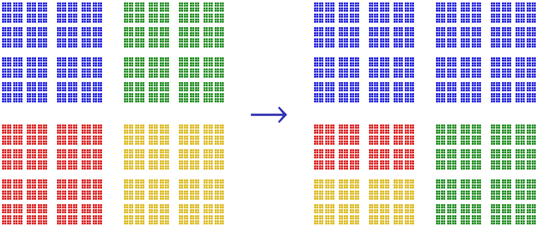
# Twisted Brin–Thompson Groups



Jim Belk

University of St Andrews

June 26, 2020

# Introduction

This is joint work!

Matt Zaremsky

University at Albany

# Main Result

**Theorem (B.–Zaremsky 2020)**

*Every finitely generated group embeds quasi-isometrically as a subgroup of a finitely generated simple group.*

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**Strategy:** Let $G$ be a group of permutations of a set $S$.
We construct a simple group $SV_G$ and an embedding

$$G \longrightarrow SV_G.$$

If $G$ is f.g. and acts transitively on $S$, then $SV_G$ is f.g. and the embedding is quasi-isometric.

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If $G$ is f.g. and acts transitively on $S$, then $SV_G$ is f.g. and the embedding is quasi-isometric.

**Note:** In fact, $SV_G$ is generated by two elements of finite order, one of which has order two.

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**Prior Work:**

1. **Hall (1974):** Every f.g. group embeds into a f.g. simple group.
2. **Goryushkin (1974), Schupp (1976):** Every f.g. group embeds into a two-generated simple group.
3. **Bridson (1998):** Every f.g. group quasi-isometrically embeds into a f.g. group with no proper finite-index subgroups.

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## Theorem (B.–Zaremsky 2020)

*There exists a simple group of type* $\mathrm{F}_\infty$ *that has all of the following as subgroups:*

1. *All right-angled Artin groups,*
2. *All finitely-generated Coxeter groups,*
3. *All surface groups,*
4. *All graph braid groups,*
5. *All one-relator groups with torsion, and*
6. *All fundamental groups of finite-volume hyperbolic 3-manifolds.*

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Also, for each $n \in \mathbb{N}$ we construct a simple group of type $\mathrm{F}_{n-1}$ but not of type $\mathrm{F}_n$.

Skipper, Witzel, and Zaremsky found the first such family in 2019.

Such groups must be pairwise non-quasi-isometric, reproving a 2009 result of Caprace and Réemy.

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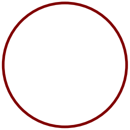

$T$ acts on the circle.

$V$ acts on the Cantor set.

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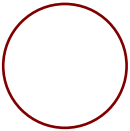

*T* acts on the circle.
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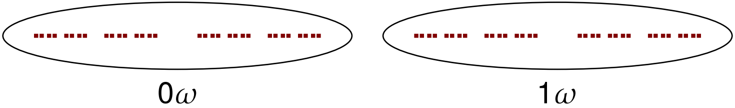

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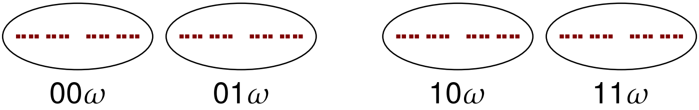

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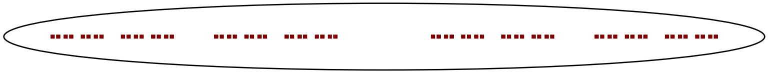

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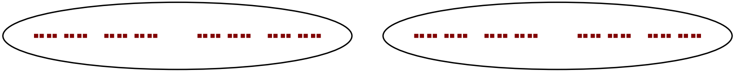

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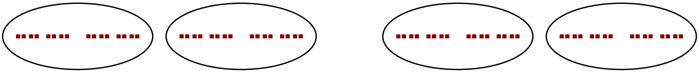

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The ***Cantor set*** $C$ is the infinite product space $\{0,1\}^\omega$.

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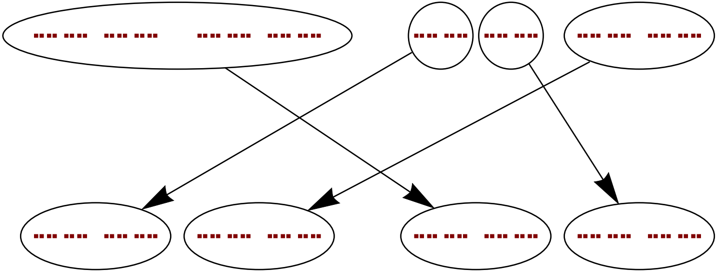
# Definition of $V$

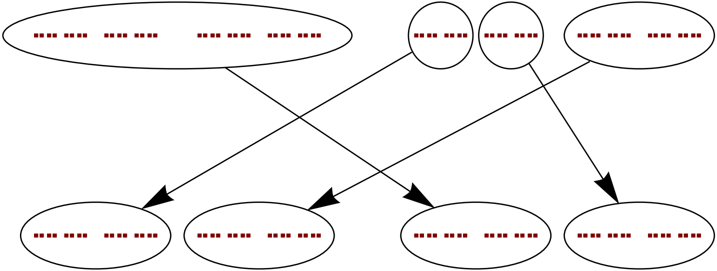
A ***dyadic rearrangement*** of $C$ is a homeomorphism that maps "linearly" between the pieces of two dyadic subdivisions.



The group of all such homeomorphisms is ***Thompson's group V***.

# Definition of $V$

Each piece maps by a ***prefix replacement***.



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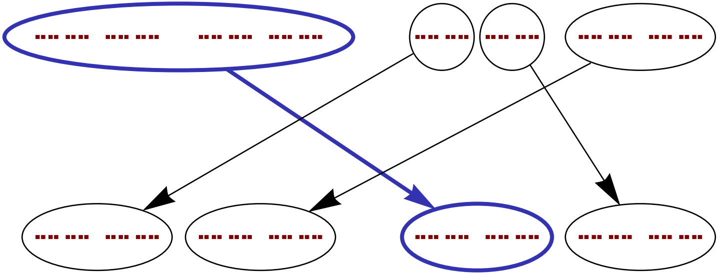

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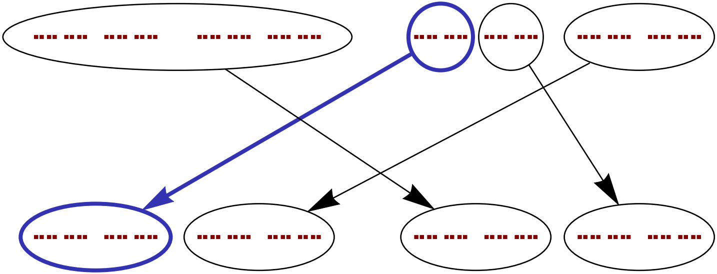

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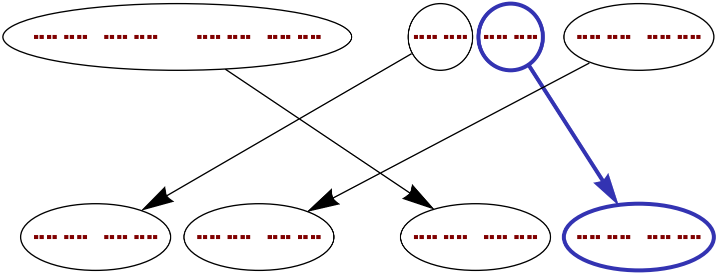

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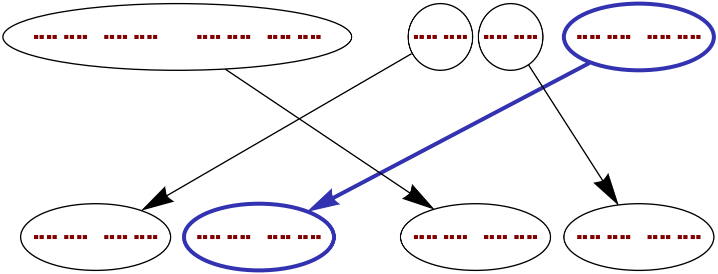

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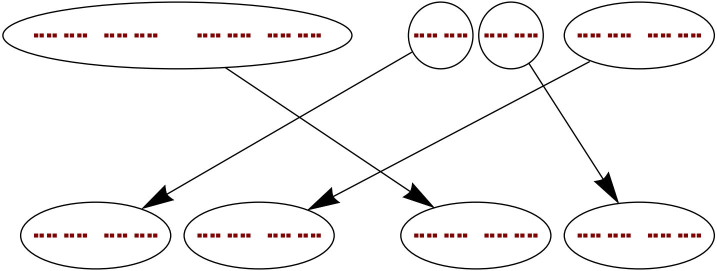

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# Brin–Thompson Groups

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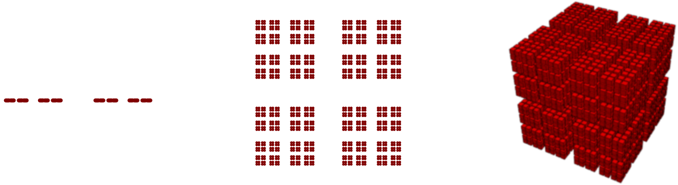

$V$ $\qquad$ $2V$ $\qquad$ $3V$ $\qquad$ $\cdots$

They are “higher-dimensional” versions of Thompson’s group $V$.

# Definition of $sV$

Brin's group $2V$ acts on the ***Cantor Square*** $C \times C$.

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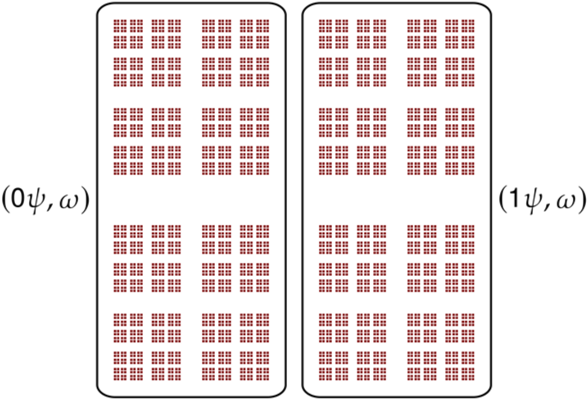

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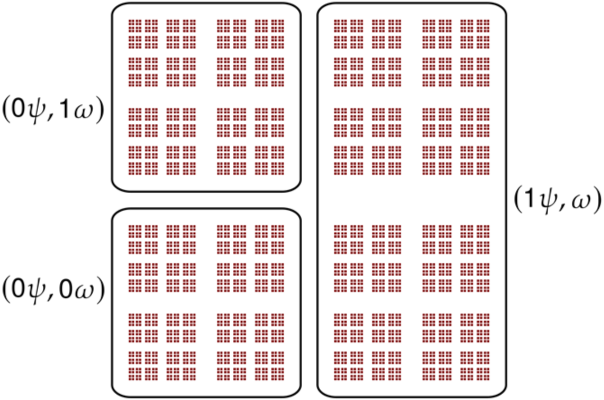

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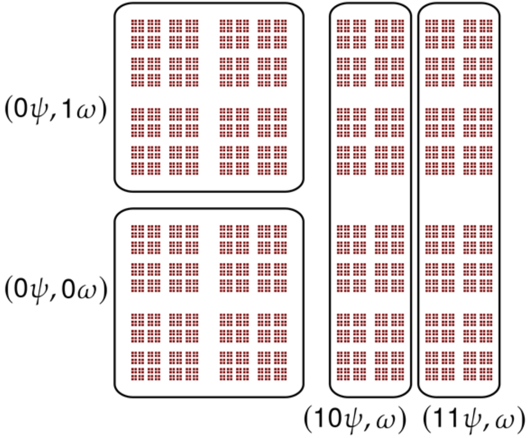

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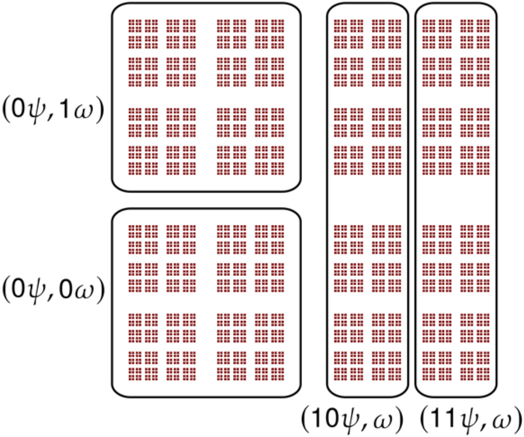
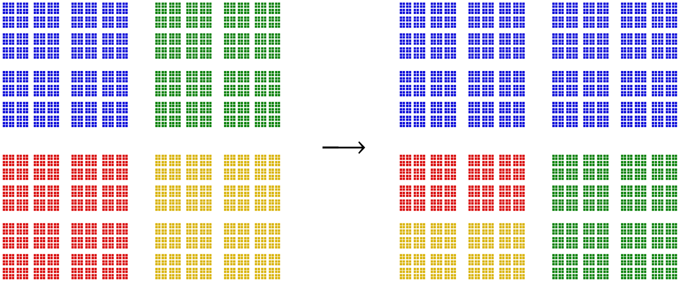



This is a ***dyadic subdivision*** of $C \times C$.

# Definition of $sV$

Homeomorphisms act piecewise by prefix pair replacements:



$(0\psi, 1\omega) \mapsto (\psi, 1\omega)$  $(1\psi, 1\omega) \mapsto (1\psi, 0\omega)$

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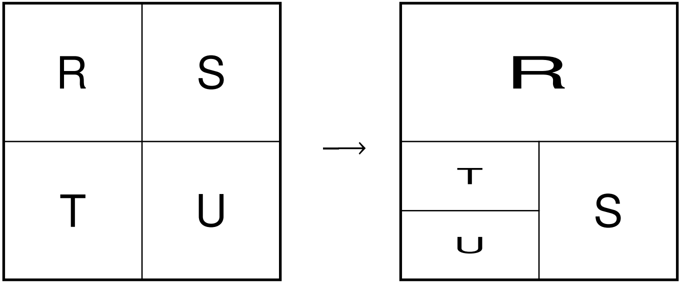
This is the ***pattern pair*** for the same element.

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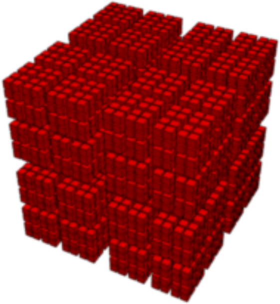
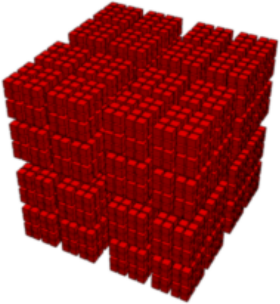
# Definition of $sV$

In general, $sV$ acts by homeomorphisms on $C^s$.



$\longrightarrow$



There is also an $\omega V$ that acts by homeomorphisms on $C^\omega$, but it isn't finitely generated.

# Properties of $sV$

For $s$ finite, the groups $sV$

- Are finitely presented and simple,
  (Brin 2005 and 2010)
- Are non-isomorphic for different values of $s$,
  (Bleak, Lanoue 2010)
- Have type $\mathrm{F}_\infty$,
  (Kochloukova et al. 2013 and Fluch et al. 2013)
- Have the Haagerup property and Serre's property FA, and
  (Kato 2015)
- Have unsolvable torsion problem for $s \geq 2$.
  (Belk, Bleak 2017)

# Properties of $sV$

### Theorem (B.–Bleak–Matucci 2018)

*The group $\omega V$ has all of the following as subgroups:*

1. *All right-angled Artin groups,*
2. *All finitely-generated Coxeter groups,*
3. *All surface groups,*
4. *All graph braid groups,*
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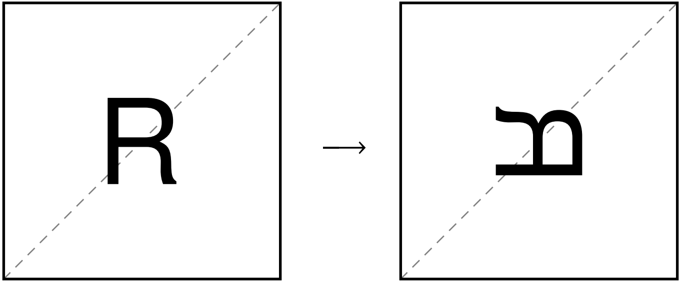
# Adding Twists

# The Diagonal Flip

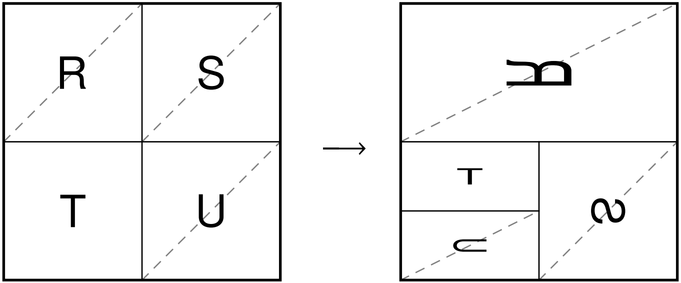
The Cantor square has a ***diagonal flip*** that switches the two coordinates.



$$(\psi, \omega) \mapsto (\omega, \psi)$$

# Twisted $2V$

***Twisted 2V*** is similar to $2V$, except rectangles are allowed to diagonally flip.

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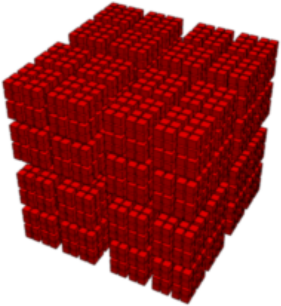
# Twisted $3V$'s

A ***twist*** of the Cantor cube $C^3$ is any permutation of the three coordinates.



Each nontrivial subgroup $G \leq \Sigma_3$ determines a ***twisted group*** $3V_G$.

# General Construction

Any set $S$ has a Brin–Thompson group $SV$ that acts on $C^S$.

Any group $G$ of permutations of $S$ determines a ***twisted Brin–Thompson group*** $SV_G$.

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## Example

Consider the action of $\mathbb{Z}$ on itself by translation (so $G = S = \mathbb{Z}$). This defines a twisted group

$$SV_G \curvearrowright C^{\mathbb{Z}}$$

This group contains $\omega V$ but is finitely generated!

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## Sketch of Proof.

Let $G_{\text{half}}$ be the copy of $G$ supported on any one half of $C^S$.

## Lemma

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**Problem:** $SV$ isn't finitely generated.

# Generating $SV$

Each pair of elements of $S$ defines a copy of $2V$ in $SV$.

## Lemma

*For any connected graph with vertex set $S$, the corresponding copies of $2V$ generate $SV$.*

## Proof.

This uses the work of Hennig and Matucci (2012). $\square$

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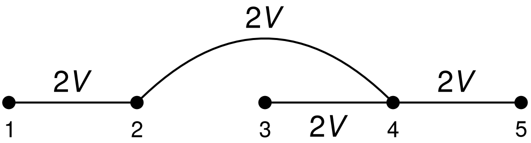
For example, $5V$ is generated by four copies of $2V$:



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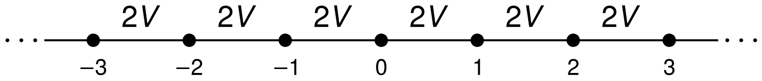
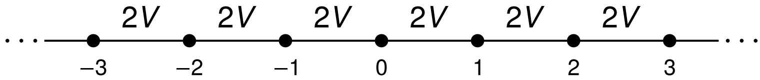
This uses the work of Hennig and Matucci (2012). □

If $S = \mathbb{Z}$, then $SV$ is generated by the following copies of $2V$:



# Generating $SV$

But conjugation by twists permutes the copies of $2V$.




**Conclusion:** If $G = S = \mathbb{Z}$, then $SV_G$ is generated by $G_{\text{half}}$, $G$, and one copy of $2V$.

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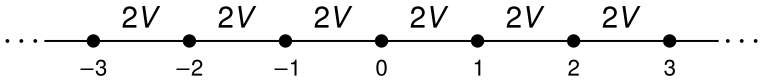


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## Theorem (B.–Zaremsky 2020)

*If $G$ is finitely generated and has finitely many orbits in $S$, then $SV_G$ is generated by $G_{\text{half}}$, $G$, and finitely many copies of $2V$.*

# Simplicity

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**Previous results**:

- Thompson proved that $T$ and $V$ are simple. These were the first known examples of infinite, finitely-presented simple groups.
- $F$ is not simple, but its commutator subgroup $[F, F]$ is.
- Brin proved that $2V$ is simple in 2004, and extended this result to all $sV$'s in 2010.
- Many other Thompson-like groups have been considered, and typically they at least have simple commutator subgroup.

Matui (2015) and Bleak–Elliott–Hyde (2020) have both developed good technology for proving such things.

# Bleak–Elliott-Hyde

Let $H \leq \mathrm{Homeo}(C)$ be finitely generated. We say that $H$ is:

1. **Vigorous** if for all nonempty, disjoint clopen sets $A, B_1, B_2 \subset C$ there exists an $h \in A$ with $h|_A = \mathrm{id}$ and $h(B_1) \subset B_2$.
2. **Generated by elements of small support** if $H$ is generated by its elements that are supported on proper clopen subsets of $C$.
3. **Perfect** if $[H, H] = H$.

## Theorem (Bleak–Elliott–Hyde 2020)

*Suppose $H$ satisfies (1), (2), and (3) above. Then*

1. *$H$ is simple, and*
2. *There exist $h_1, h_2 \in H$ with $|h_1| = 2$ and $|h_2| < \infty$ such that $\langle h_1, h_2 \rangle = H$.*

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Since $C^S \approx C$, we can apply this to $SV_G$. We have:

- $SV_G$ satisfies (1) since $SV$ does.
- $SV_G$ satisfies (2) since it is generated by $SV$ and $G_{\mathrm{half}}$.
- $SV_G$ satisfies (3) since $SV = [SV, SV]$ and $G_{\mathrm{half}} \leq [SV_G, SV_G]$.

# Conclusion

If we use $S = G$, we get an embedding

$$G \longrightarrow SV_G$$

of any finitely-generated group $G$ into a two-generated simple group.

This recovers the theorems of Hall (1974) and Goryushkin (1974), and part of the results of Schupp (1976).

# Quasi-Isometric Embeddings

# Word Length

Let $G$ be a group with finite generating set $X$, and $\Gamma(G, X)$ be the associated Cayley graph.

The ***word metric*** on $G$ is defined by

$$d(g, h) = \text{length of the shortest path in } \Gamma(G, X) \text{ from } g \text{ to } h$$

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Let $G$ be a group with finite generating set $X$, and $\Gamma(G, X)$ be the associated Cayley graph.

The ***word metric*** on $G$ is defined by

$$d(g, h) = \text{length of the shortest path in } \Gamma(G, X) \text{ from } g \text{ to } h$$

**Note:** Two different generating sets $X_1, X_2$ give two different word metrics $d_1, d_2$, but

$$\frac{1}{C} d_1(g, h) \ \leq \ d_2(g, h) \ \leq \ C\, d_1(g, h)$$

for some constant $C$.

# Quasi-Isometrically Embedded Subgroups

Let $H \leq G$ be finitely-generated groups.

The inclusion $H \longrightarrow G$ is a ***quasi-isometric embedding*** if

$$d_H(h_1, h_2) \;\leq\; C\, d_G(h_1, h_2)$$

for some constant $C$ (and all $h_1, h_2 \in H$).

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## Theorem (B.–Zaremsky 2020)

*If $G$ is finitely-generated, the inclusion*

$$G \longrightarrow SV_G$$

*is a quasi-isometric embedding.*

# Quasi-Retracting

Let $G$ be a group with finite generating set $X$.

Let $H \leq G$ be finitely generated.

## Proposition

*Suppose there exists a function* $r \colon G \longrightarrow H$ *such that* $r|_H = \mathrm{id}$ *and*

$$d_H\big(r(xg), r(g)\big) \leq C$$

*for some constant* $C$ *and all* $g \in G$ *and* $x \in X$*. Then the inclusion* $H \longrightarrow G$ *is a quasi-isometric embedding.*

Such a function $r$ is an example of a ***quasi-retraction***, and $H$ is a ***quasi-retract*** of $G$.

# Quasi-Retracting $SV_G$ Onto $G$

### Theorem (B.–Zaremsky 2020)

*If $G$ is finitely-generated, the inclusion $G \longrightarrow SV_G$ is a quasi-isometric embedding.*

### Sketch of Proof.

Fix a point $p \in C^S$. Define $r\colon SV_G \longrightarrow G$ by

$$r(f) = \text{the twist of } f \text{ near } p.$$

Since $SV_G$ is generated by $G$, $G_{\text{half}}$, and copies of $2V$, we have

$$d_G\big(r(xf), r(f)\big) \leq 1$$

for every $f \in SV_G$ and every generator $x$ of $SV_G$. $\square$

# Finiteness Properties

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Let $G$ be a group.

A ***K*(*G*, 1)-*complex*** is a connected CW complex $X$ such that:

1. $\pi_1(X) \cong G$, and
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We say that $G$ has ***type* $\mathbf{F}_n$** if there exists a $K(G,1)$-complex whose $n$-skeleton has finitely many cells.

$$\begin{aligned} \text{type } F_1 &= \text{finitely generated} \\ \text{type } F_2 &= \text{finitely presented} \\ &\vdots \end{aligned}$$

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We say that $G$ has ***type*** $\mathbf{F}_\infty$ if it has type $\mathrm{F}_n$ for all $n$.

# Finiteness Properties

In 1983, Brown and Geoghegan proved that Thompson's group $F$ is of type $\mathrm{F}_\infty$.

Kenneth Brown

Ross Geoghegan

Brown later generalized this to $T$ and $V$ (1987), using a method now known as ***Brown's criterion***.

# Type $F_n$ Using Actions

## Proposition

*Let $G$ be a group acting rigidly on a CW complex $X$, and let $n \geq 1$. Suppose that:*

1. *$X$ is $(n-1)$-connected and has finitely many orbits of cells.*
2. *For each $0 \leq k \leq n$, the stabilizer of each $k$-cell in $X$ is of type $F_{n-k}$.*

*Then $G$ has type $F_n$.*

The trouble is finding a complex $X$ that satisfies (1). How do we ensure that $X$ is $(n-1)$-connected?

Contractible complexes are $(n-1)$-connected, but they usually have infinitely many orbits of cells.

# Brown's Criterion

Brown's idea is to use a *chain* of complexes:

$$X_1 \subset X_2 \subset X_3 \subset \cdots$$

We make sure that:

1. Each $X_n$ has finitely many orbits of cells, and
2. The union $X = \bigcup_{k=1}^{\infty} X_k$ is contractible.

Since the $X_k$ are "converging" to a contractible space, they ought to be $(n-1)$-connected for large $k$. We just need to show that the induced homomorphisms

$$\pi_i(X_k) \longrightarrow \pi_i(X_{k+1})$$

are isomorphisms for large $k$ (and $0 \leq i \leq n-1$).

# Discrete Morse Theory

In 1996, Bestvina and Brady introduced powerful methods for analysing the homomorphisms $\pi_i(X_k) \to \pi_i(X_{k+1})$.

Mladen Bestvina

Noel Brady

They showed how to understand such homomorphisms by considering the connectivity of the ***descending links***.

# Finiteness Properties

The combination of Brown's criterion and Bestvina–Brady discrete Morse theory has become standard in the study of finiteness properties.

Fluch, Marschler, Witzel, and Zaremsky used this combination to prove the following.

### Theorem (FMWZ 2013)

*For s finite, the Brin–Thompson group $sV$ has type $\mathrm{F}_\infty$.*

Their chain of CW complexes $X_1 \subset X_2 \subset \cdots$ was obtained from a filtration of a deformation retraction of the simplicial realization of a certain poset derived from a natural groupoid that contains $sV$.

# Finiteness Properties

We use similar methods to analyse the finiteness properties of the $SV_G$'s, though the descending link analysis is much more difficult.

### Theorem (B.-Zaremsky 2020)

*Let $n \geq 1$, and suppose that:*

1. *For each $k \geq 1$, the action of $G$ on $S^k$ has finitely many orbits.*
2. *The group $G$ has type $\mathrm{F}_n$, and*
3. *The stabilizer in $G$ of any finite subset of $S$ has type $\mathrm{F}_n$.*

*Then $SV_G$ has type $\mathrm{F}_n$.*

Condition (1) says that $G$ is ***oligomorphic***.

# Consequences

### Theorem (B.-Zaremsky 2020)

*If $G$ and $S$ are finite, then $SV_G$ is a simple group of type $\mathrm{F}_\infty$.*

It would be interesting to classify such groups up to isomorphism. They are not isomorphic to the $sV$'s by the work of Bleak and Lanoue (2010).

### Theorem (B.-Zaremsky 2020)

*If $G$ is Thompson's group $F$ and $S$ is the set of dyadic rationals in $(0,1)$, then $SV_G$ is a simple group of type $\mathrm{F}_\infty$ that contains $\omega V$.*

# Examples

Alonso (1994) proved that a quasi-retract of a group of type $\mathrm{F}_n$ must have type $\mathrm{F}_n$. Thus:

$$G \text{ is not } \mathrm{F}_n \quad \Rightarrow \quad SV_G \text{ is not } \mathrm{F}_n.$$

## Theorem (B.-Zaremsky 2020)

*If $G$ is Houghton's group $H_n$ and $S = \{1, \ldots, n\} \times \mathbb{N}$, then $SV_G$ is a simple group of type $\mathrm{F}_{n-1}$ but not of type $\mathrm{F}_n$.*

# Questions

1. Does every hyperbolic group embed in a type $F_\infty$ simple group? (B.–Bleak–Matucci–Zaresmky conjecture yes.)

2. (Higman) Does every finitely presented group with solvable word problem embed (quasi-isometrically?) in a finitely presented simple group?

3. Can braid groups be embedded in finitely presented simple groups? Mapping class groups? $\mathrm{Out}(F_n)$?

4. (Leary) Does every finitely presented group embed in a type $F_\infty$ group?

The End