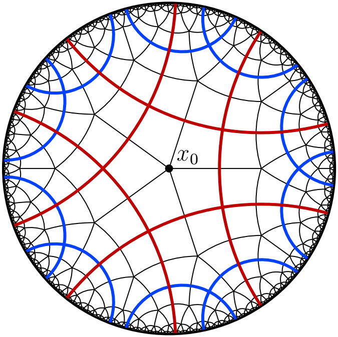
Rational Actions of Hyperbolic Groups



Jim Belk, University of St Andrews

joint with Collin Bleak and Francesco Matucci

Homeomorphisms of the Cantor Set

The ***Cantor set*** C is the space $\{0, 1\}^\omega$ of all infinite binary sequences.

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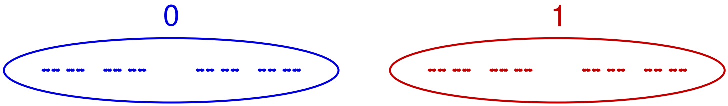

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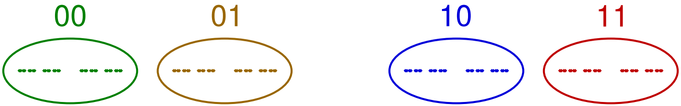

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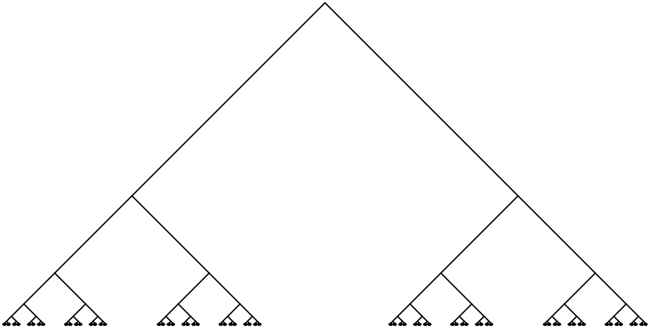

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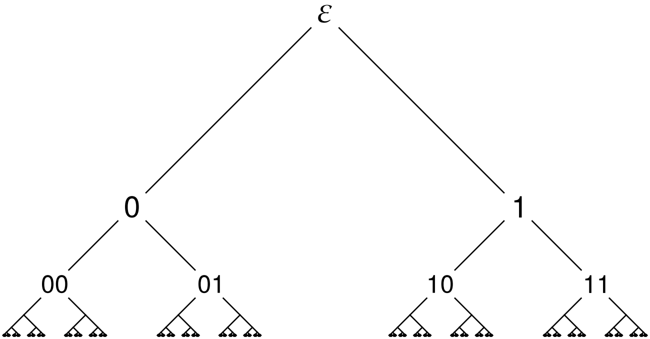

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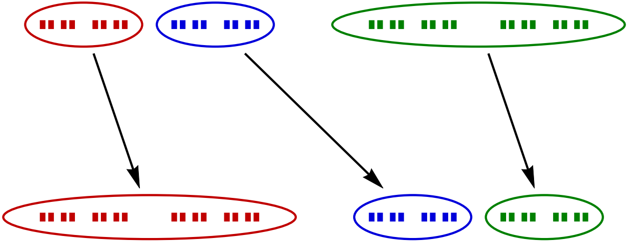
Theorem (Anderson 1958)

The full group $\mathrm{Homeo}(C)$ *of homeomorphisms of C is an uncountable simple group.*

$\mathrm{Homeo}(C)$ has many interesting subgroups.

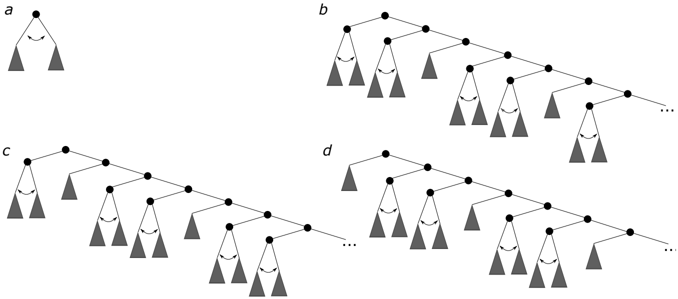
Homeomorphisms of the Cantor Set

Thompson's groups *F*, *T*, and *V*



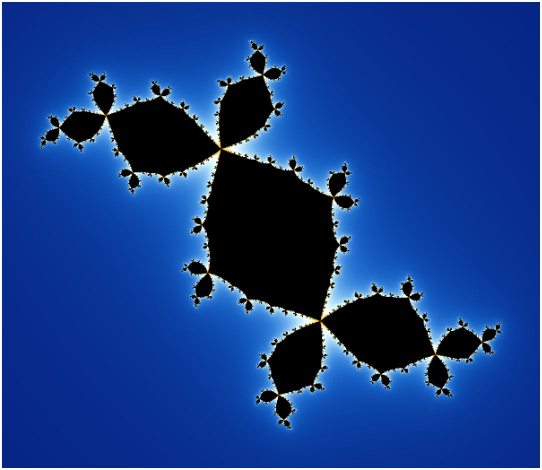
Homeomorphisms of the Cantor Set

Grigorchuk's group



Homeomorphisms of the Cantor Set

Iterated monodromy groups

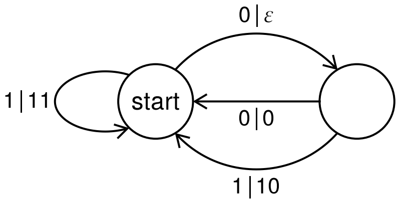


Homeomorphisms of the Cantor Set

Definition (Grigorchuk, Nekrashevych, Sushchanskiĭ)

A homeomorphism of C is ***rational*** if it can be defined by a finite-state automaton.

The group of all such homeomorphisms is the ***rational group*** $\mathcal{R}$.



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Theorem (GNS 2000)

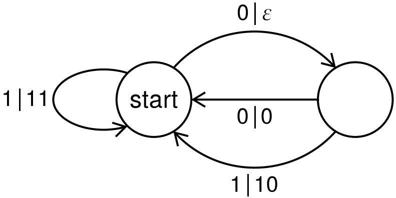
Subgroups of $\mathcal{R}$ include:

1. *Thompson's groups F, T, and V.*
2. *Grigorchuk's group and other self-similar groups.*
3. *The automorphism group of a full shift (one or two-sided) over a finite alphabet.*

Automata

Automata

A ***finite-state automaton*** is a machine for processing binary strings.



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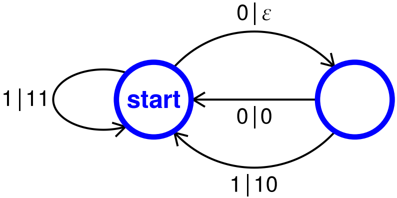

It has finitely many ***states***, one of which is the ***start state***.

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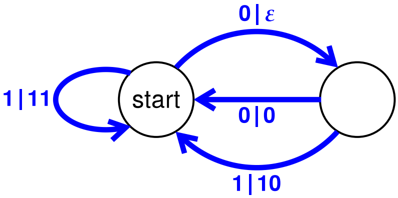

There are ***transitions*** between the states:

$$\xrightarrow{\;p\;|\;q\;} \quad \text{input } p \text{ and output } q.$$

The ***input*** must be 0 or 1, but the ***output*** can be any binary string.

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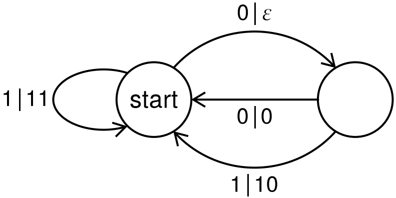

Input String:

1 1 0 1 0 0 1 0 0 0 $\cdots$

Output String:

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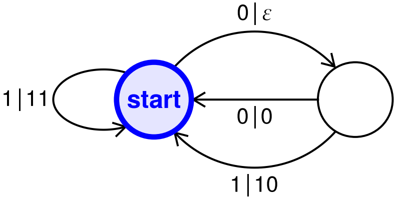

Input String:

$$\mid 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad \cdots$$

Output String:

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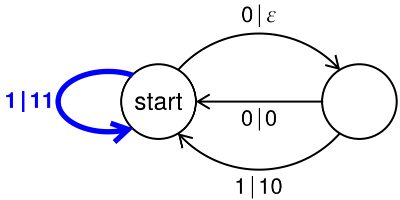

Input String:

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Output String:

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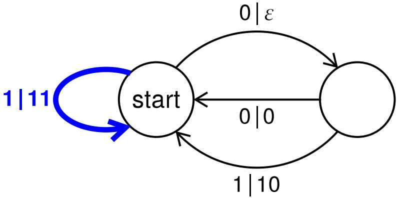

Input String:

1 | 1 0 1 0 0 1 0 0 0 $\cdots$

Output String:

1 1

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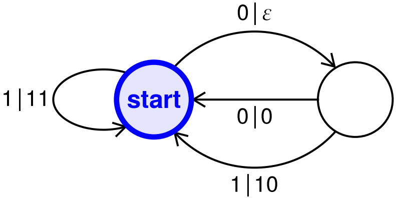

Input String:

1 | 1 0 1 0 0 1 0 0 0 ⋯

Output String:

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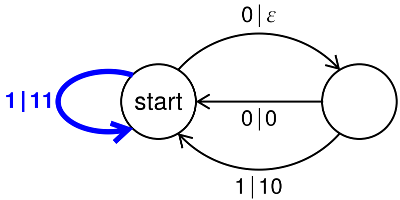

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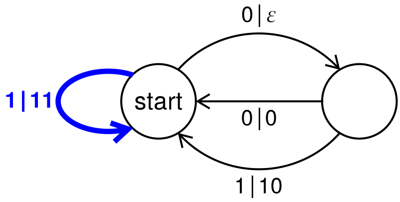

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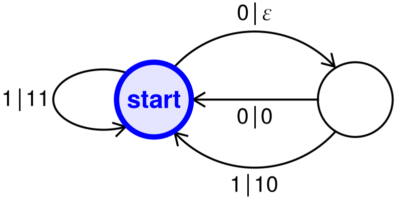

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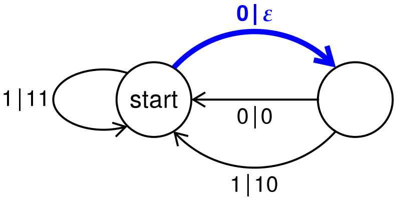

Input String:

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Output String:

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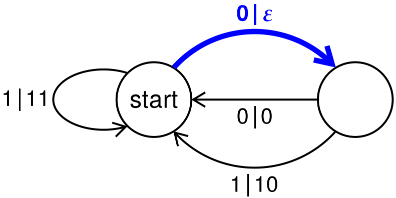

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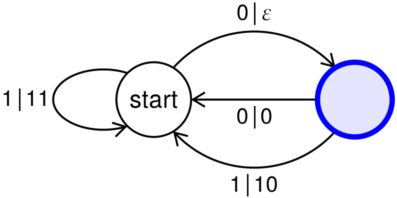

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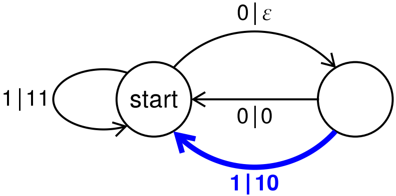

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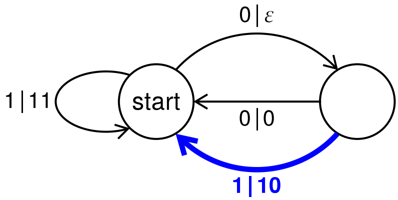

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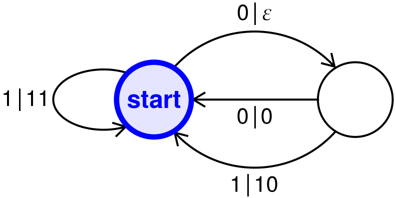

Input String:

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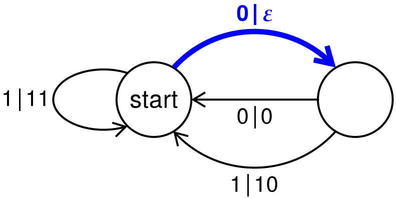

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Output String:

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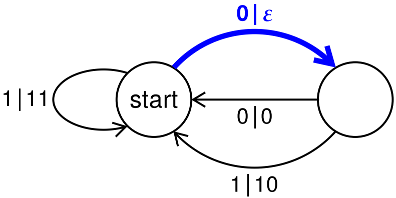

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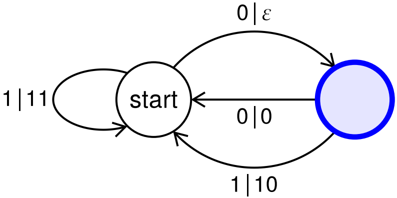

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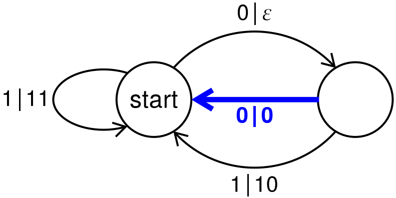

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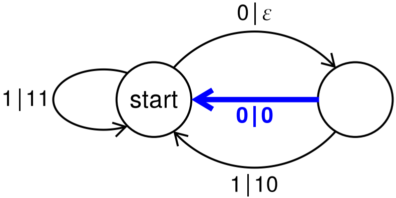

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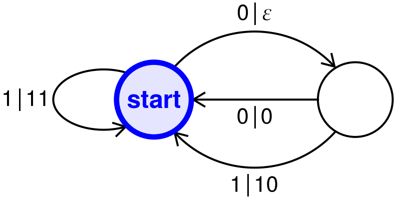

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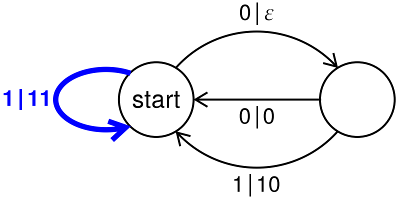

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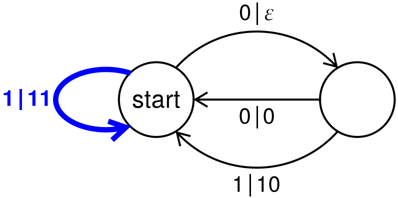

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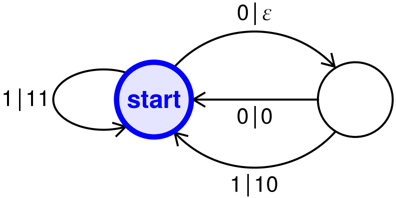

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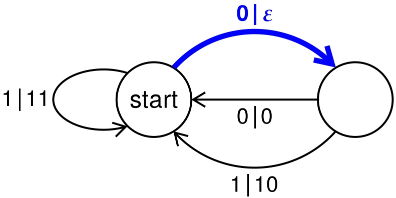

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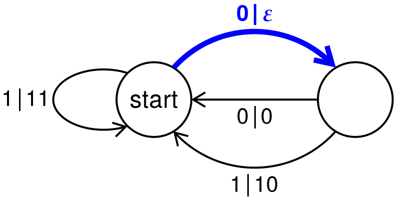

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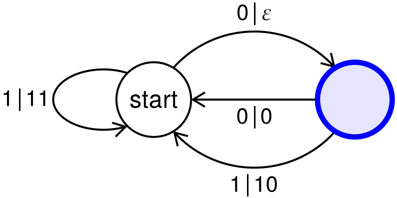

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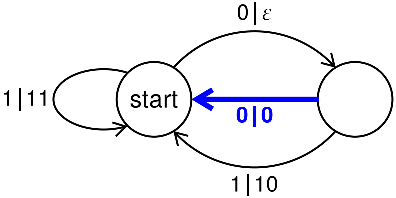

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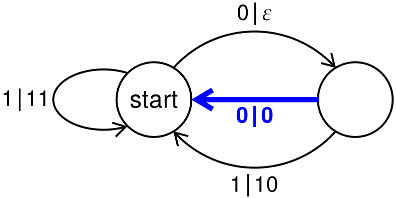

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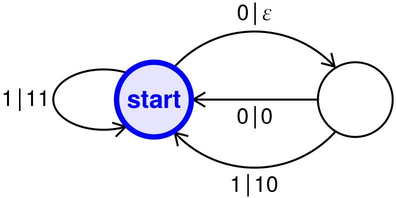

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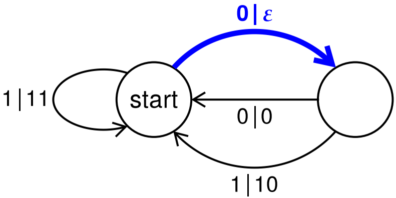

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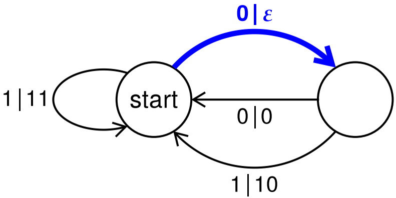

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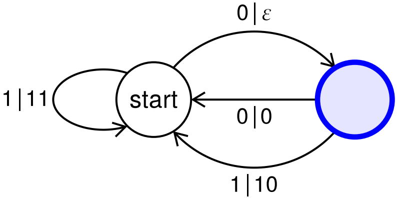

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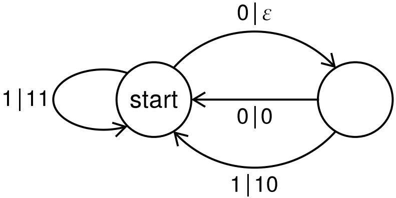

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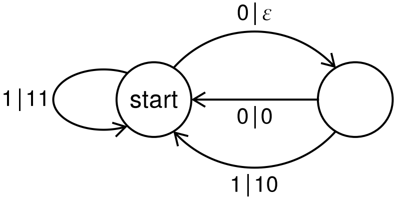


This defines a ***rational function***

$$f\colon \{0,1\}^\omega \longrightarrow \{0,1\}^\omega$$

Such a function is a homeomorphism as long as it is bijective.

The Rational Group

Definition (Grigorchuk, Nekrashevych, Sushchanskiĭ)

The group of all rational homeomorphisms of $\{0,1\}^\omega$ is the ***rational group*** $\mathcal{R}$.

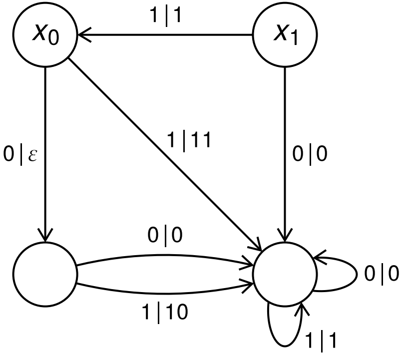
Theorem (B, Hyde, Matucci 2017)

- *$\mathcal{R}$ is simple.*
- *$\mathcal{R}$ is not finitely generated.*

An ***automata group*** is any finitely generated subgroup of $\mathcal{R}$.

Example: Thompson's Group F

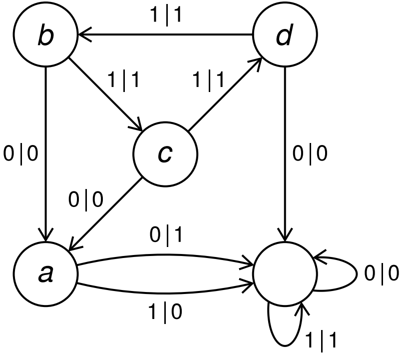
The following automaton defines two rational homeomorphisms x_0 and x_1:



The group $\langle x_0, x_1 \rangle$ is ***Thompson's group F***.

Example: Grigorchuk's Group

The following automaton defines four rational homeomorphisms a, b, c, and d:

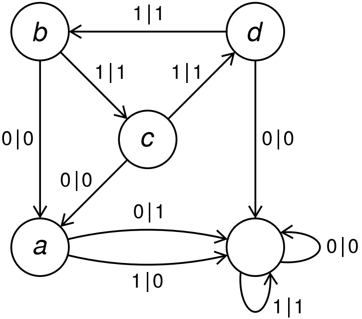


The group $\langle a, b, c, d \rangle$ is ***Grigorchuk's group***.

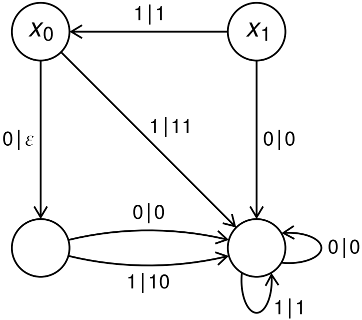
Two Notes

1. Synchronous vs. Asynchronous

Grigorchuk's group is a ***synchronous*** automata group, but Thompson's group F is ***asynchronous***.



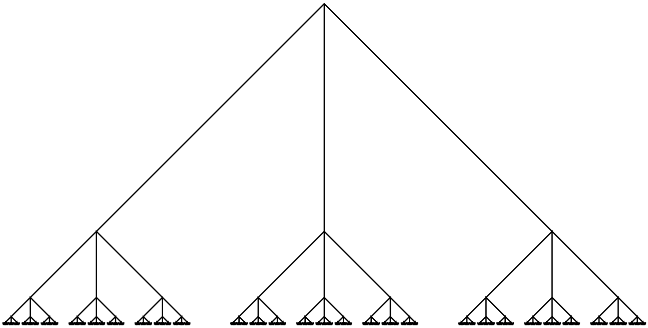
synchronous



asynchronous

2. Other Finite Alphabets

Grigorchuk, Nekrashevych, and Sushchanskiĭ also considered rational homeomorphisms with respect to larger alphabets.



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Example

For $n = 3$, define a homeomorphism $h\colon \{1, 2, 3\}^\omega \to \{0, 1\}^\omega$ by

$$1 \mapsto 00, \qquad 2 \mapsto 01, \qquad 3 \mapsto 1.$$

Then conjugation by h is an isomorphism from $\mathcal{R}_3$ to $\mathcal{R}$.

Rational Embeddings

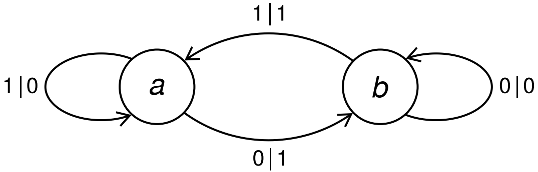
Rational Embeddings

Question

Which finitely generated groups G can be realized as automata groups?

Example (The Lamplighter Group)

Grigorchuk and Zuk (2000) realized the group $\mathbb{Z}_2 \wr \mathbb{Z}$ using synchronous automata:



Here $\langle a, b \rangle \cong \mathbb{Z}_2 \wr \mathbb{Z}$.

Rational Embeddings

Theorem (Brunner and Sidki 1998)

$\mathrm{GL}(n, \mathbb{Z})$ *embeds into* $\mathcal{R}$ *for all* $n \geq 1$.

Theorem (Silva and Steinberg 2005)

The generalized lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ *embed into* $\mathcal{R}$.

Theorem (Bartholdi and Šunić 2006)

The Baumslag-Solitar groups $\mathrm{BS}(1, n)$ *embed into* $\mathcal{R}$.

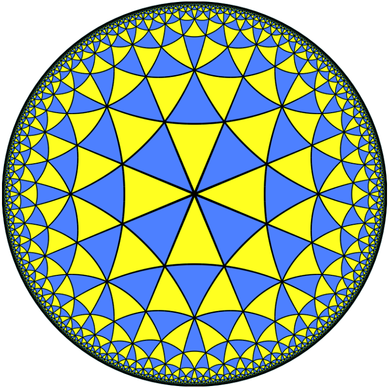
Theorem (B, Bleak 2014)

The higher-dimensional Thompson groups nV *embed into* $\mathcal{R}$.

Rational Embeddings

Main Theorem (B, Bleak, and Matucci 2018)

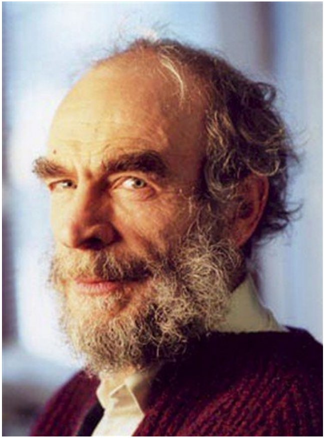
All hyperbolic groups embed into $\mathcal{R}$.



Hyperbolic Groups

Hyperbolic Groups

In the 1980's, Mikhail Gromov transformed the study of infinite discrete groups using ideas from Riemannian geometry.



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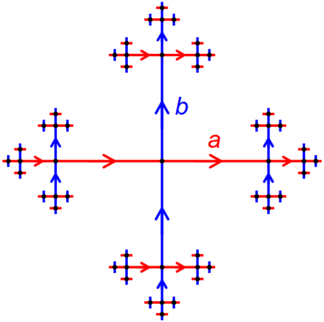
A group G is viewed geometrically as a ***Cayley graph***. Shortest-length paths in the graph are ***geodesics***.



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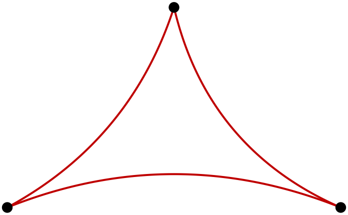
A group G is viewed geometrically as a ***Cayley graph***. Shortest-length paths in the graph are ***geodesics***.

Certain "large-scale" properties of manifolds also make sense for graphs (and hence groups).

Hyperbolic Groups

Let Γ be a locally finite graph, and let $\delta > 0$.

Gromov's δ-thin triangles condition:



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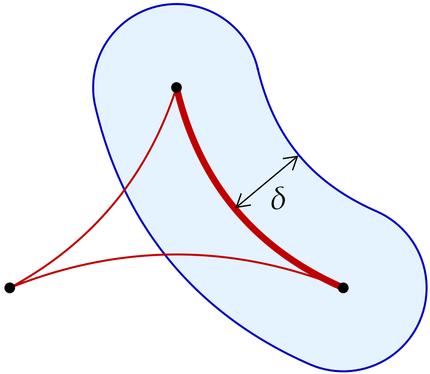

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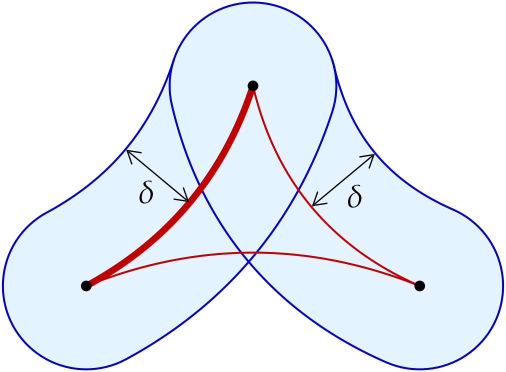

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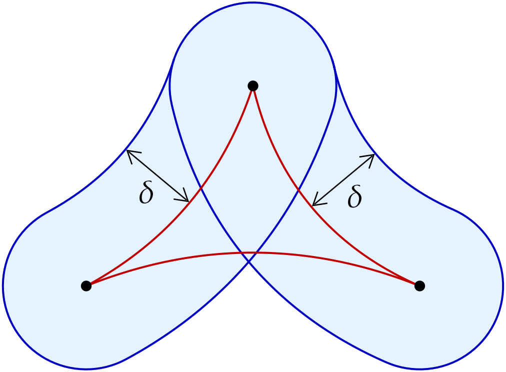

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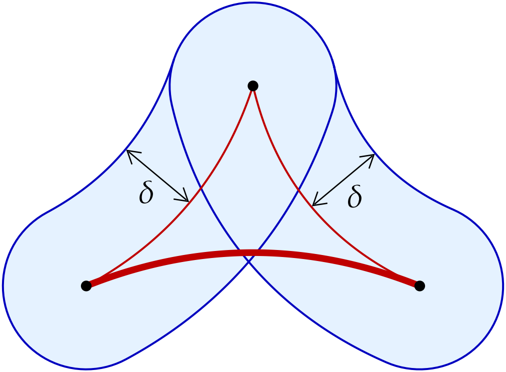

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Let Γ be a locally finite graph, and let $\delta > 0$.

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Definition (Gromov)

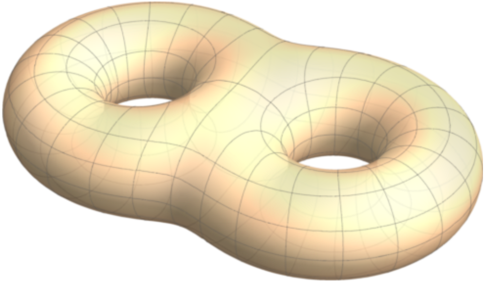
A finitely-generated group G is ***hyperbolic*** if its Cayley graph is hyperbolic.

Note: This does not depend on the generating set.

Hyperbolic Groups

Examples

- Fundamental groups of negatively curved compact manifolds.



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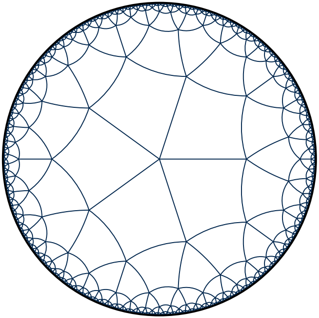
- Symmetry groups of hyperbolic tessellations.



Hyperbolic Groups

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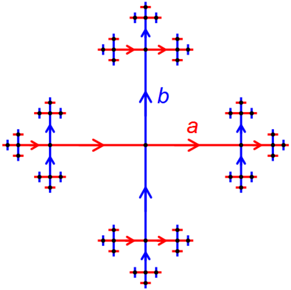
- Symmetry groups of hyperbolic tessellations.
- Free groups, free products of finite groups, etc.



Hyperbolic Groups

Examples

- Fundamental groups of negatively curved compact manifolds.
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- Free groups, free products of finite groups, etc.

Principle (Gromov)

“Almost all” finitely presented groups are hyperbolic.

Hyperbolic Boundaries

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Every hyperbolic group G has a ***boundary at infinity*** ∂G.

Properties:

1. ∂G is compact and metrizable.
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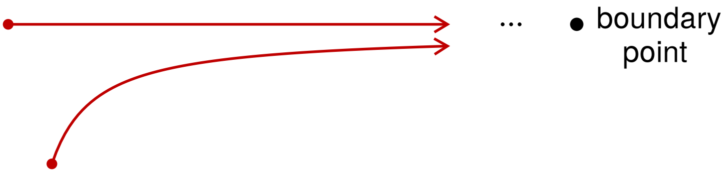

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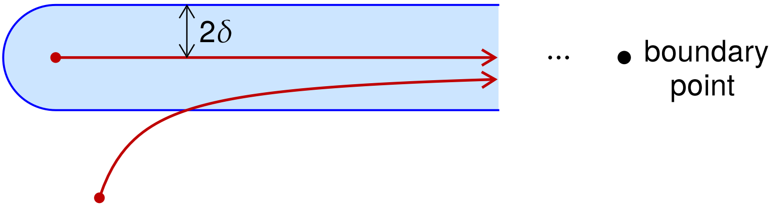
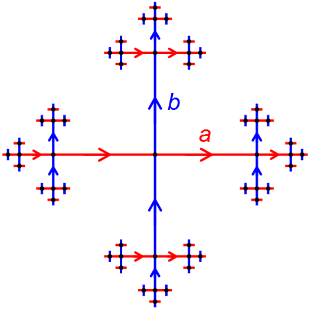


Hyperbolic Boundaries

Examples

- If G is a free group, then ∂G is its Cantor set of leaves.



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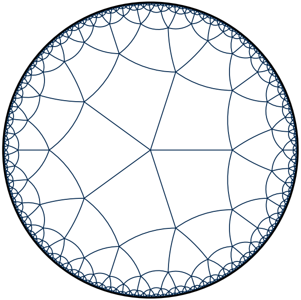
- If G is a free group, then ∂G is its Cantor set of leaves.
- If G is the symmetry group of a hyperbolic tessellation, then ∂G is a circle.



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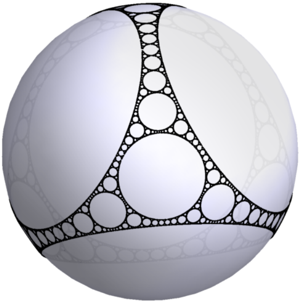
- If G is the symmetry group of a hyperbolic tessellation, then ∂G is a circle.
- For a "typical" hyperbolic group, the boundary is a fractal.



Rational Actions

Rational Actions

Let X be a compact metrizable space.

Definition (Binary Addresses)

A ***system of binary addresses*** for X is a quotient map

$$q\colon \{0,1\}^\omega \to X.$$

For example, the usual binary number system defines a quotient map

$$q\colon \{0,1\}^\omega \to [0,1].$$

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Theorem (Well-Known)

Every compact metrizable space has a system of binary addresses.

Rational Actions

Let G be a group acting by homeomorphisms on a compact metrizable space X.

Definition (Rational Action)

The action of G on X is ***rational*** if there exists a quotient map $q\colon \{0,1\}^\omega \to X$ and a homomorphism $\varphi\colon G \to \mathcal{R}$ such that the diagram

$$\begin{tikzcd}
\{0,1\}^\omega \arrow[r, "\varphi(g)"] \arrow[d, two heads, "q"'] & \{0,1\}^\omega \arrow[d, two heads, "q"] \\
X \arrow[r, "g"'] & X
\end{tikzcd}$$

commutes for all $g \in G$.

Rational Actions

Observation: *A group G embeds into $\mathcal{R}$ if and only if G acts faithfully and rationally on some compact metrizable space.*

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Rational Actions

Main Theorem (B, Bleak, Matucci 2018)

Let G be a hyperbolic group. Then the action of G on ∂G is rational.

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Note: As long as the action of G on ∂G is faithful, it follows that G embeds into $\mathcal{R}$.

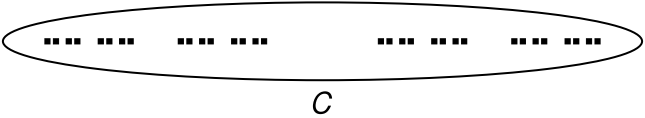
Note: Sometimes the action isn't faithful, but G always does act faithfully on $\partial(G * \mathbb{Z})$, and thus always embeds into $\mathcal{R}$.

Self-Similar Trees

Some Terminology

By definition, $\mathcal{R}$ acts on the binary Cantor set $C = \{0, 1\}^{\omega}$.

Each finite binary sequence α corresponds to a ***branch*** C_α of C.



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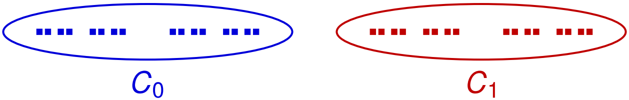

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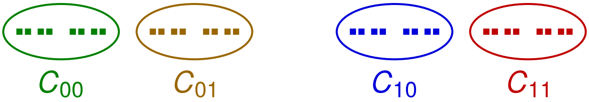

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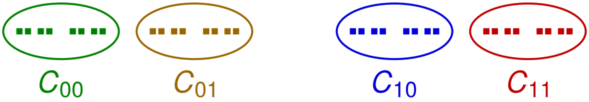
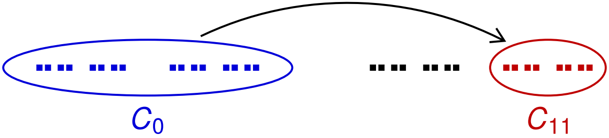


Any two branches of C have a ***canonical homeomorphism*** between them.



Geometric Characterization of $\mathcal{R}$

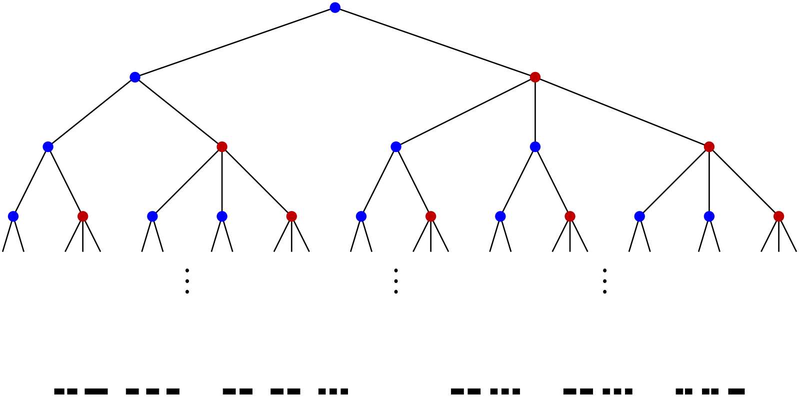
Theorem (Grigorchuk, Nekrashevych, Sushchanskiĭ)

A homeomorphism $F : C \longrightarrow C$ is rational if and only if it has finitely many local actions on the branches of C.

$$
\begin{tikzcd}
C_\alpha \arrow[r, hook, "F"] \arrow[d, "\text{canonical}"'] & C_{\alpha'} \arrow[d, "\text{canonical}"] \\
C_\beta \arrow[r, hook, "F"'] & C_{\beta'}
\end{tikzcd}
$$

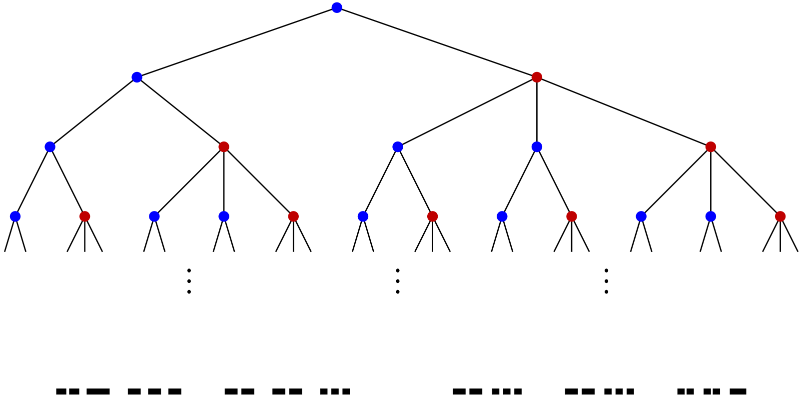
Self-Similar Trees

We will need to use a more general class of Cantor spaces.



Self-Similar Trees

A ***self-similar tree*** has finitely many ***types*** of vertices:



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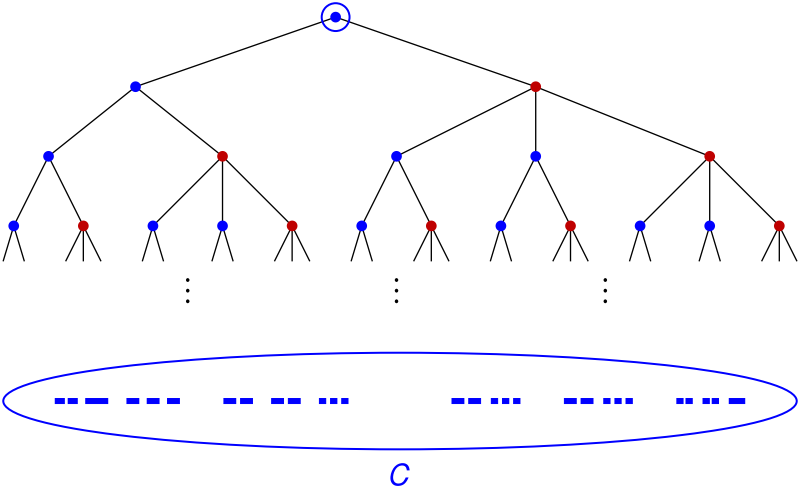

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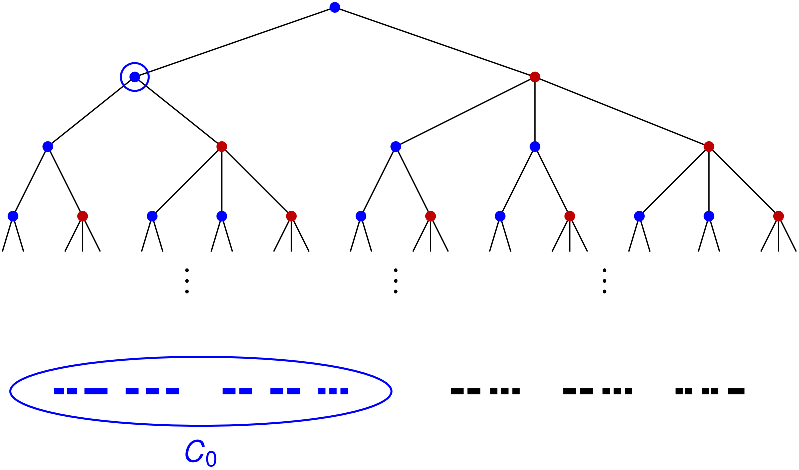

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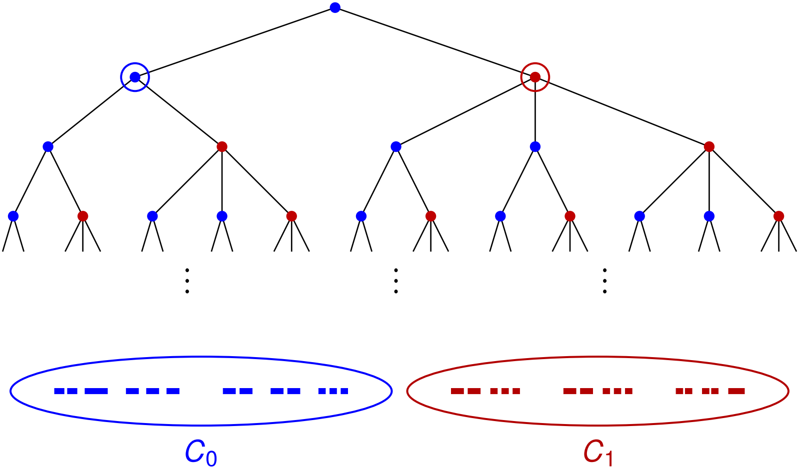

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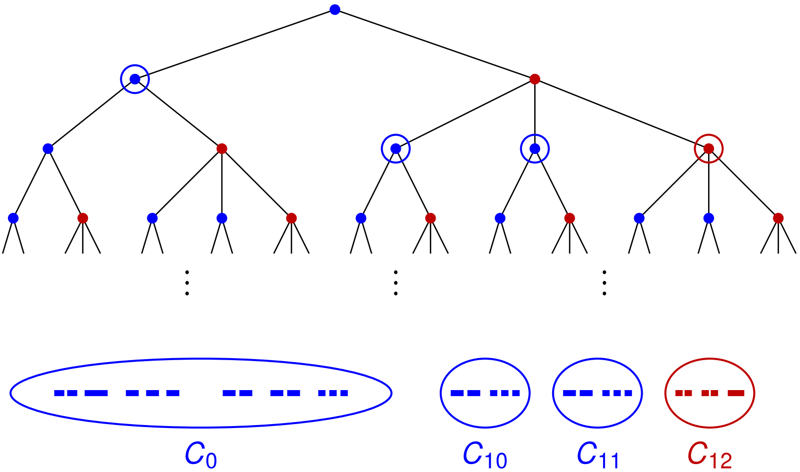

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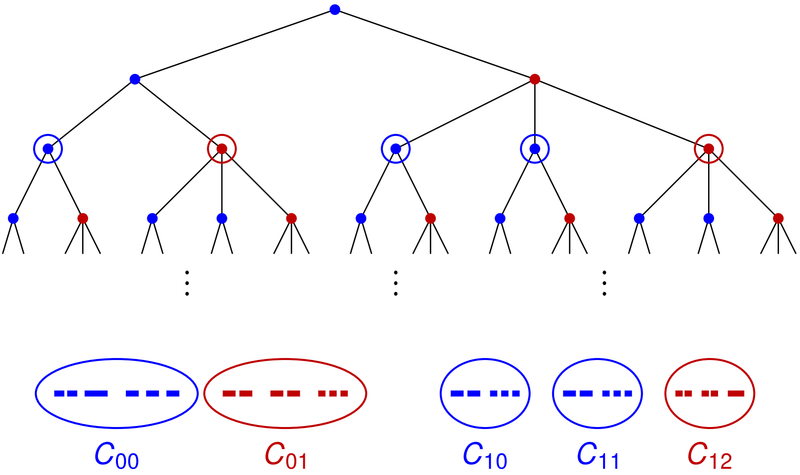
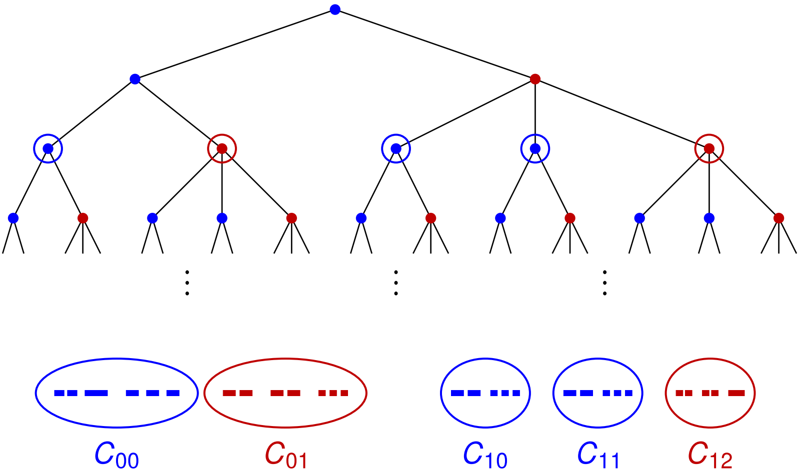


Self-Similar Trees

We also allow finite sets of canonical homeomorphisms.



Self-Similar Trees

Definition (Rational Homeomorphism)

Let C be the space of leaves of a self-similar tree. A homeomorphism $F\colon C \to C$ is ***rational*** if it has finitely many different local actions.

Here F has the ***same local action*** on C_1 and C_2 if there exist

$$C_1' \supseteq F(C_1) \qquad \text{and} \qquad C_2' \supseteq F(C_2)$$

and canonical homeomorphisms φ and ψ making the following diagram commute:

$$\begin{tikzcd} C_1 \arrow[r, "F"] \arrow[d, "\varphi"'] & C_1' \arrow[d, "\psi"] \\ C_2 \arrow[r, "F"'] & C_2' \end{tikzcd}$$

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Theorem (B, Bleak, Matucci)

As long as C has no isolated points, the group $\mathcal{R}_C$ of rational homeomorphisms of C is isomorphic to $\mathcal{R}$.

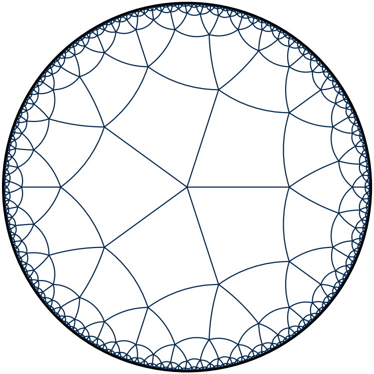
Indeed, $\mathcal{R}_C$ is conjugate to $\mathcal{R}$ by a homeomorphism $C \to \{0,1\}^\omega$.

The Tree of Atoms

The Tree of Atoms

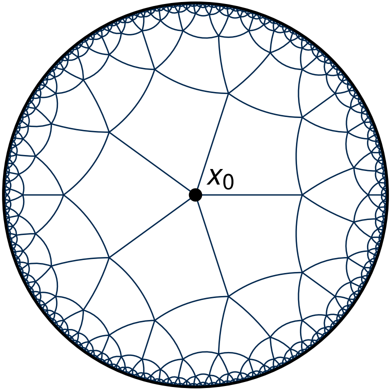
Let Γ be any hyperbolic graph (e.g. a Cayley graph).

We will construct a collection of subsets of Γ called “atoms”.



The Tree of Atoms

Fix a ***base vertex*** x_0 of Γ.



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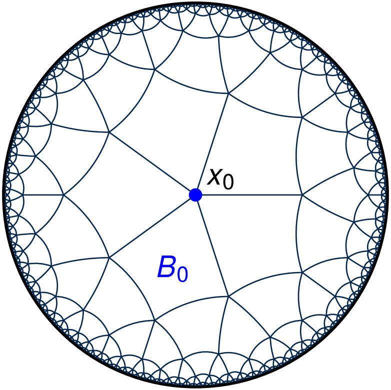
Fix a ***base vertex*** x_0 of Γ. Let B_n be the n-ball centered at x_0.



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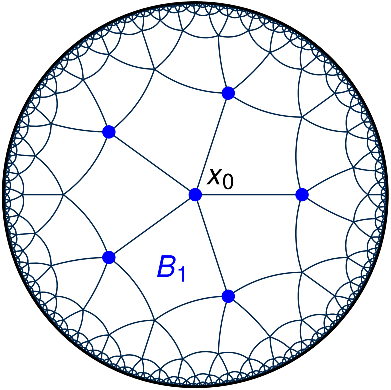

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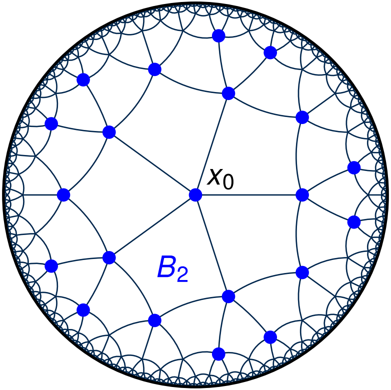

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For each vertex v of Γ, define $\bar{d}_v : \Gamma \longrightarrow \mathbb{R}$ by

$$\bar{d}_v(x) = d(x, v) - d(x_0, v)$$

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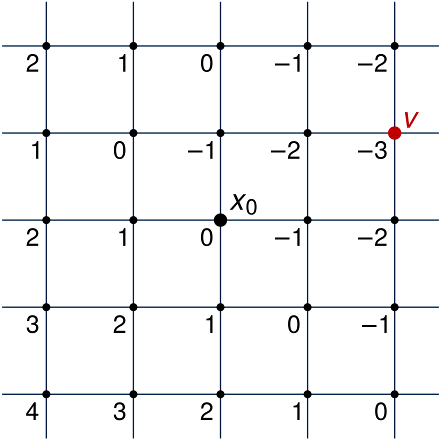

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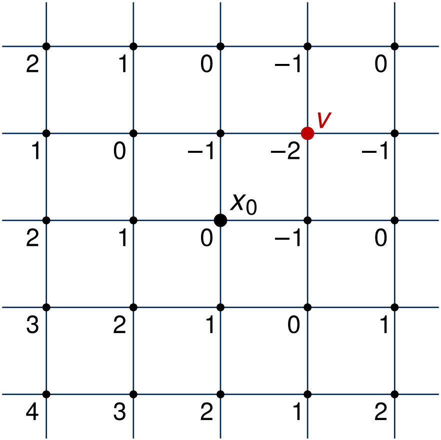

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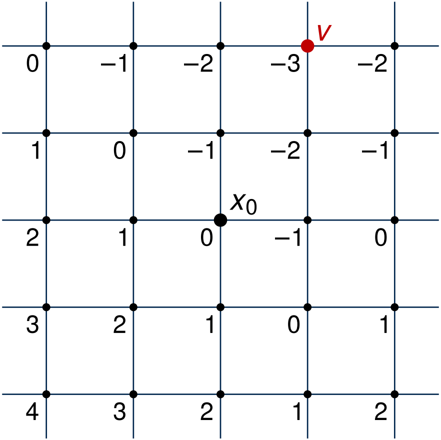

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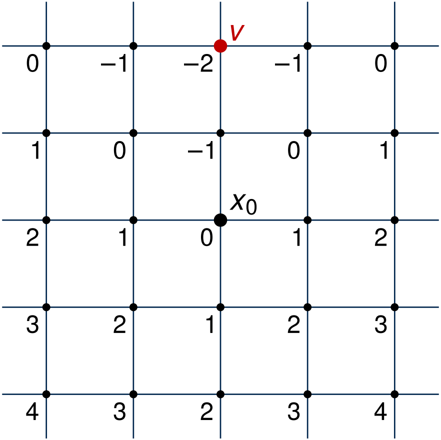

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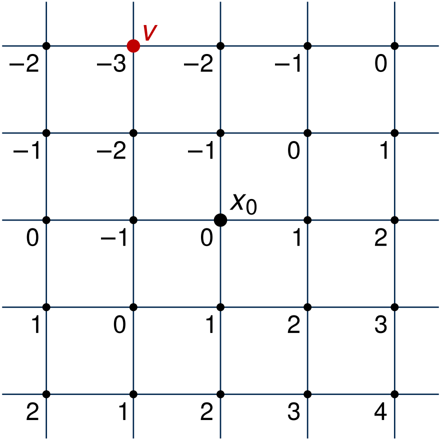

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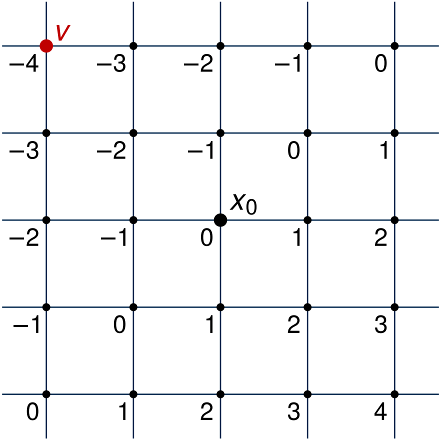

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Put $v \sim w$ if $\bar{d}_v$ and $\bar{d}_w$ agree on the n-ball B_n.

The equivalence classes are the ***level n atoms***.

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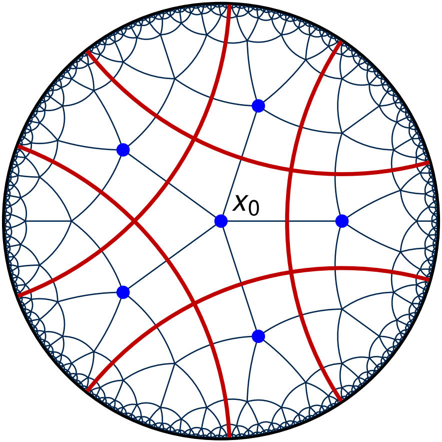

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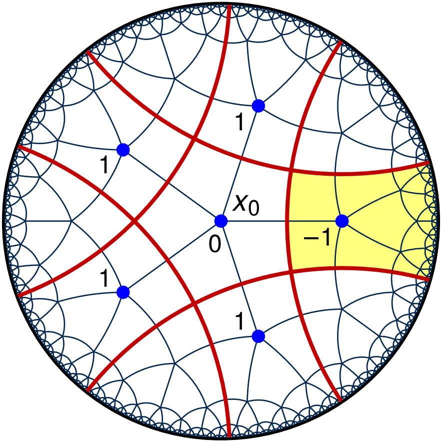

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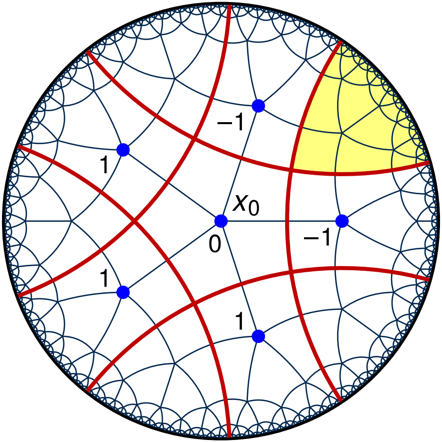

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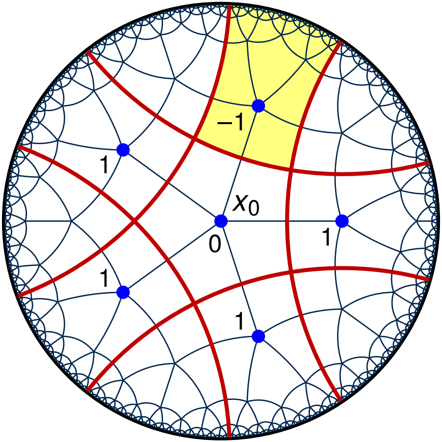

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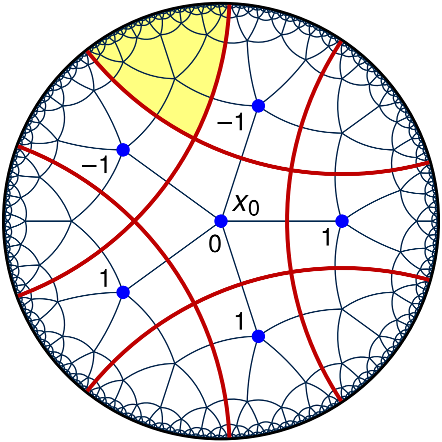

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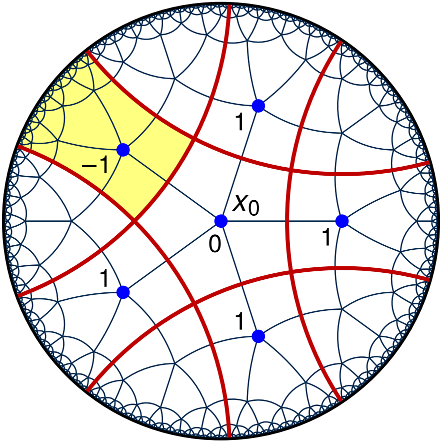

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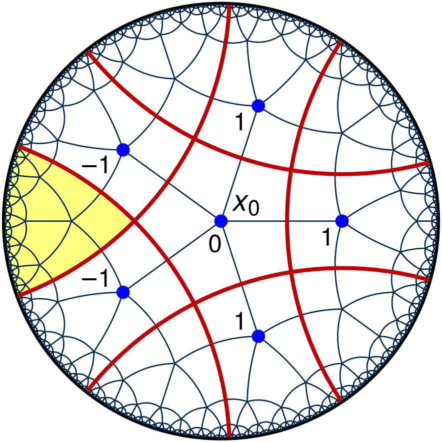

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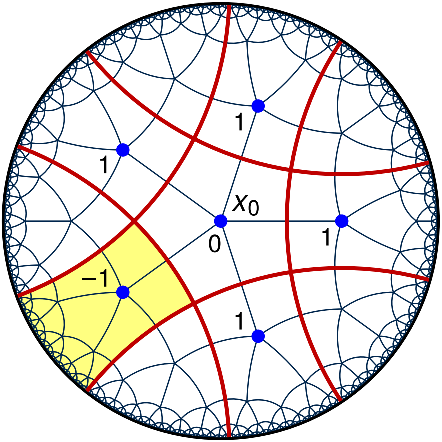

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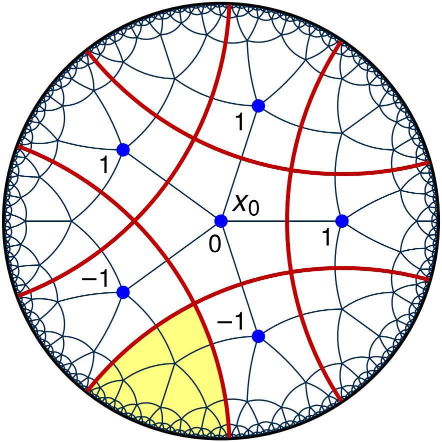

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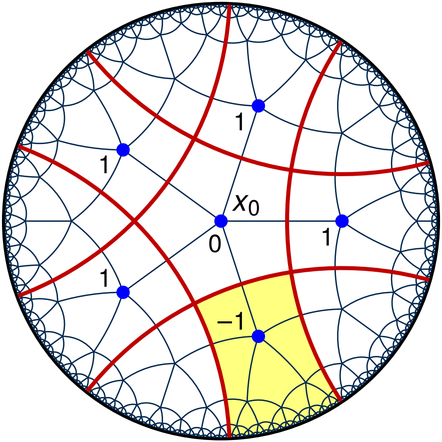

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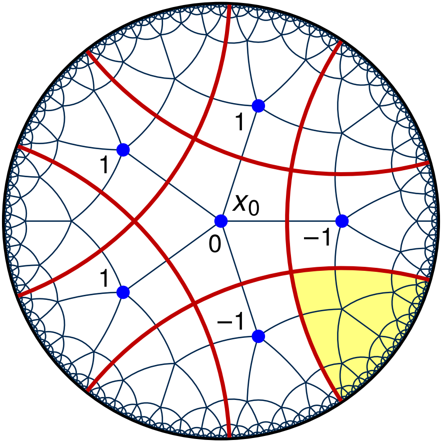

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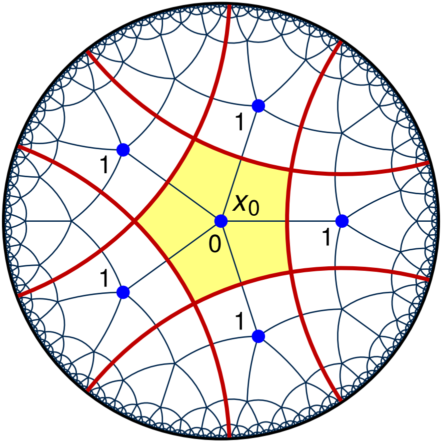
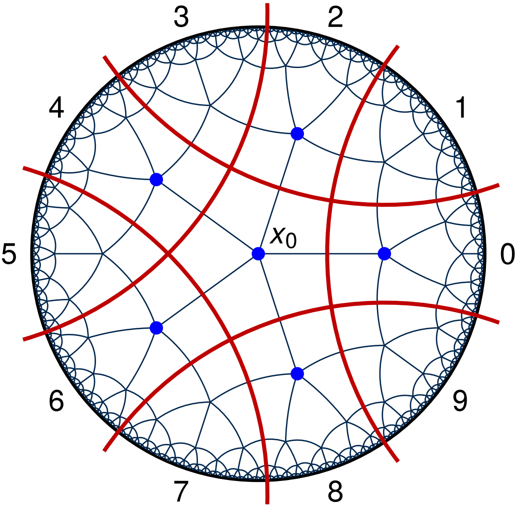


The Tree of Atoms

The infinite atoms at each level determine a subdivision of $\partial\Gamma$.



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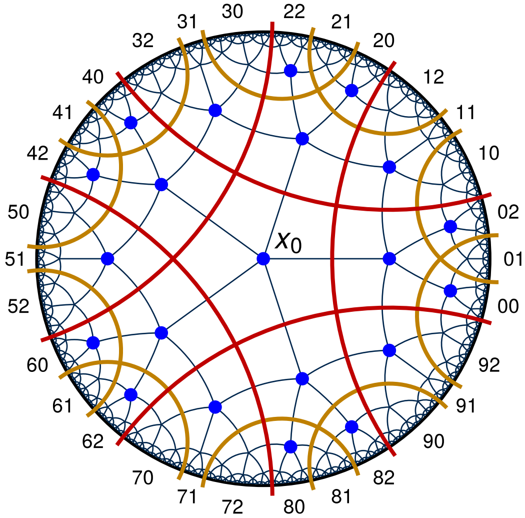


The Tree of Atoms

Let G be a hyperbolic group.

Theorem (B, Bleak, Matucci)

The collection of infinite atoms in G has the structure of a self-similar tree.

The Cantor space of leaves is the ***horofunction boundary*** $\partial_h G$ of G, which has ∂G as a quotient.

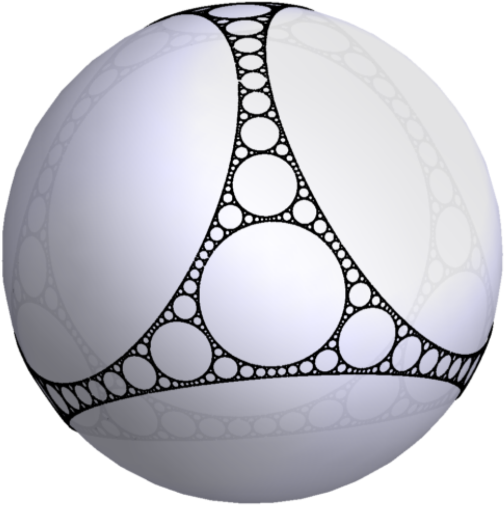
Theorem (B, Bleak, Matucci)

Elements of G act as rational homeomorphisms of $\partial_h G$.

The proofs make essential use of hyperbolicity.

Another Example

The group $\left\langle r_1, r_2, r_3, r_4 \;\middle|\; r_i^2 = 1, (r_i r_j)^6 = 1 \right\rangle$ has boundary homeomorphic to the Sierpiński carpet.



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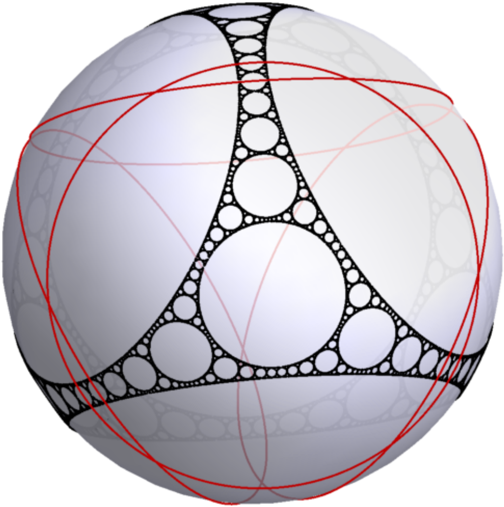

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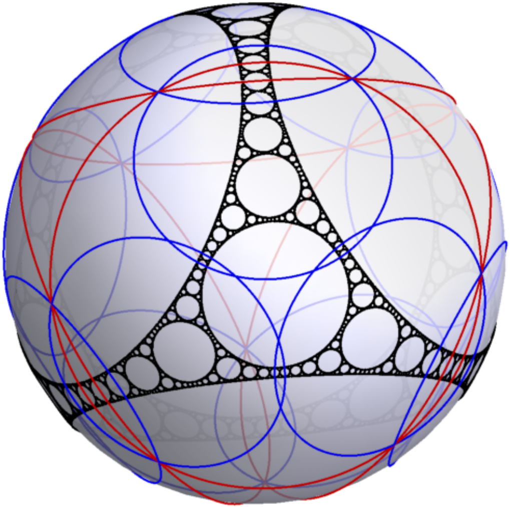

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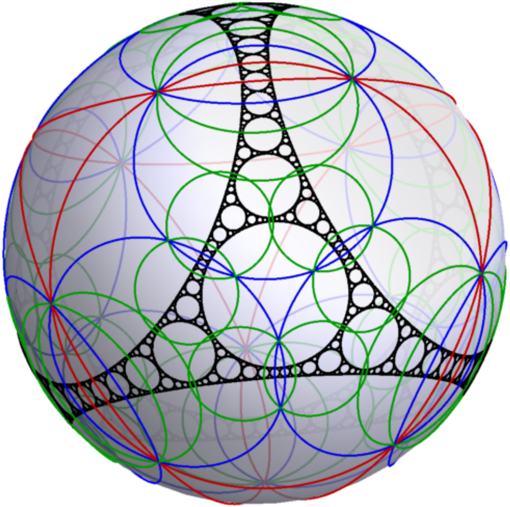

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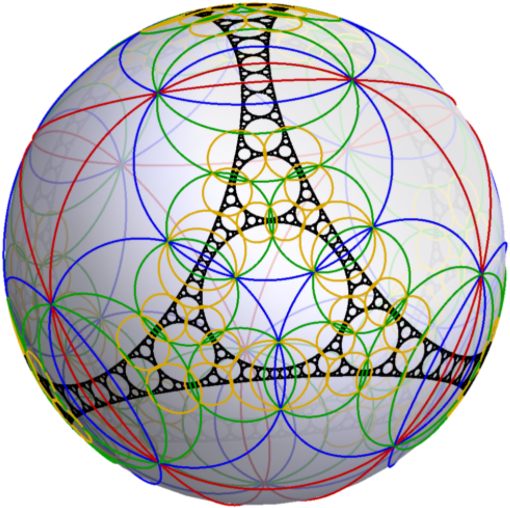

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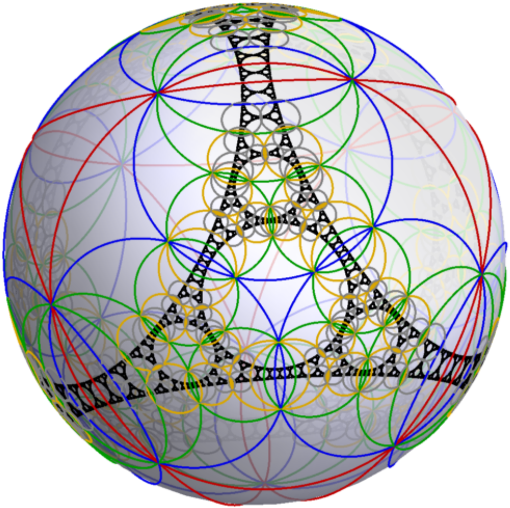

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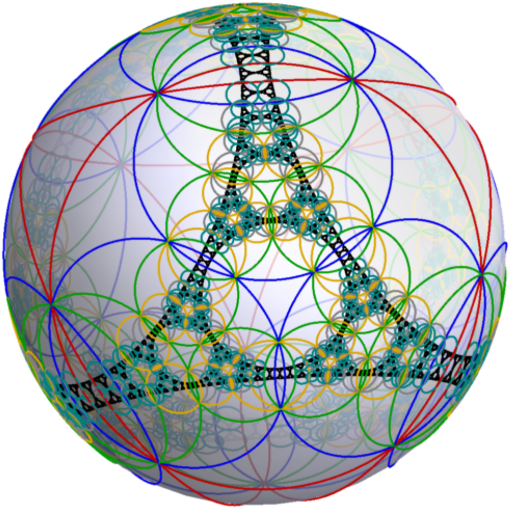

Questions

1. Do any of the following classes of groups embed into $\mathcal{R}$?
   - CAT(0) groups
   - Braid groups
   - Mapping class groups
   - $\mathrm{Out}(F_n)$
2. Does there exist a finitely generated subgroup of $\mathcal{R}$ that contains all hyperbolic groups?
3. Is there an efficient algorithm to compute automata for the generators of a hyperbolic group?