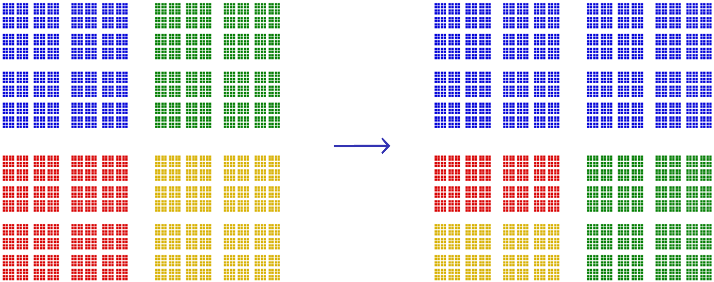
Embeddings into Topological Full Groups



Jim Belk, University of Glasgow

Collaborators

Collin Bleak	Francesco Matucci	Matthew Zaremsky
U. of St Andrews	U. of Milano–Bicocca	SUNY at Albany

The Word Problem

Let G be a group with finite generating set $S = \{s_1, \dots, s_r\}$.

The Word Problem in G (Dehn 1911)

Given a word $w = s_{i_1}^{k_1} \cdots s_{i_n}^{k_n}$, decide whether w represents the identity in G.

G has ***solvable word problem*** if there exists an algorithmic solution.

The Word Problem

Let G be a group with finite generating set $S = \{s_1, \ldots, s_r\}$.

The Word Problem in G (Dehn 1911)

Given a word $w = s_{i_1}^{k_1} \cdots s_{i_n}^{k_n}$, decide whether w represents the identity in G.

G has ***solvable word problem*** if there exists an algorithmic solution.

Example: Any finitely generated subgroup of $\mathrm{GL}_n(\mathbb{Z})$.

The Word Problem

Let G be a group with finite generating set $S = \{s_1, \ldots, s_r\}$.

The Word Problem in G (Dehn 1911)

Given a word $w = s_{i_1}^{k_1} \cdots s_{i_n}^{k_n}$*, decide whether* w *represents the identity in* G*.*

G has ***solvable word problem*** if there exists an algorithmic solution.

Example: Any finitely generated subgroup of $\mathrm{GL}_n(\mathbb{Z})$.

In general, having solvable word problem is inherited by finitely generated subgroups.

The Word Problem

The word problem has two parts:

1. If $w = 1$, can we determine this in finite time?
2. If $w \neq 1$, can we determine this in finite time?

The Word Problem

The word problem has two parts:

1. If $w = 1$, can we determine this in finite time?
2. If $w \neq 1$, can we determine this in finite time?

Part (1) is solvable whenever G is finitely presented

$$G = \langle s_1, \ldots, s_r \mid R_1, \ldots, R_n \rangle.$$

The Word Problem

The word problem has two parts:

1. If $w = 1$, can we determine this in finite time?
2. If $w \neq 1$, can we determine this in finite time?

Part (1) is solvable whenever G is finitely presented

$$G = \langle s_1, \ldots, s_r \mid R_1, \ldots, R_n \rangle.$$

Indeed part (1) is solvable iff G is ***computably presented***

$$G = \langle s_1, \ldots, s_r \mid R_1, R_2, R_3, \ldots \rangle.$$

The Word Problem

The word problem has two parts:

1. If $w = 1$, can we determine this in finite time?
2. If $w \neq 1$, can we determine this in finite time?

Part (1) is solvable whenever G is finitely presented

$$G = \langle s_1, \ldots, s_r \mid R_1, \ldots, R_n \rangle.$$

Indeed part (1) is solvable iff G is ***computably presented***

$$G = \langle s_1, \ldots, s_r \mid R_1, R_2, R_3, \ldots \rangle.$$

So the trick is solving part (2).

An Observation

Proposition (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

An Observation

Proposition (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Proof.

Given a presentation $\langle s_1, \ldots s_m \mid R_1, \ldots R_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1	**Search #2**
Search for a proof that $$w = 1$$ using the relations $R_1, \ldots, R_n$.	Search for a proof that $$s_1 = \cdots = s_m = 1$$ using $w = 1$ and $R_1, \ldots, R_n$.

Eventually one of the searches terminates. □

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

Higman's Embedding Theorem (1961)

Let G be a finitely generated group. Then:

G is computably presented	$\Leftrightarrow$	*G embeds into a finitely presented group*

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

Theorem (Boone–Higman 1974)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a computably presented simple group*

Thompson's Group V

Thompson's Group V

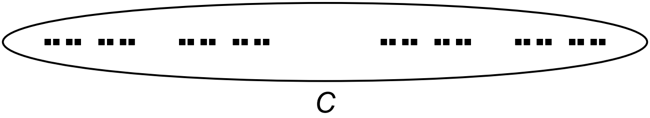
Thompson's group V is the largest of three groups defined by Richard J. Thompson in the 1960's.

Richard J. Thompson, 2004

Thompson's Group V

Let C be the Cantor space $\{0, 1\}^\omega$.

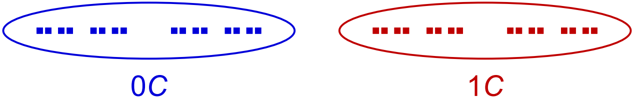
Each finite binary sequence α determines a ***cone*** αC.



Thompson's Group V

Let C be the Cantor space $\{0, 1\}^\omega$.

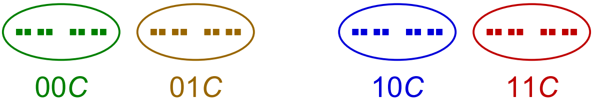
Each finite binary sequence α determines a ***cone*** αC.



Thompson's Group V

Let C be the Cantor space $\{0, 1\}^\omega$.

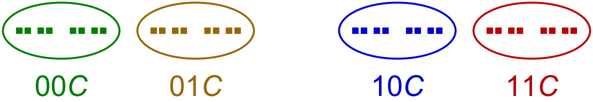
Each finite binary sequence α determines a ***cone*** αC.



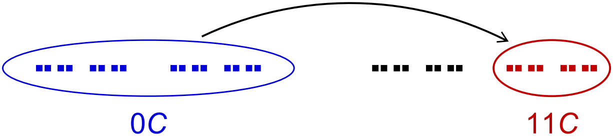
Thompson's Group V

Let C be the Cantor space $\{0, 1\}^\omega$.

Each finite binary sequence α determines a ***cone*** αC.

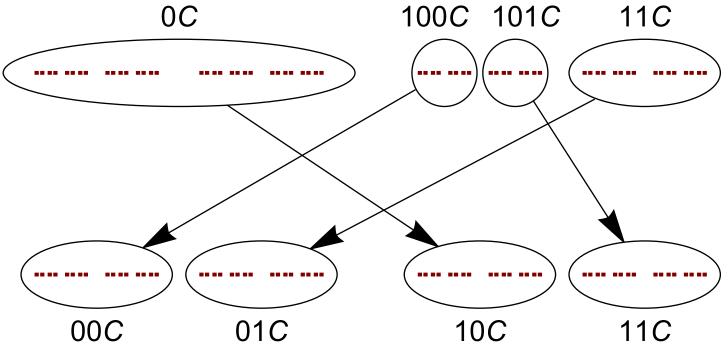


There is a ***prefix-replacement*** homeomorphism between any two cones.



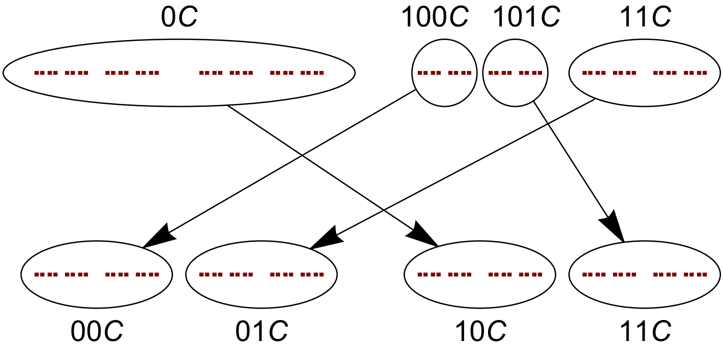
Thompson's Group V

Elements of ***Thompson's group V*** map the cones of one partition to the cones of another by prefix replacement.



Thompson's Group V

Elements of ***Thompson's group V*** map the cones of one partition to the cones of another by prefix replacement.



(Thompson 1965, Higman 1974) V is finitely presented and simple.

Subgroups of V

The following groups embed into V:

1. All finite groups, free groups, free abelian groups, $\bigoplus_\omega V$.
2. (Higman 1974) Locally finite groups, e.g. $\mathbb{Q}/\mathbb{Z}$.
3. (Röver 1999) Free products of finitely many finite groups.
4. (Guba–Sapir 1999) $\mathbb{Z} \wr \mathbb{Z},\ (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z},\ ((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \mathbb{Z},\ \ldots$
5. (Bleak–Salazar-Díaz 2013) $V \wr A$ and $V * A$, where A is any finite group or $A \in \{\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\}$.

Open Question: One-ended hyperbolic groups?

Non-Subgroups of V

The following groups do *not* embed into V:

1. (Higman 1974) $GL_3(\mathbb{Z})$.
2. (Higman 1974) Torsion-free abelian groups that are not free abelian, e.g. $\mathbb{Q}$.
3. (Röver 1999) Groups of Burnside type (i.e. infinite, finitely generated torsion groups).
4. (Bleak–Salazar-Díaz 2013) The free product $\mathbb{Z} * \mathbb{Z}^2$, and hence braid groups and mapping class groups.
5. (Corwin 2013) The restricted wreath product $\mathbb{Z} \wr \mathbb{Z}^2$.

Making Finitely Presented Simple Groups

Groupoids of Germs

Let X be a Cantor space, and consider all triples

$$\{(x,h,y) \mid x,y \in X,\ h \in \text{Homeo}(X),\ h(x) = y\}.$$

Write $(x,h,y) \sim (x,h',y)$ if h and h' agree near x.

The equivalence classes $[x,h,y]$ are ***germs*** on X. They form a (very non-Hausdorff) étale groupoid germs(X).

Groupoids of Germs

Let X be a Cantor space, and consider all triples

$$\{(x,h,y) \mid x,y \in X,\, h \in \text{Homeo}(X),\, h(x) = y\}.$$

Write $(x,h,y) \sim (x,h',y)$ if h and h' agree near x.

The equivalence classes $[x,h,y]$ are ***germs*** on X. They form a (very non-Hausdorff) étale groupoid $\text{germs}(X)$.

Any subgroupoid of $\text{germs}(X)$ that contains the unit space is a ***groupoid of germs*** on X.

Groupoids of Germs

Let X be a Cantor space, and consider all triples

$$\{(x,h,y) \mid x,y \in X,\ h \in \mathrm{Homeo}(X),\ h(x) = y\}.$$

Write $(x,h,y) \sim (x,h',y)$ if h and h' agree near x.

The equivalence classes $[x,h,y]$ are ***germs*** on X. They form a (very non-Hausdorff) étale groupoid germs(X).

Any subgroupoid of germs(X) that contains the unit space is a ***groupoid of germs*** on X.

Note: Groupoids of germs are ***effective***, i.e. the interior of $\{g \in \mathcal{G} \mid s(g) = r(g)\}$ is $\mathcal{G}^{(0)}$.

Example: The Full Shift

For example, let $C = \{0, 1\}^\omega$, and let $\sigma : C \to C$ be the shift map

$$\sigma(0\psi) = \sigma(1\psi) = \psi.$$

Let $\mathcal{G}_2$ be the groupoid *generated* by all germs $[x, \sigma, y]$.

Example: The Full Shift

For example, let $C = \{0, 1\}^\omega$, and let $\sigma\colon C \to C$ be the shift map

$$\sigma(0\psi) = \sigma(1\psi) = \psi.$$

Let $\mathcal{G}_2$ be the groupoid *generated* by all germs $[x, \sigma, y]$.

Then the elements of $\mathcal{G}_2$ are all germs of all prefix replacements $\alpha C \to \beta C$.

Example: The Full Shift

For example, let $C = \{0, 1\}^\omega$, and let $\sigma \colon C \to C$ be the shift map

$$\sigma(0\psi) = \sigma(1\psi) = \psi.$$

Let $\mathcal{G}_2$ be the groupoid *generated* by all germs $[x, \sigma, y]$.

Then the elements of $\mathcal{G}_2$ are all germs of all prefix replacements $\alpha C \to \beta C$.

Note: The reduced C^*-algebra $C^*_r(\mathcal{G}_2)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$. More generally $C^*_r(\mathcal{G}_n) \cong \mathcal{O}_n$.

Topological Full Groups

Definition (Matui 2015)

Let X be a Cantor space and let $\mathcal{G}$ be a groupoid of germs on X. The corresponding ***topological full group*** is

$$[\![G]\!] = \{g \in \mathrm{Homeo}(X) \mid [x, g, g(x)] \in \mathcal{G} \text{ for all } x \in X\}.$$

Topological Full Groups

Definition (Matui 2015)

Let X be a Cantor space and let $\mathcal{G}$ be a groupoid of germs on X. The corresponding ***topological full group*** is

$$[\![G]\!] = \{g \in \mathrm{Homeo}(X) \mid [x, g, g(x)] \in \mathcal{G} \text{ for all } x \in X\}.$$

Example

If $\mathcal{G}_2$ is the groupoid generated by germs of the full shift, then $[\![\mathcal{G}_2]\!] = V$.

Topological Full Groups

Definition (Matui 2015)

Let X be a Cantor space and let $\mathcal{G}$ be a groupoid of germs on X. The corresponding ***topological full group*** is

$$[\![G]\!] = \{g \in \text{Homeo}(X) \mid [x, g, g(x)] \in \mathcal{G} \text{ for all } x \in X\}.$$

Example

If $\mathcal{G}_2$ is the groupoid generated by germs of the full shift, then $[\![\mathcal{G}_2]\!] = V$.

Most known finitely presented simple groups are topological full groups.

Example: Brin–Thompson groups

$\mathcal{G}_2 \times \mathcal{G}_2$ is a groupoid of germs on $C \times C$. The group $[\![\mathcal{G}_2 \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group 2V*** (Brin 2004).

Example: Brin–Thompson groups

$\mathcal{G}_2 \times \mathcal{G}_2$ is a groupoid of germs on $C \times C$. The group $[\![\mathcal{G}_2 \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group*** $\boldsymbol{2V}$ (Brin 2004).

Example: Brin–Thompson groups

$\mathcal{G}_2 \times \mathcal{G}_2$ is a groupoid of germs on $C \times C$. The group $[[\mathcal{G}_2 \times \mathcal{G}_2]]$ is the ***Brin–Thompson group*** $\boldsymbol{2V}$ (Brin 2004).

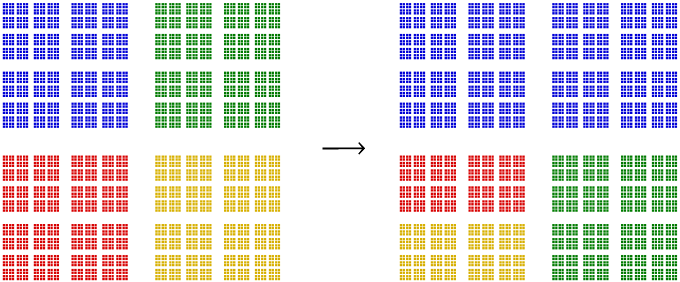
Matthew Brin

Example: Brin–Thompson groups

$\mathcal{G}_2 \times \mathcal{G}_2$ is a groupoid of germs on $C \times C$. The group $[\![\mathcal{G}_2 \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group 2V*** (Brin 2004).

Example: Brin–Thompson groups

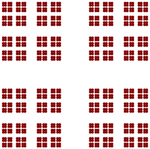
$\mathcal{G}_2 \times \mathcal{G}_2$ is a groupoid of germs on $C \times C$. The group $[[\mathcal{G}_2 \times \mathcal{G}_2]]$ is the ***Brin–Thompson group 2V*** (Brin 2004).



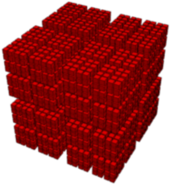
Example: Brin–Thompson groups

More generally, $[\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group nV***.

V



$2V$



$3V$

$\cdots$

Example: Brin–Thompson groups

More generally, $[\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group nV***.

Example: Brin–Thompson groups

More generally, $[\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group nV***.

Theorem (Brin 2009, Hennig–Matucci 2012)

All of the groups nV are finitely presented and simple.

Example: Brin–Thompson groups

More generally, $[\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group nV***.

Theorem (Brin 2009, Hennig–Matucci 2012)

All of the groups nV are finitely presented and simple.

Note: For $n \geq 2$, these groups have:

1. Unsolvable order problem (B–Bleak 2017), and
2. Unsolvable conjugacy problem (Salo 2020[arXiv]).

Example: Brin–Thompson groups

The subgroup structure of the nV's seems to be very rich.

Theorem (B–Bleak–Matucci 2020, Salo 2021[arXiv])

For $n \geq 2$, every right angled Artin group

$$A = \langle x_1, \ldots, x_d \mid [x_{i_1}, x_{j_1}] = \cdots = [x_{i_m}, x_{j_m}] = 1 \rangle$$

embeds into nV.

Example: Brin–Thompson groups

The subgroup structure of the nV's seems to be very rich.

Theorem (B–Bleak–Matucci 2020, Salo 2021[arXiv])

For $n \geq 2$, every right angled Artin group

$$A = \langle x_1, \ldots, x_d \mid [x_{i_1}, x_{j_1}] = \cdots = [x_{i_m}, x_{j_m}] = 1 \rangle$$

embeds into nV.

Corollary

For $n \geq 2$, the following groups embed into nV:

1. *All finitely generated Coxeter groups.*
2. *Many hyperbolic groups.*
3. *Many fundamental groups of 3-manifolds.*

Finding More Examples

How can we find more étale groupoids $\mathcal{G}$ for which $[\![\mathcal{G}]\!]$ is simple?

Finding More Examples

How can we find more étale groupoids $\mathcal{G}$ for which $[[\mathcal{G}]]$ is simple?

Theorem (Matui 2015)

If $\mathcal{G}$ is minimal and purely infinite, then the commutator subgroup $[[\mathcal{G}]]'$ is simple.

Finding More Examples

How can we find more étale groupoids $\mathcal{G}$ for which $[\![\mathcal{G}]\!]$ is simple?

Theorem (Matui 2015)

If $\mathcal{G}$ is minimal and purely infinite, then the commutator subgroup $[\![\mathcal{G}]\!]'$ is simple.

Here a groupoid of germs $\mathcal{G}$ over a Cantor space X is:

1. ***Minimal*** if every $\mathcal{G}$-orbit is dense in X.
2. ***Purely infinite*** if for every clopen $E \subsetneq X$, there exist $g, h \in [\![\mathcal{G}]\!]$ so that $g(E)$ and $h(E)$ are disjoint subsets of E.

Example: Scott's Groups

Let's make a finitely presented simple group that contains $\mathrm{GL}_n(\mathbb{Z})$.

Example: Scott's Groups

Let's make a finitely presented simple group that contains $\mathrm{GL}_n(\mathbb{Z})$.

1. We need a Cantor space, so consider $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright \mathbb{Z}_{(2)}^n$.

Example: Scott's Groups

Let's make a finitely presented simple group that contains $\mathrm{GL}_n(\mathbb{Z})$.

1. We need a Cantor space, so consider $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright \mathbb{Z}_{(2)}^n$.
2. This action isn't minimal, so use $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$.

Example: Scott's Groups

Let's make a finitely presented simple group that contains $\mathrm{GL}_n(\mathbb{Z})$.

1. We need a Cantor space, so consider $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright \mathbb{Z}_{(2)}^n$.
2. This action isn't minimal, so use $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$.
3. This isn't purely infinite, so throw in the germs for $\mathbf{v} \mapsto 2\mathbf{v}$.

Example: Scott's Groups

Let's make a finitely presented simple group that contains $\mathrm{GL}_n(\mathbb{Z})$.

1. We need a Cantor space, so consider $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright \mathbb{Z}_{(2)}^n$.
2. This action isn't minimal, so use $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$.
3. This isn't purely infinite, so throw in the germs for $\mathbf{v} \mapsto 2\mathbf{v}$.

Let $\mathcal{G}$ be the groupoid generated by germs of elements of $\mathrm{Aff}(\mathbb{Z}^n)$ and $\mathbf{v} \mapsto 2\mathbf{v}$.

Theorem (Scott 1984)

The group $[\![\mathcal{G}]\!]$ is finitely presented, simple, and contains $\mathrm{GL}_n(\mathbb{Z})$.

Example: Scott's Groups

Let's make a finitely presented simple group that contains $\mathrm{GL}_n(\mathbb{Z})$.

1. We need a Cantor space, so consider $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright \mathbb{Z}_{(2)}^n$.
2. This action isn't minimal, so use $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$.
3. This isn't purely infinite, so throw in the germs for $\mathbf{v} \mapsto 2\mathbf{v}$.

Let $\mathcal{G}$ be the groupoid generated by germs of elements of $\mathrm{Aff}(\mathbb{Z}^n)$ and $\mathbf{v} \mapsto 2\mathbf{v}$.

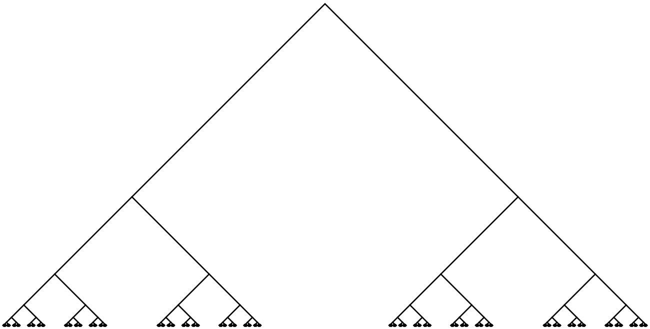
Theorem (Scott 1984)

The group $[[\mathcal{G}]]$ is finitely presented, simple, and contains $\mathrm{GL}_n(\mathbb{Z})$.

In particular, Boone–Higman holds for groups of polynomial growth.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

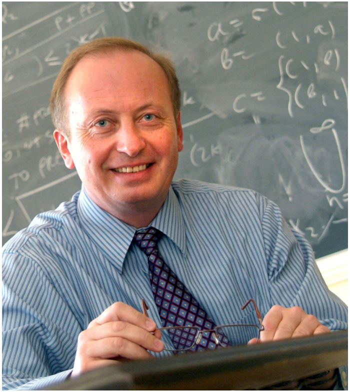


Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.



Rostislav Grigorchuk

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

It is famous because:

1. (Grigorchuk 1980) It is a concrete example of a Burnside group (infinite, finitely generated, torsion).
2. (Grigorchuk 1983) It was the first known example of a group of intermediate growth.
3. (Grigorchuk 1983) It was the first known example of an amenable group which is not elementary amenable.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

It acts by homeomorphisms on the Cantor space $\partial T_2 = \{0, 1\}^\omega$.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

It acts by homeomorphisms on the Cantor space $\partial T_2 = \{0, 1\}^\omega$.

Problem: The action is measure-preserving.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

It acts by homeomorphisms on the Cantor space $\partial T_2 = \{0, 1\}^\omega$.

Problem: The action is measure-preserving.

Solution: Throw in the germs of the shift map σ.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

It acts by homeomorphisms on the Cantor space $\partial T_2 = \{0,1\}^\omega$.

Problem: The action is measure-preserving.

Solution: Throw in the germs of the shift map σ.

Theorem (Röver 1999)

If $\mathcal{G}$ is the resulting étale groupoid, then $[\![\mathcal{G}]\!]$ is a finitely presented simple group.

Example: Röver's Group

The ***Grigorchuk group*** is a certain finitely generated group of automorphisms of an infinite rooted binary tree T_2.

It acts by homeomorphisms on the Cantor space $\partial T_2 = \{0,1\}^\omega$.

Problem: The action is measure-preserving.

Solution: Throw in the germs of the shift map σ.

Theorem (Röver 1999)

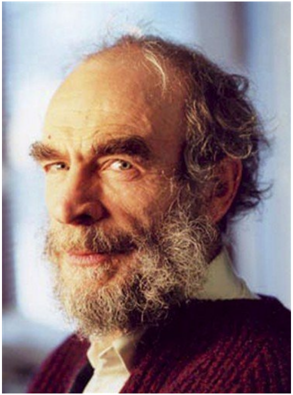
If $\mathcal{G}$ is the resulting étale groupoid, then $[[\mathcal{G}]]$ is a finitely presented simple group.

This was later generalized by Nekrashevych (2004) to a class of self-similar groups.

Hyperbolic Groups

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



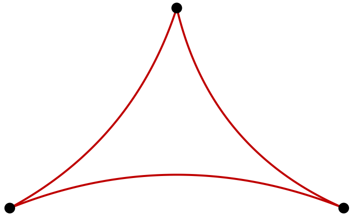
Misha Gromov

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.

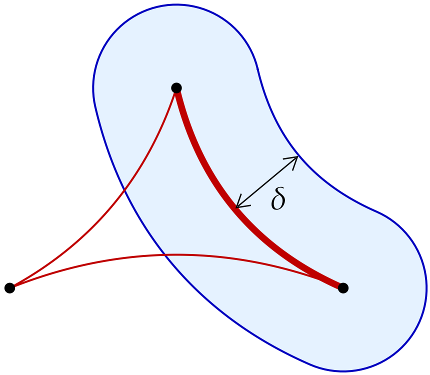
Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



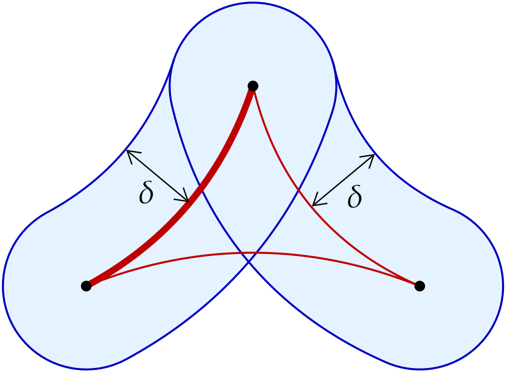
Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



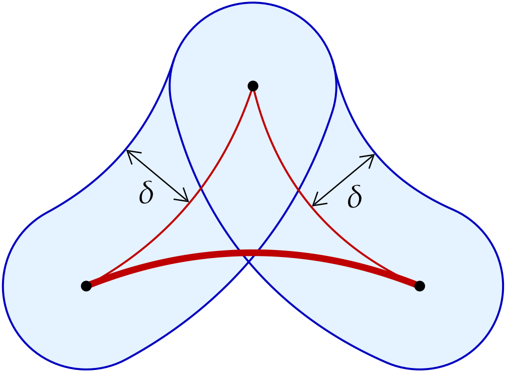
Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



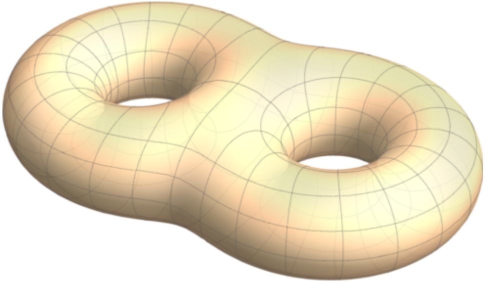
Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.

For example, the fundamental group of any compact hyperbolic manifold is a hyperbolic group.



Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.

For example, the fundamental group of any compact hyperbolic manifold is a hyperbolic group.

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.

For example, the fundamental group of any compact hyperbolic manifold is a hyperbolic group.

In a certain precise sense, almost every finitely presented group is hyperbolic (Ol'Shanskii 1991).

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.

For example, the fundamental group of any compact hyperbolic manifold is a hyperbolic group.

In a certain precise sense, almost every finitely presented group is hyperbolic (Ol'Shanskii 1991).

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Outline of the Proof

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Let G be a hyperbolic group.

1. Find a Cantor space on which G acts by homeomorphisms, and let $\mathcal{G}$ be the étale groupoid of germs.
2. Prove that the topological full group $[[\mathcal{G}]]$ is finitely presented.
3. If $[[\mathcal{G}]]$ is not simple, embed it into a larger finitely presented simple group.

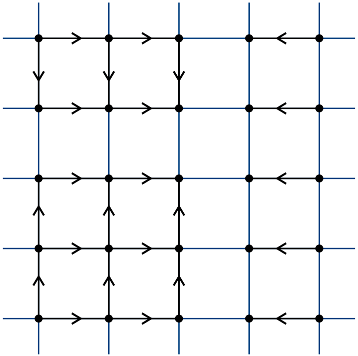
Step 1: Making a Cantor Space

Every connected graph Γ has a ***horofunction boundary*** $\partial_h\Gamma$, which is compact, totally disconnected, and metrizable.

Step 1: Making a Cantor Space

Every connected graph Γ has a ***horofunction boundary*** $\partial_h\Gamma$, which is compact, totally disconnected, and metrizable.

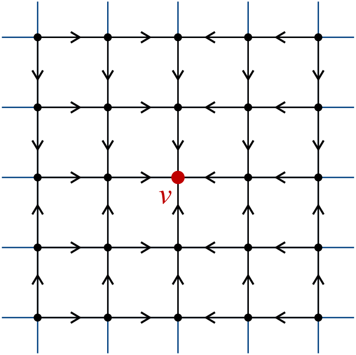
A ***vector field*** on Γ is any assignment of orientations to some of the edges of Γ.



Step 1: Making a Cantor Space

Every connected graph Γ has a ***horofunction boundary*** $\partial_h\Gamma$, which is compact, totally disconnected, and metrizable.

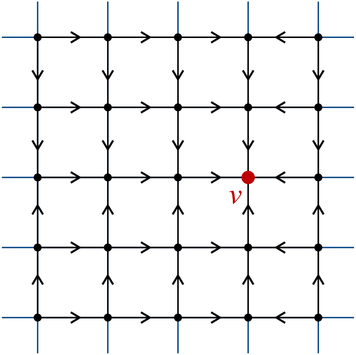
A ***vector field*** on Γ is any assignment of orientations to some of the edges of Γ. Every vertex v determines a ***principal vector field***.



Step 1: Making a Cantor Space

Every connected graph Γ has a ***horofunction boundary*** $\partial_h\Gamma$, which is compact, totally disconnected, and metrizable.

A ***vector field*** on Γ is any assignment of orientations to some of the edges of Γ. Every vertex v determines a ***principal vector field***.



Step 1: Making a Cantor Space

Every connected graph Γ has a ***horofunction boundary*** $\partial_h\Gamma$, which is compact, totally disconnected, and metrizable.

A ***vector field*** on Γ is any assignment of orientations to some of the edges of Γ. Every vertex v determines a ***principal vector field***.

Step 1: Making a Cantor Space

Every connected graph Γ has a ***horofunction boundary*** $\partial_h \Gamma$, which is compact, totally disconnected, and metrizable.

A ***vector field*** on Γ is any assignment of orientations to some of the edges of Γ. Every vertex v determines a ***principal vector field***.

The space $\mathrm{Vec}(\Gamma)$ of vector fields on Γ is a Cantor space, and principal vector fields determine an injection $V(\Gamma) \to \mathrm{Vec}(\Gamma)$.

Step 1: Making a Cantor Space

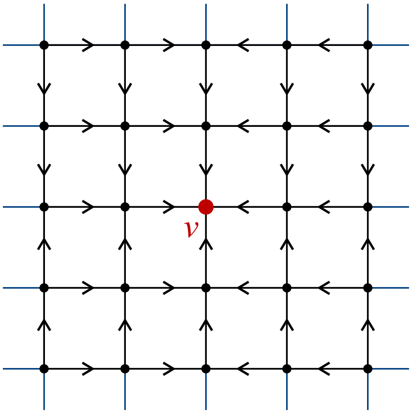
Every connected graph Γ has a ***horofunction boundary*** $\partial_h\Gamma$, which is compact, totally disconnected, and metrizable.

A ***vector field*** on Γ is any assignment of orientations to some of the edges of Γ. Every vertex v determines a ***principal vector field***.

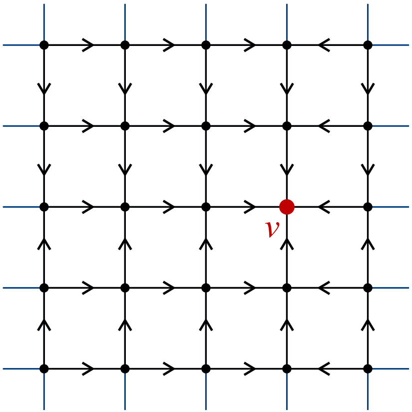
The space $\mathrm{Vec}(\Gamma)$ of vector fields on Γ is a Cantor space, and principal vector fields determine an injection $V(\Gamma) \to \mathrm{Vec}(\Gamma)$.

The ***horofunction boundary*** of Γ is the set of accumulation points of $V(\Gamma)$ in $\mathrm{vec}(\Gamma)$.

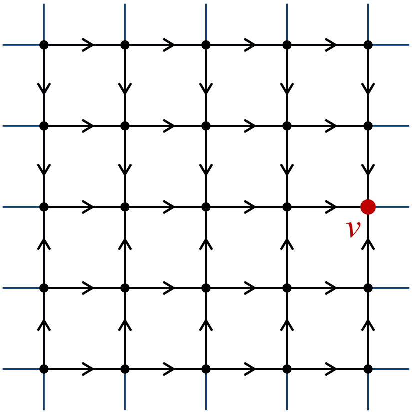
Step 1: Making a Cantor Space



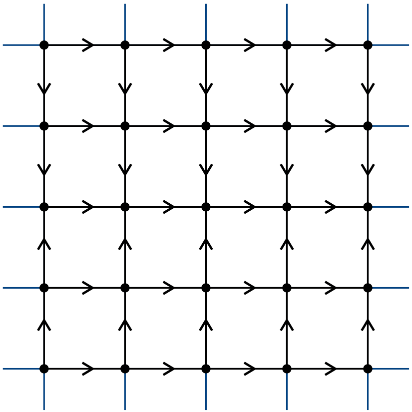
Step 1: Making a Cantor Space



Step 1: Making a Cantor Space

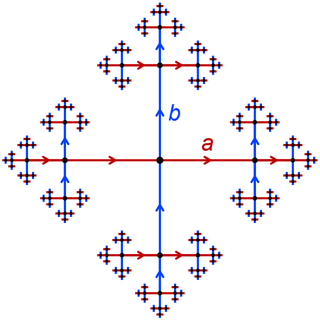


Step 1: Making a Cantor Space



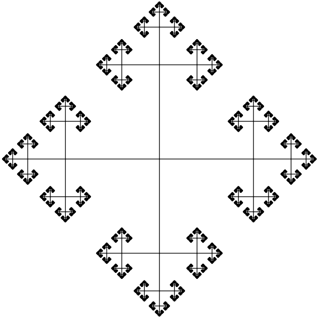
Step 1: Making a Cantor Space

For “most” hyperbolic groups G, the horofunction boundary $\partial_h G$ is a Cantor space, and G acts faithfully on $\partial_h G$.



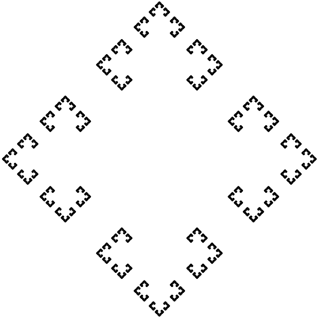
Step 1: Making a Cantor Space

For "most" hyperbolic groups G, the horofunction boundary $\partial_h G$ is a Cantor space, and G acts faithfully on $\partial_h G$.



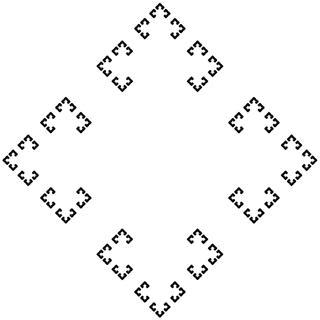
Step 1: Making a Cantor Space

For "most" hyperbolic groups G, the horofunction boundary $\partial_h G$ is a Cantor space, and G acts faithfully on $\partial_h G$.



Step 1: Making a Cantor Space

For “most” hyperbolic groups G, the horofunction boundary $\partial_h G$ is a Cantor space, and G acts faithfully on $\partial_h G$.

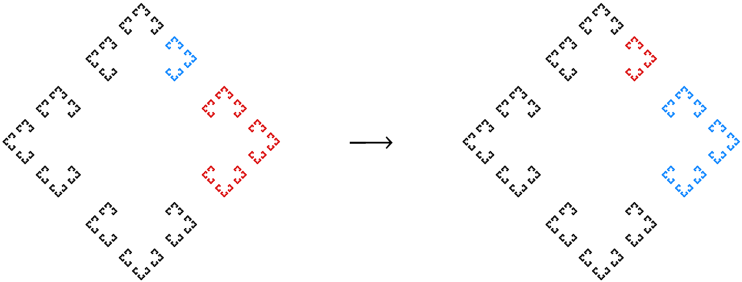


If not, replacing G by $G * \mathbb{Z}$ fixes the problem.

Step 2: Proving Finite Presentability

Let $\mathcal{G}$ be the étale groupoid of germs of elements of G acting on $\partial_h G$.

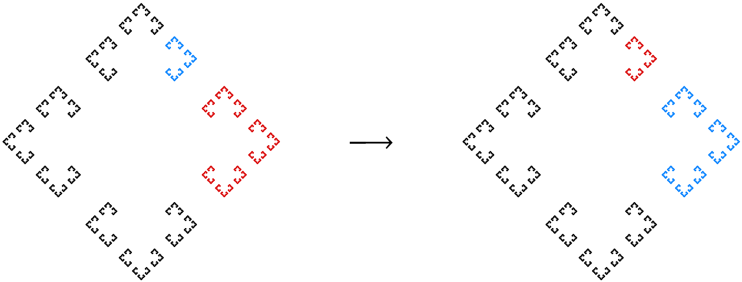
We wish to prove that $[[\mathcal{G}]]$ is finitely presented.



Step 2: Proving Finite Presentability

Let $\mathcal{G}$ be the étale groupoid of germs of elements of G acting on $\partial_h G$.

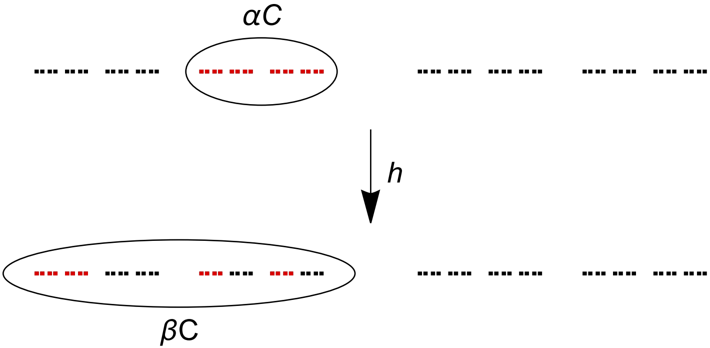
We wish to prove that $[[\mathcal{G}]]$ is finitely presented.



Strategy: Show that the action is ***rational***.

Step 2: Proving Finite Presentability

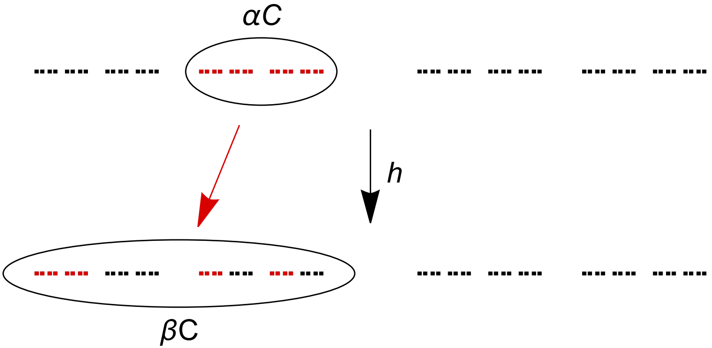
Let h be a homeomorphism of $C = \{0, 1\}^\omega$, let αC be a cone, and let βC be the smallest cone that contains $h(\alpha C)$.



Step 2: Proving Finite Presentability

Let h be a homeomorphism of $C = \{0, 1\}^\omega$, let αC be a cone, and let βC be the smallest cone that contains $h(\alpha C)$.

The injection $C \to C$ determined by the mapping $\alpha C \to \beta C$ called the ***local action*** of h on αC.



Step 2: Proving Finite Presentability

Let h be a homeomorphism of $C = \{0, 1\}^{\omega}$, let αC be a cone, and let βC be the smallest cone that contains $h(\alpha C)$.

The injection $C \to C$ determined by the mapping $\alpha C \to \beta C$ called the ***local action*** of h on αC.

Step 2: Proving Finite Presentability

Let h be a homeomorphism of $C = \{0, 1\}^\omega$, let αC be a cone, and let βC be the smallest cone that contains $h(\alpha C)$.

The injection $C \to C$ determined by the mapping $\alpha C \to \beta C$ called the ***local action*** of h on αC.

The homeomorphism h is ***rational*** if it has only finitely many different local actions.

Step 2: Proving Finite Presentability

Let h be a homeomorphism of $C = \{0,1\}^\omega$, let αC be a cone, and let βC be the smallest cone that contains $h(\alpha C)$.

The injection $C \to C$ determined by the mapping $\alpha C \to \beta C$ called the ***local action*** of h on αC.

The homeomorphism h is ***rational*** if it has only finitely many different local actions.

Theorem (Grigorchuk–Nekrashevych–Sushchanskiĭ 2000)

The group of rational homeomorphisms of C is a topological full group $[\![\mathcal{R}_2]\!]$.

Step 2: Proving Finite Presentability

Let G be a group of rational homeomorphisms of $C = \{0, 1\}^{\omega}$.

A local action $C \longrightarrow C$ lies in the ***nucleus*** for G if it appears infinitely often in some $g \in G$.

G is ***contracting*** if its nucleus of local actions is finite.

Step 2: Proving Finite Presentability

Let G be a group of rational homeomorphisms of $C = \{0, 1\}^\omega$.

A local action $C \to C$ lies in the ***nucleus*** for G if it appears infinitely often in some $g \in G$.

G is ***contracting*** if its nucleus of local actions is finite.

Theorem (Nekrashevych 2017, BBMZ 2023)

If $\mathcal{G}$ is a groupoid of rational germs that contains $\mathcal{G}_2$ and $[\![\mathcal{G}]\!]$ is contracting, then $[\![\mathcal{G}]\!]$ is finitely presented.

Step 2: Proving Finite Presentability

Let G be a group of rational homeomorphisms of $C = \{0,1\}^\omega$.

A local action $C \to C$ lies in the ***nucleus*** for G if it appears infinitely often in some $g \in G$.

G is ***contracting*** if its nucleus of local actions is finite.

Theorem (Nekrashevych 2017, BBMZ 2023)

If $\mathcal{G}$ is a groupoid of rational germs that contains $\mathcal{G}_2$ and $[\![\mathcal{G}]\!]$ is contracting, then $[\![\mathcal{G}]\!]$ is finitely presented.

Note: This theorem continues to hold if we replace C by an irreducible SFT.

Step 2: Proving Finite Presentability

Theorem (B–Bleak–Matucci–Zaremsky)

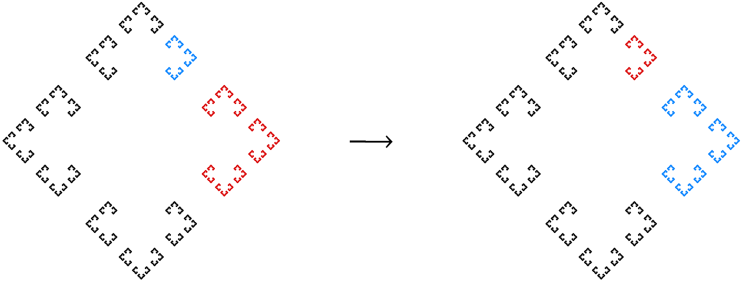
Let G be a hyperbolic group with $\partial_h G$ a Cantor space. Then there exists an irreducible SFT X and a homeomorphism $\partial_h G \to X$ that conjugates $[\![\mathcal{G}]\!]$ to a contracting group of rational homeomorphisms.

This proves that $[\![\mathcal{G}]\!]$ is finitely presented.

Step 3: Embedding in a Larger Group

Let G be a hyperbolic group with $\partial_h G$ a Cantor space.

Even though G embeds in $[\![\mathcal{G}]\!]$, we only know that $[\![\mathcal{G}]\!]'$ is simple.



Step 3: Embedding in a Larger Group

Let G be a hyperbolic group with $\partial_h G$ a Cantor space.

Even though G embeds in $[\![\mathcal{G}]\!]$, we only know that $[\![\mathcal{G}]\!]'$ is simple.

Step 3: Embedding in a Larger Group

Let G be a hyperbolic group with $\partial_h G$ a Cantor space.

Even though G embeds in $[\![\mathcal{G}]\!]$, we only know that $[\![\mathcal{G}]\!]'$ is simple.

Theorem (B–Zaremsky 2022)

Under mild hypotheses, if $[\![\mathcal{G}]\!]$ is any finitely presented topological full group, then $[\![\mathcal{G}]\!]$ embeds in a finitely presented simple group.

Step 3: Embedding in a Larger Group

Let G be a hyperbolic group with $\partial_h G$ a Cantor space.

Even though G embeds in $[\![\mathcal{G}]\!]$, we only know that $[\![\mathcal{G}]\!]'$ is simple.

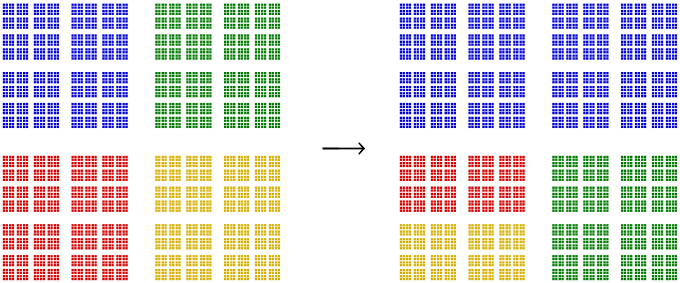
Theorem (B–Zaremsky 2022)

Under mild hypotheses, if $[\![\mathcal{G}]\!]$ is any finitely presented topological full group, then $[\![\mathcal{G}]\!]$ embeds in a finitely presented simple group.

This uses ***twisted Brin–Thompson groups***.

Step 3: Embedding in a Larger Group

Recall that $nV = [\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group***, which acts on $C^n = C \times \cdots \times C$.



Step 3: Embedding in a Larger Group

Recall that $nV = [\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group***, which acts on $C^n = C \times \cdots \times C$.

Step 3: Embedding in a Larger Group

Recall that $nV = [\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group***, which acts on $C^n = C \times \cdots \times C$.

If $H \leq S_n$, then H acts on C^n by permutation of coordinates.

Step 3: Embedding in a Larger Group

Recall that $nV = [\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group***, which acts on $C^n = C \times \cdots \times C$.

If $H \leq S_n$, then H acts on C^n by permutation of coordinates.

Let $\mathcal{G}$ be the groupoid generated by $\mathcal{G}_2 \times \cdots \times \mathcal{G}_2$ and the germs of elements of H. Then $[\![\mathcal{G}]\!]$ is a ***twisted Brin–Thompson group***.

Step 3: Embedding in a Larger Group

Recall that $nV = [\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group***, which acts on $C^n = C \times \cdots \times C$.

If $H \leq S_n$, then H acts on C^n by permutation of coordinates.

Let $\mathcal{G}$ be the groupoid generated by $\mathcal{G}_2 \times \cdots \times \mathcal{G}_2$ and the germs of elements of H. Then $[\![\mathcal{G}]\!]$ is a ***twisted Brin–Thompson group***.

Theorem (B–Zaremsky 2022)

Twisted Brin–Thompson groups are always simple.

Step 3: Embedding in a Larger Group

Recall that $nV = [\![\mathcal{G}_2 \times \cdots \times \mathcal{G}_2]\!]$ is the ***Brin–Thompson group***, which acts on $C^n = C \times \cdots \times C$.

If $H \leq S_n$, then H acts on C^n by permutation of coordinates.

Let $\mathcal{G}$ be the groupoid generated by $\mathcal{G}_2 \times \cdots \times \mathcal{G}_2$ and the germs of elements of H. Then $[\![\mathcal{G}]\!]$ is a ***twisted Brin–Thompson group***.

Theorem (B–Zaremsky 2022)

Twisted Brin–Thompson groups are always simple.

Note: There's no reason n needs to be finite. All of this still works if H is a group of permutations of a countably infinite set.

Step 3: Embedding in a Larger Group

Theorem (B–Zaremsky 2022, Zaremsky 2023)

Let H be a group of permutations of a countable set A, and suppose:

1. *H is finitely presented.*
2. *Stabilizers of finite sets in A are finitely generated, and*
3. *H has finitely many orbits on A^k for all $k \geq 1$.*

Then the corresponding twisted Brin–Thompson group is finitely presented.

In the case where H is a topological full group and A is an orbit, these conditions are almost always satisfied.

Outline of the Proof

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Let G be a hyperbolic group.

1. Find a Cantor space on which G acts by homeomorphisms, and let $\mathcal{G}$ be the étale groupoid of germs.
2. Prove that the topological full group $[\![\mathcal{G}]\!]$ is finitely presented.
3. If $[\![\mathcal{G}]\!]$ is not simple, embed it into a larger finitely presented simple group.

Open Questions

Which of the following embed into finitely presented simple groups?

- Braid groups B_n.
- Mapping class groups.
- $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$.
- Finitely presented metabelian groups.
- (Non-solvable) Baumslag–Solitar groups.
- $\mathrm{GL}_n(\mathbb{Q})$.
- Automatic groups.
- CAT(0) groups.
- Finitely presented residually finite groups.