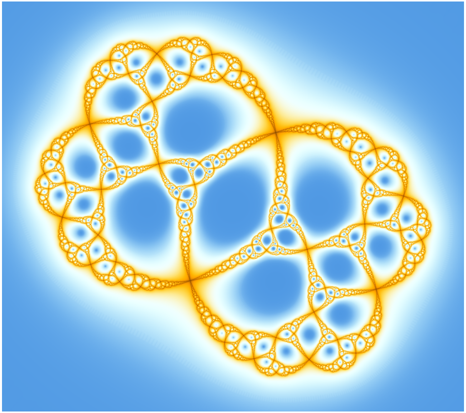

Julia set for $f(z) = \dfrac{e^{2\pi i/3}z^2 - 1}{z^2 - 1}$

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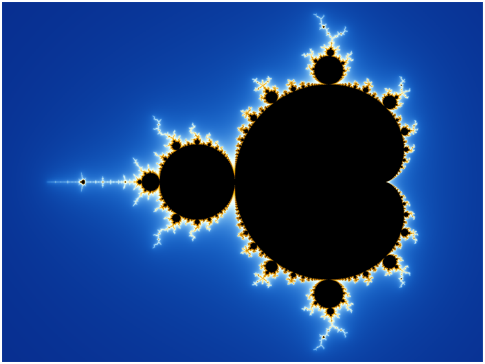

Main Results

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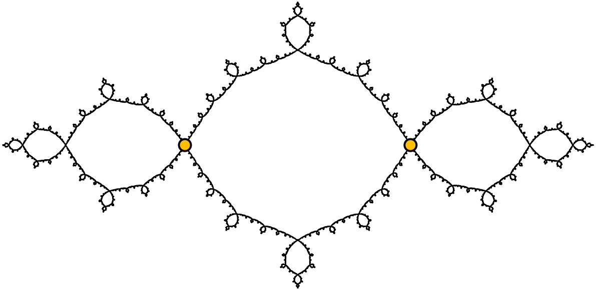
Suppose there exists a finite set $S \subset J_f$ such that $f(S) \subseteq S$ and f is one-to-one on each component of $J_f \setminus S$.



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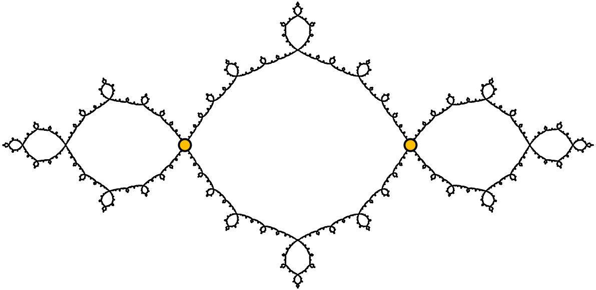

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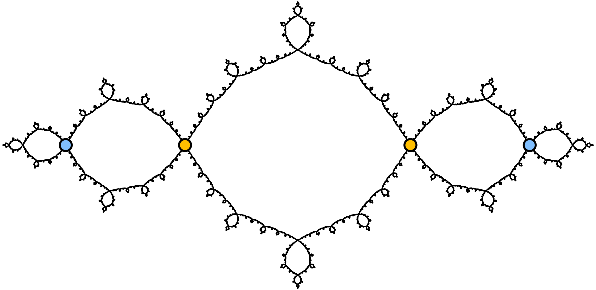

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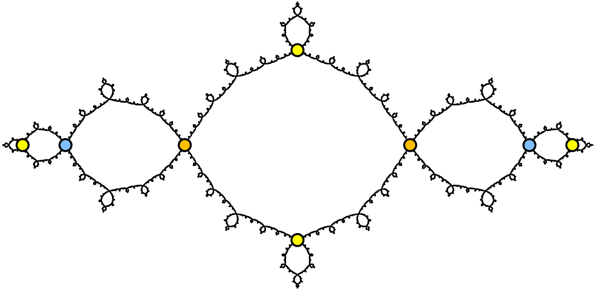

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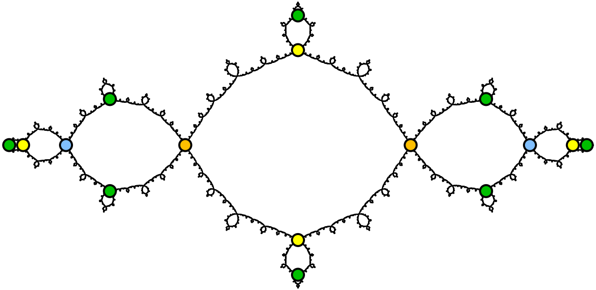

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In particular, piecewise-cellular homeomorphisms are quasisymmetries.

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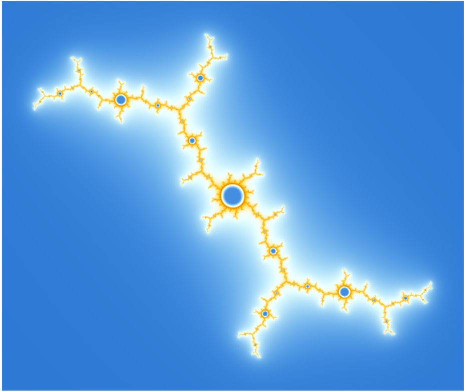

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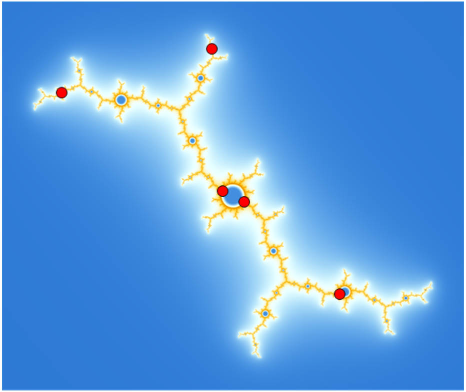

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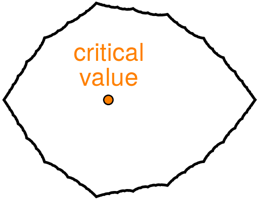

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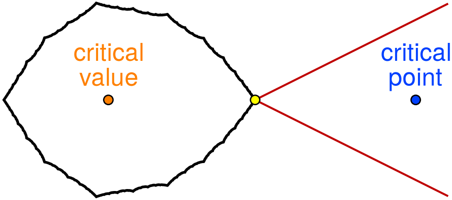

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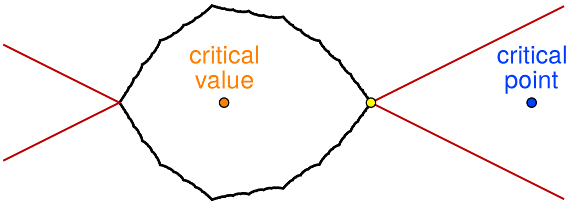

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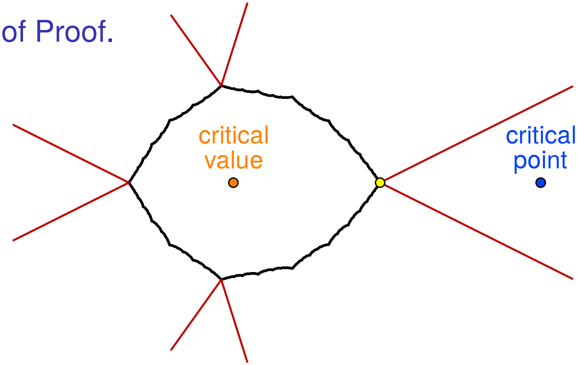

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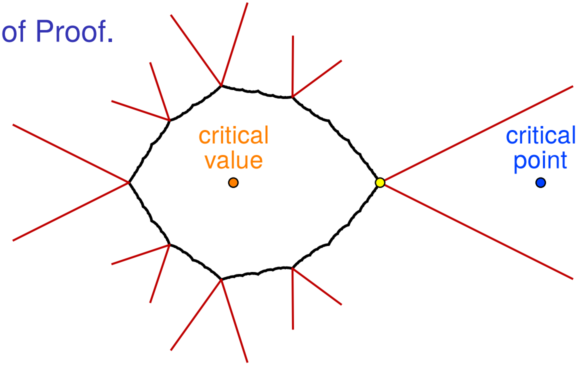

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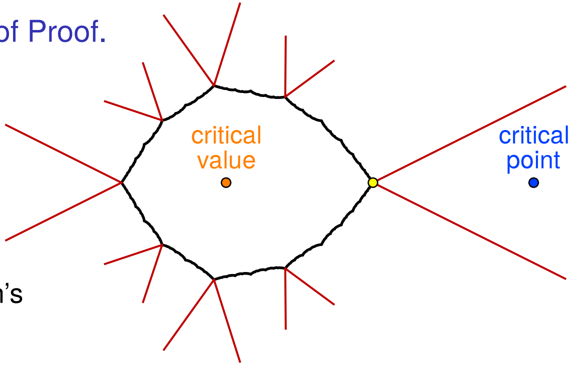

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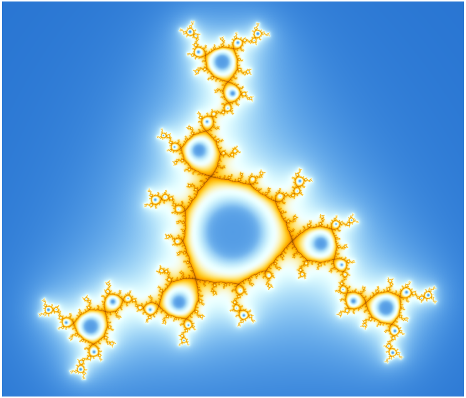

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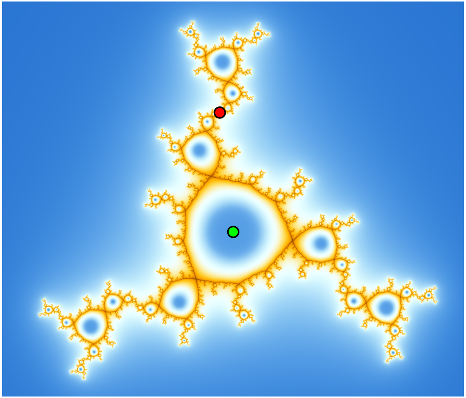

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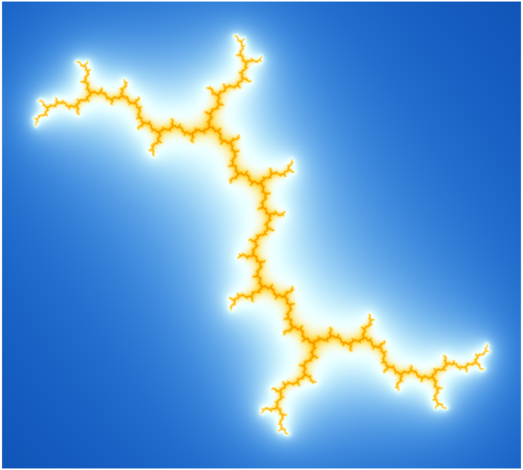
Open Questions

Can this theory be extended to the subhyperbolic case?



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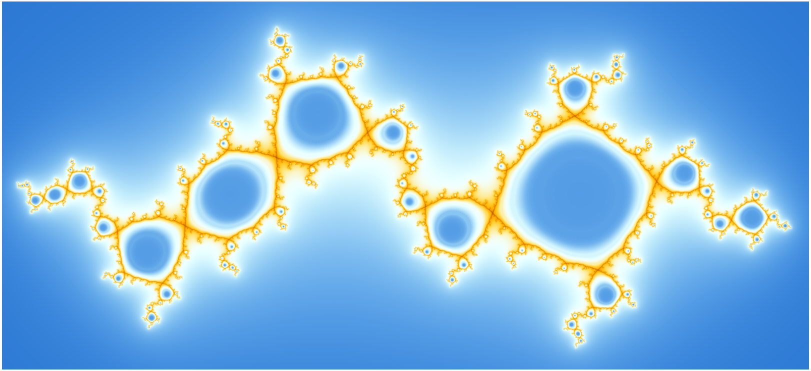
What about other hyperbolic cubic polynomials?



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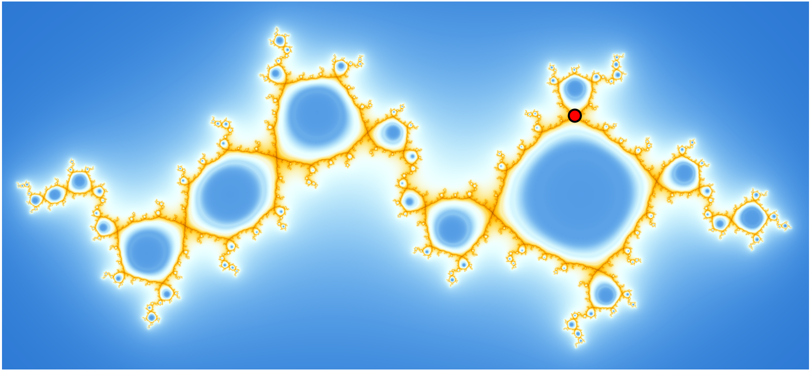

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The End