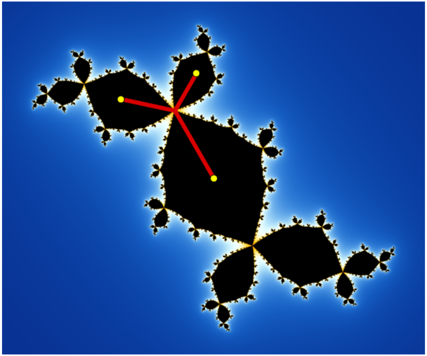
Recognizing Topological Polynomials by Lifting Trees



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Collaborators

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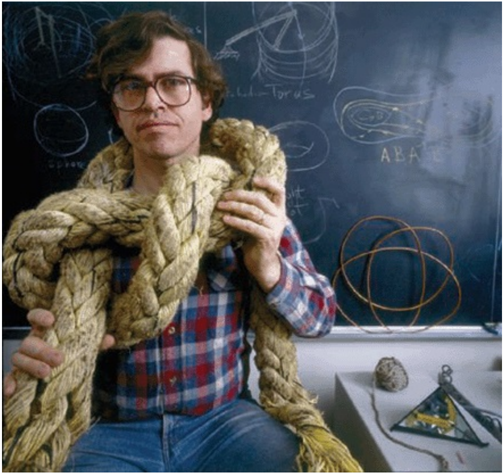
Georgia Tech

Dan Margalit,
Georgia Tech

Becca Winarski,
U. Michigan

Topological Polynomials

In the 1980's, Bill Thurston began to study complex polynomials from a topological viewpoint.



Topological Polynomials

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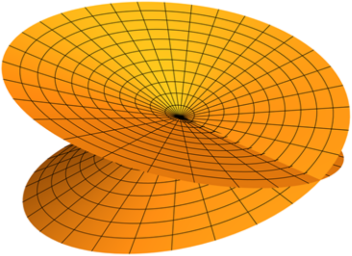
A ***topological polynomial*** is any orientation-preserving branched cover

$$f : \mathbb{C} \to \mathbb{C}$$

with finitely many branch points.

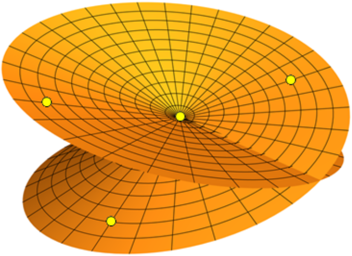


In analogy with polynomials, we refer to the branch points as ***critical points***, and their images as ***critical values***.

Marked Points

We can ***mark*** a topological polynomial by choosing a finite set $M \subset \mathbb{C}$, where

1. $f(M) \subset M$, and
2. M contains the critical values of f.



Marked Points

We can ***mark*** a topological polynomial by choosing a finite set $M \subset \mathbb{C}$, where

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The pair (f, M) is a ***marked topological polynomial***.

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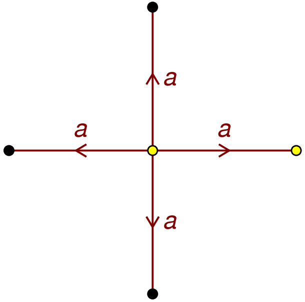
The pair (f, M) is a ***marked topological polynomial***.

Thurston's Question: Is (f, M) "topologically equivalent" to a polynomial?

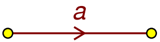
Examples

We can specify (f, M) up to isotopy by drawing

1. Any tree T containing M,
2. Its preimage $f^{-1}(T)$, and the mapping $f^{-1}(T) \to T$.



$\longrightarrow$



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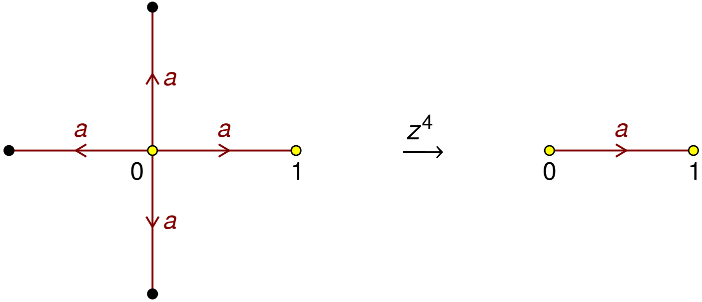

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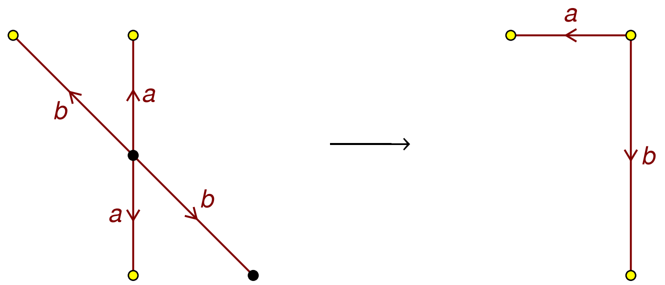

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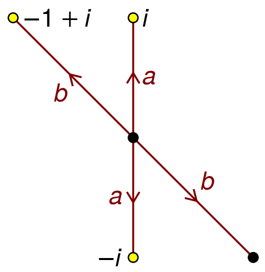
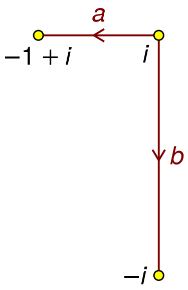


$\xrightarrow{z^2 + i}$



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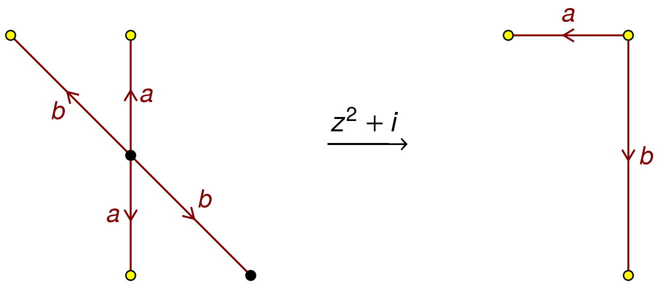

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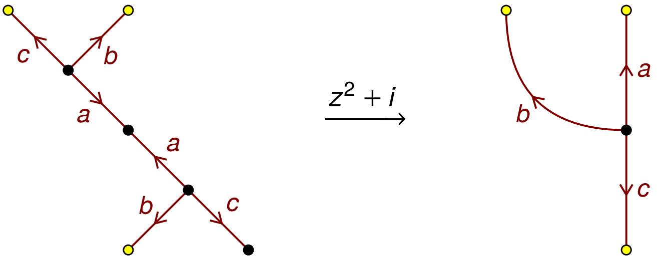

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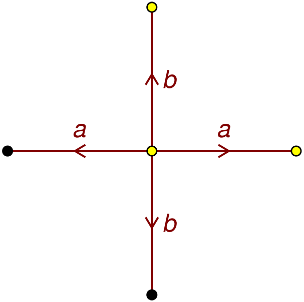

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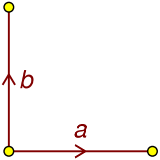
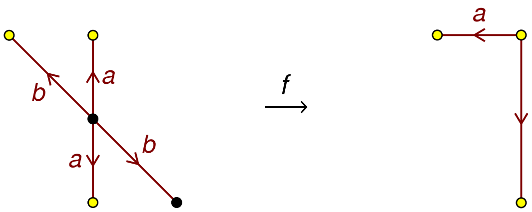
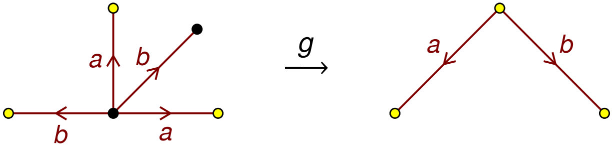


Definition of Topological Equivalence

Two marked topological polynomials are ***topologically equivalent*** if there is a homeomorphism conjugating one to the other.





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$$\begin{tikzcd}
(\mathbb{C}, M) \arrow[r, "f"] \arrow[d, "h"'] & (\mathbb{C}, M) \arrow[d, "h"'] \\
(\mathbb{C}, N) \arrow[r, "g"] & (\mathbb{C}, N)
\end{tikzcd}$$

Thurston’s Theorems

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Thurston Rigidity (1982)

Suppose that (f, M) and (g, N) are topologically equivalent:

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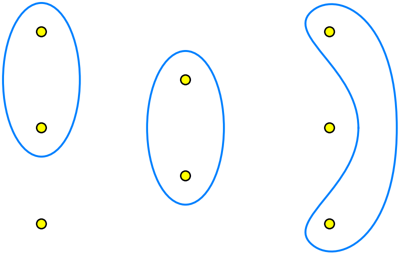
If f and g are complex polynomials, then h has the form

$$h(z) = az + b.$$

Thurston's Theorems

Thurston's Theorem (1982)

A marked polynomial (f, M) is topologically equivalent to a complex polynomial if and only if it has no ***Thurston obstruction***.



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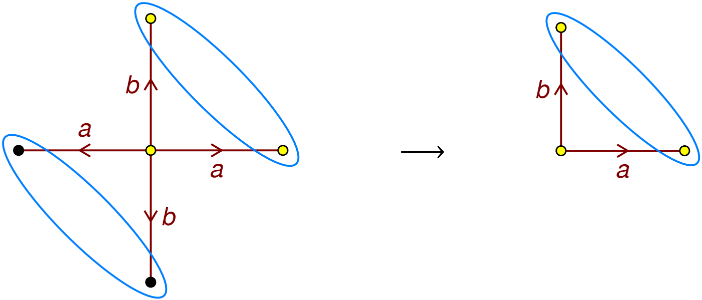
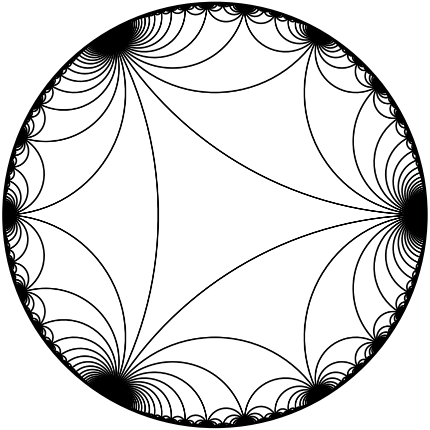


Thurston's Theorems

Thurston proved both of these theorems by analyzing the dynamics of the ***pullback map*** on Teichmüller space.



Algorithmic Question

So given a marked topological polynomial (f, M), exactly one of the following holds:

1. (f, M) is topologically equivalent to a complex polynomial, unique up to affine conjugacy.

2. (f, M) has a Thurston obstruction.

Algorithmic Question

So given a marked topological polynomial (f, M), exactly one of the following holds:

1. (f, M) is topologically equivalent to a complex polynomial, unique up to affine conjugacy.
2. (f, M) has a Thurston obstruction.

Question: How can we distinguish between these two cases? How do we actually find the complex polynomial (in case 1) or Thurston obstruction (in case 2)?

The Twisted Rabbit Problem

Hubbard (1983) observed that this is difficult even for $\deg(f) = 2$ and $|M| = 3$. This is the ***twisted rabbit problem***.

John Hubbard

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This case was solved by Bartholdi and Nekrashevych in 2006 using ***iterated monodromy groups***.

Laurent Bartholdi

Volodymyr Nekrashevych

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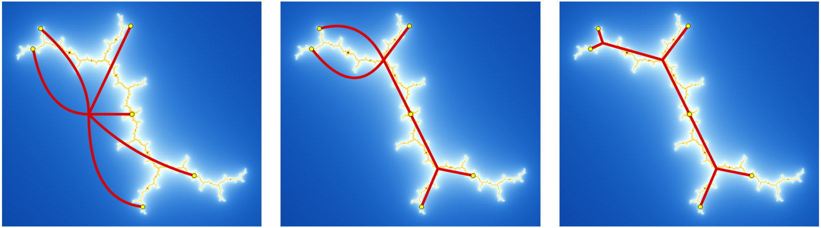
Unfortunately, Bartholdi and Nekrashevych's methods are difficult to apply for $\deg(f) \geq 3$ or $|M| \geq 4$.

Main Result

We have developed a simple geometric algorithm (the ***tree lifting algorithm***) that answers these questions much more generally.



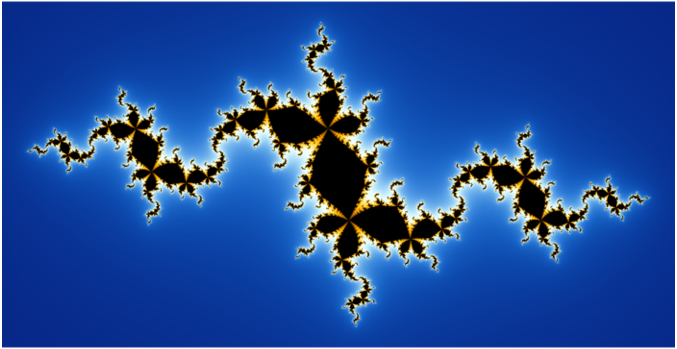
Given an (f, M), the algorithm produces either

1. The Hubbard tree for a polynomial equivalent to (f, M), or
2. The canonical Thurston obstruction for (f, M).

Some Complex Dynamics

Julia Sets

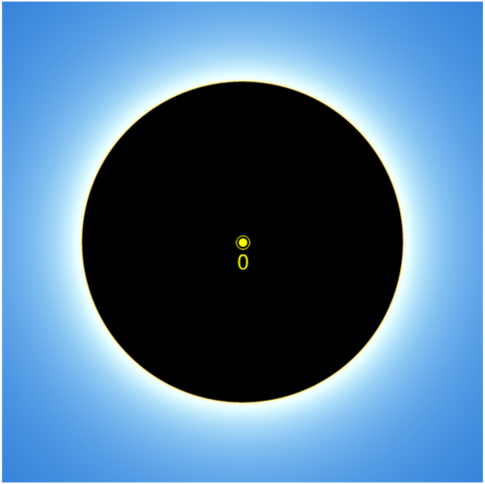
Every complex polynomial $f(z)$ of degree ≥ 2 has a ***filled Julia set***.



This is the unique maximal compact, f-invariant subset of $\mathbb{C}$.

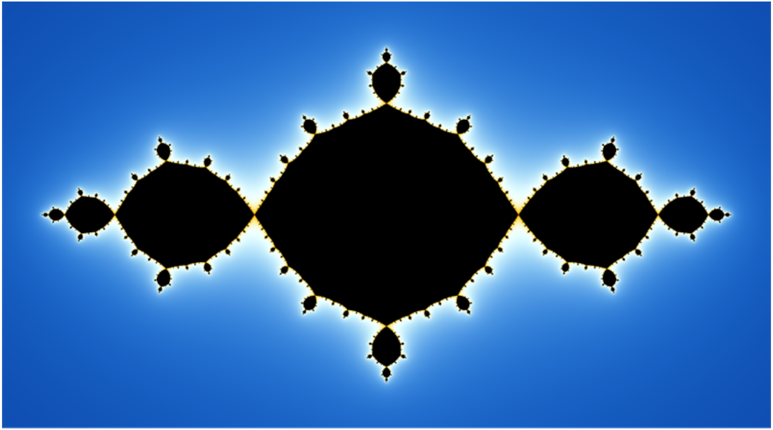
Julia Sets

For example, the filled Julia set for $f(z) = z^2$ is the closed unit disk.



Julia Sets

Here's the filled Julia set for $z^2 - 1$.



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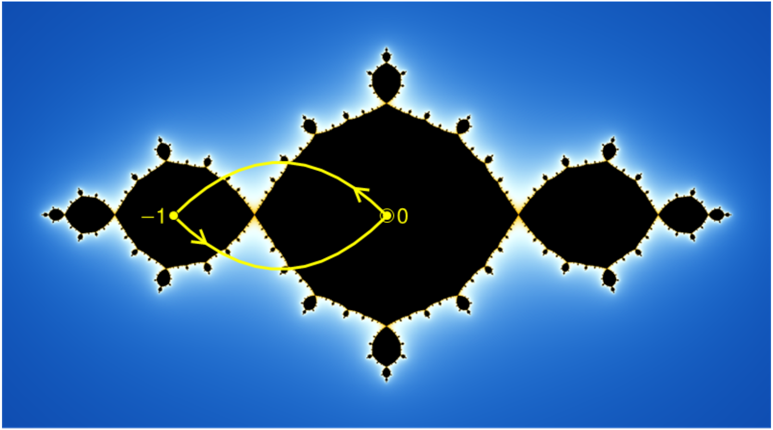

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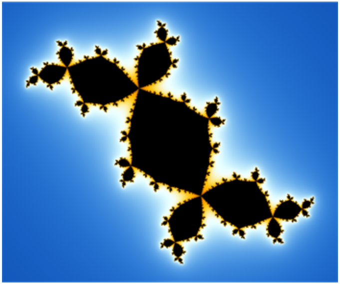
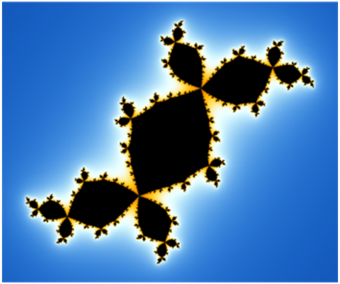


rabbit



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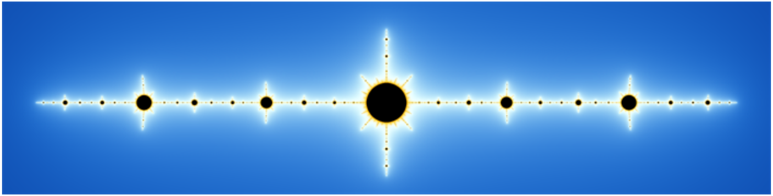

airplane

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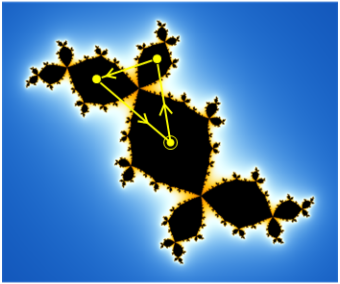

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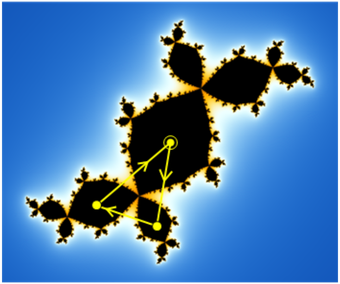

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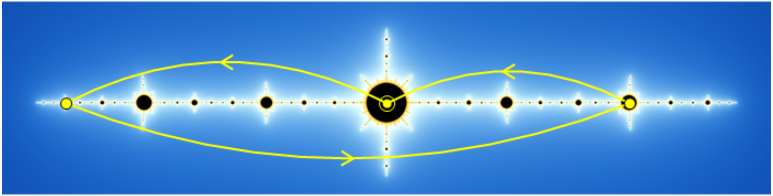
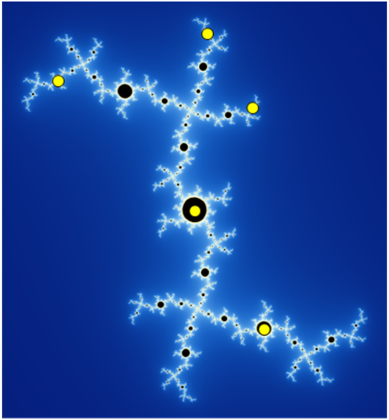


airplane

The Hubbard Tree

We can use the Julia set to define the ***Hubbard tree*** H, a special tree that contains M.



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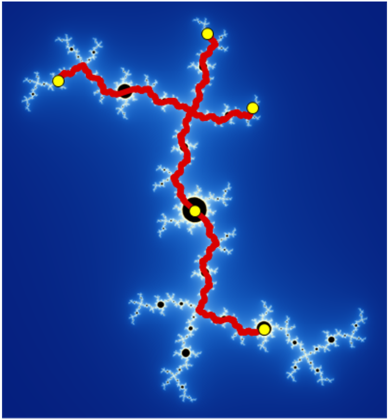

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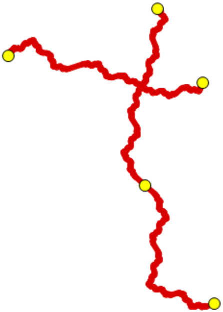

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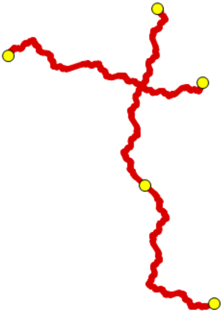

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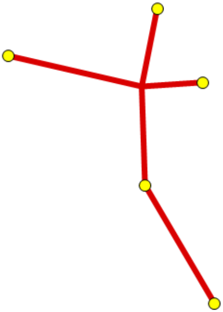

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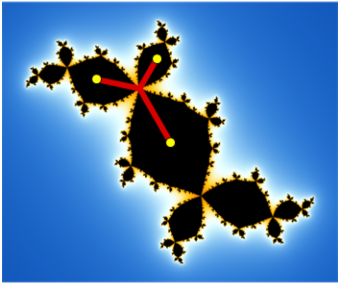

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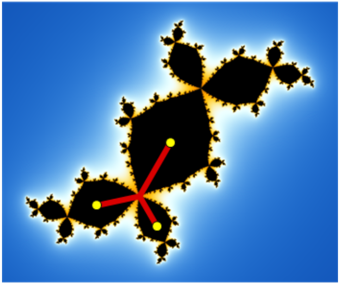

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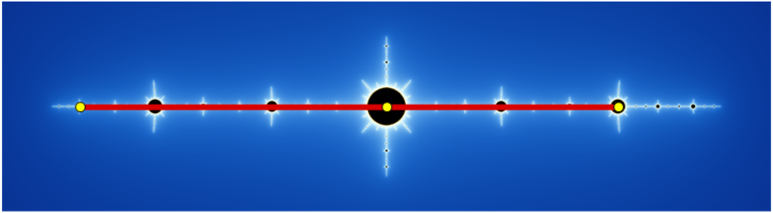

airplane

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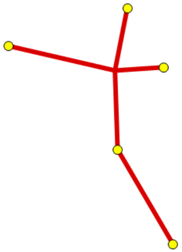
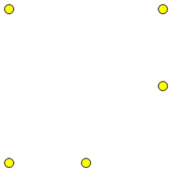
Any map that's topologically equivalent to a polynomial has a ***topological Hubbard tree***.



$\xrightarrow{\text{equivalence}}$



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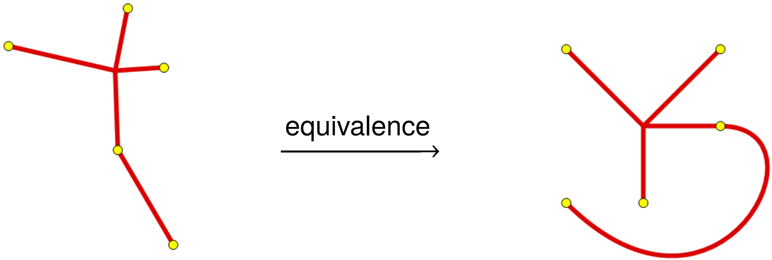

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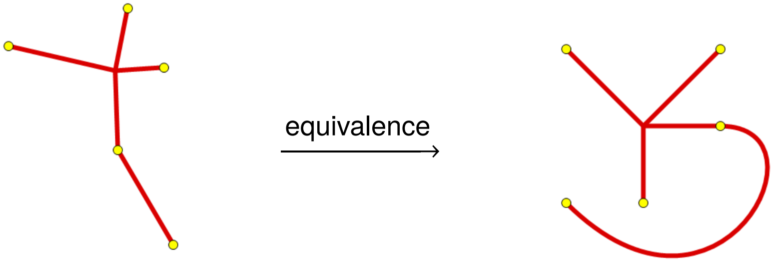

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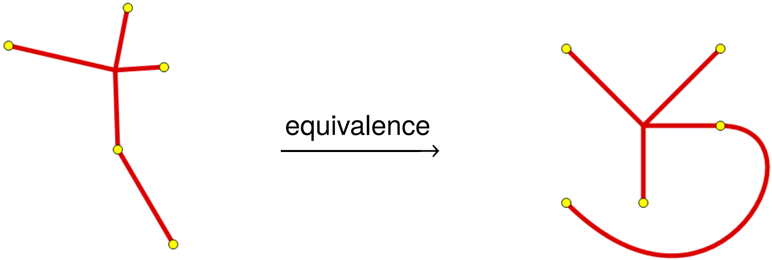

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Note: The polynomial $f(z)$ is determined by H and the mapping

$$f^{-1}(H) \longrightarrow H.$$

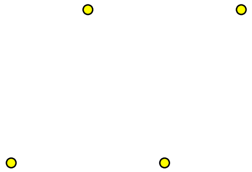
Lifting Trees

Allowed Trees

We will consider trees in $\mathbb{C}$ that satisfy the following conditions:

1. T contains M, and
2. Every leaf of T lies in M.

Isotopic trees are considered the same.



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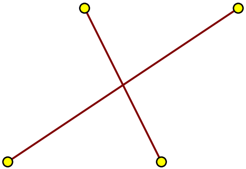

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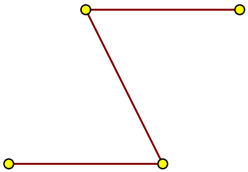

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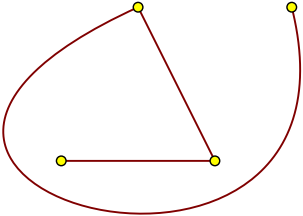

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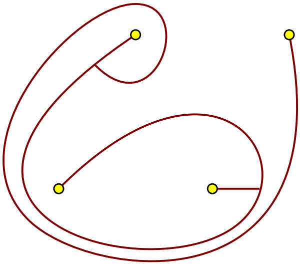
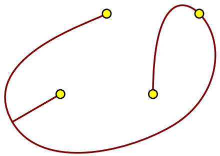
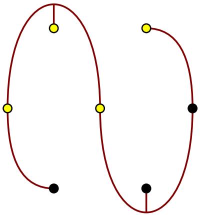


Lifting Trees

The preimage $f^{-1}(T)$ of an allowed tree is not an allowed tree.



Tree T



preimage $f^{-1}(T)$

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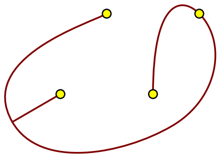

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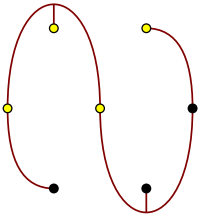
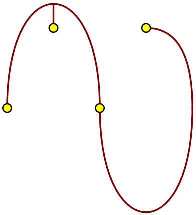


preimage $f^{-1}(T)$



Lift $\lambda_f(T)$

The ***lift*** of T is the subtree of $f^{-1}(T)$ spanned by M.

Lifting Trees

For a given M, lifting under f defines a function

$$\lambda_f : \text{allowed trees} \longrightarrow \text{allowed trees}$$

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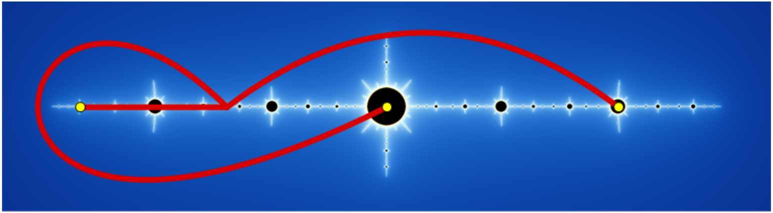
Basic Algorithm: Iterate λ_f and hope to hit the Hubbard tree.

Iterated Lifting for the Airplane

Let $f(z) \approx z^2 - 1.755$ be the airplane polynomial.



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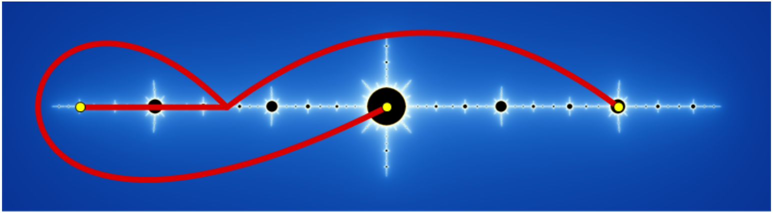

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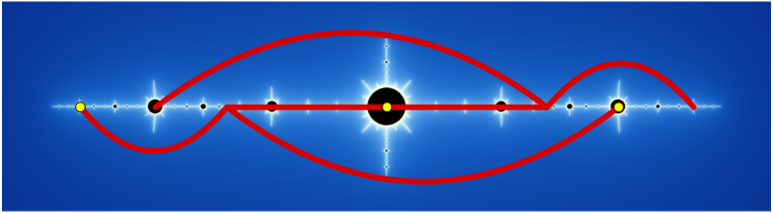

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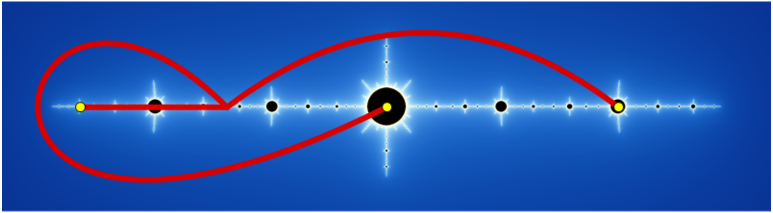

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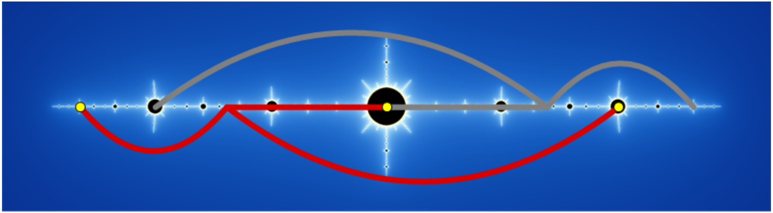

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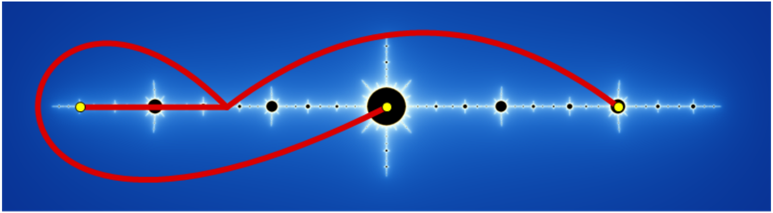

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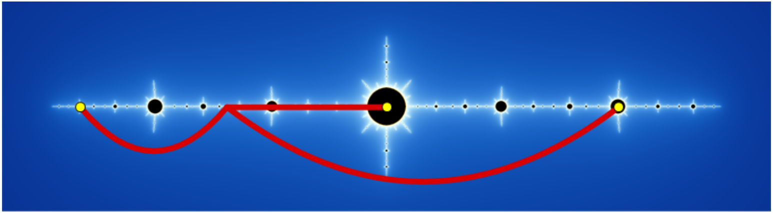

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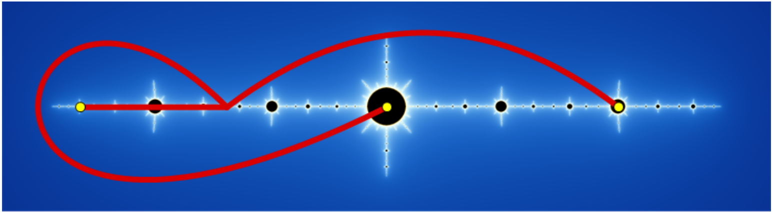

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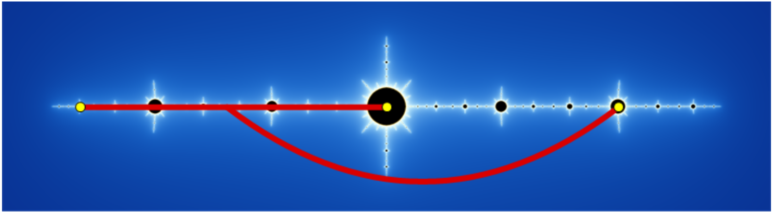

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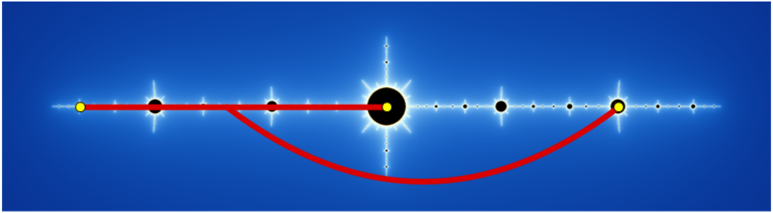

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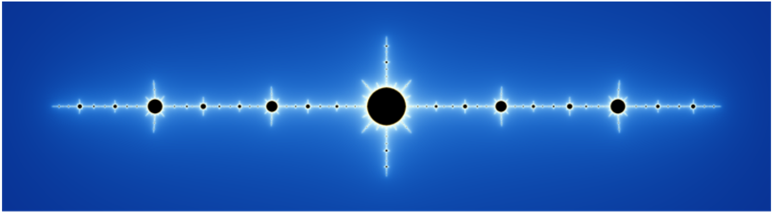

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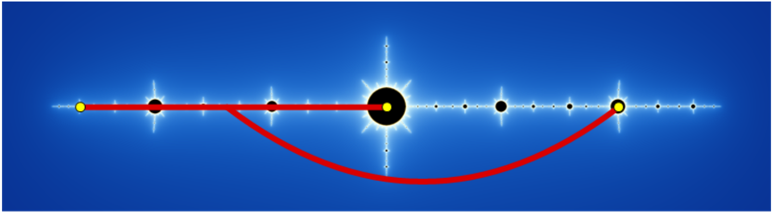

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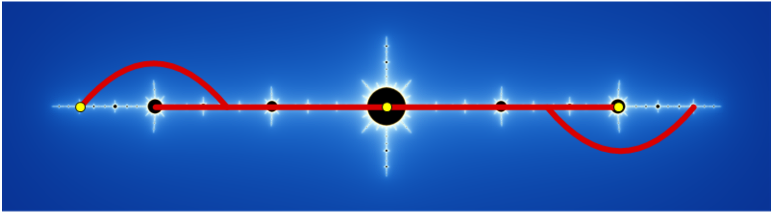

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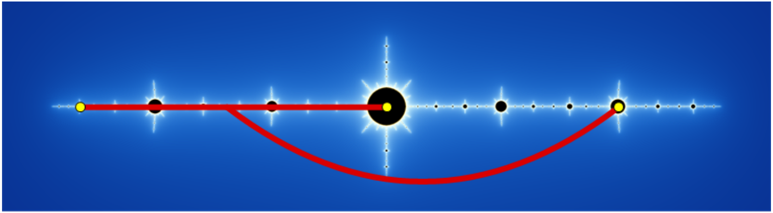

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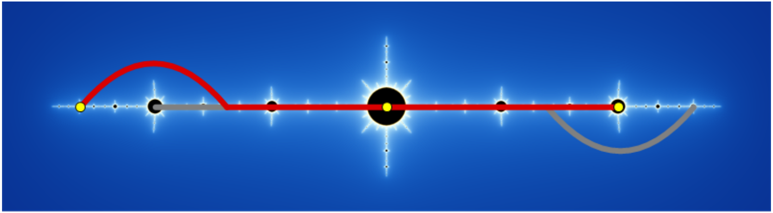

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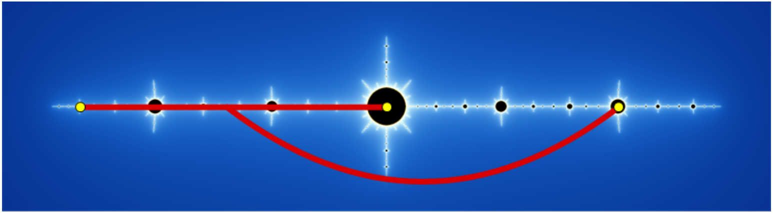

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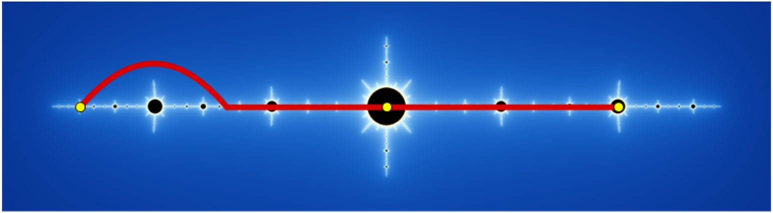

second lift T_2

Iterated Lifting for the Airplane

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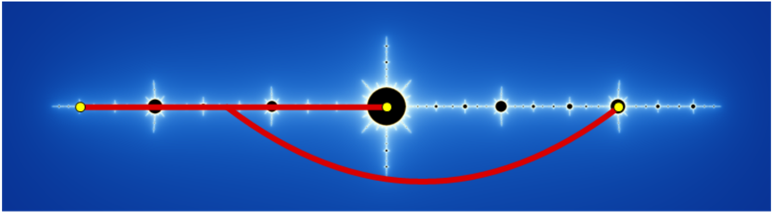

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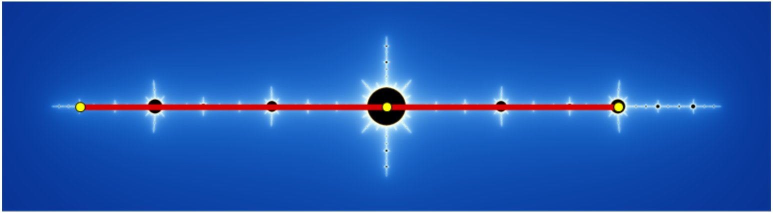

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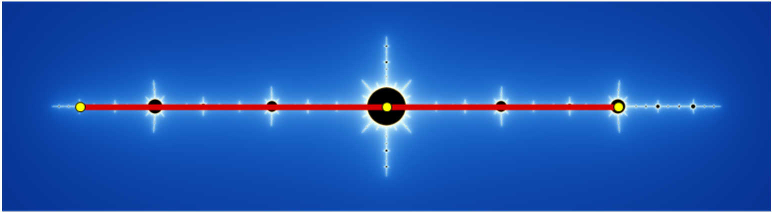

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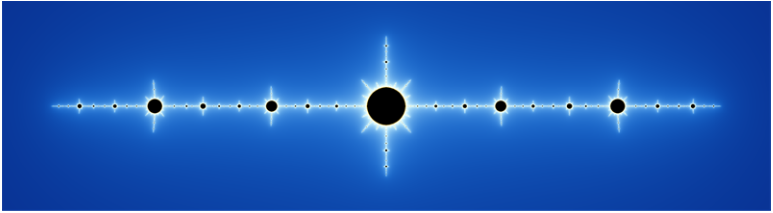

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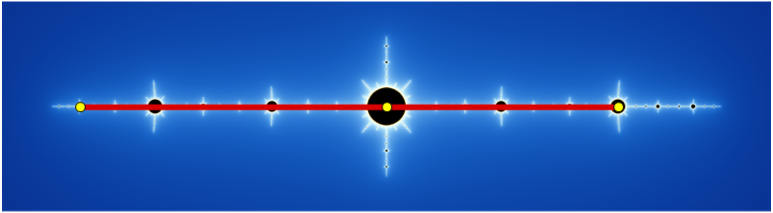

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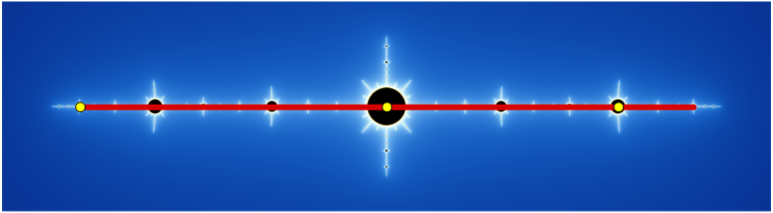

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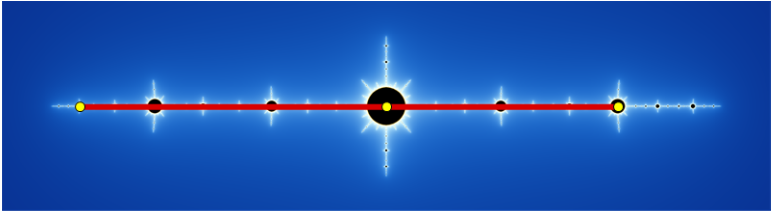

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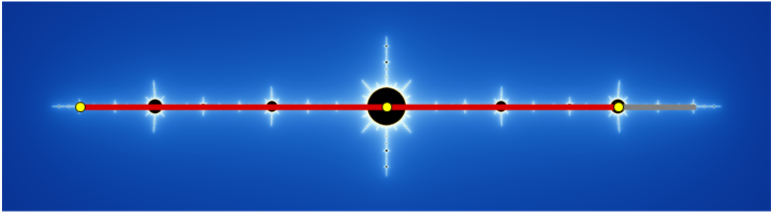

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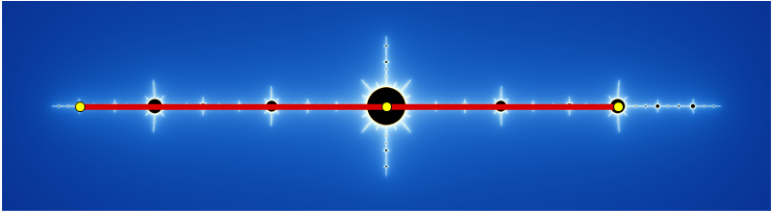

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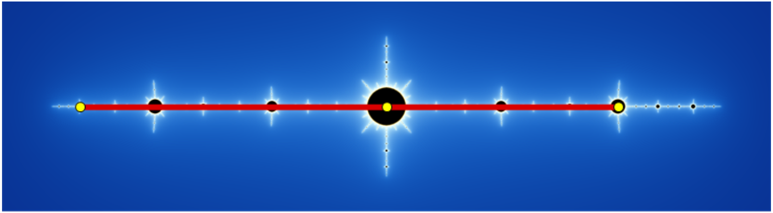

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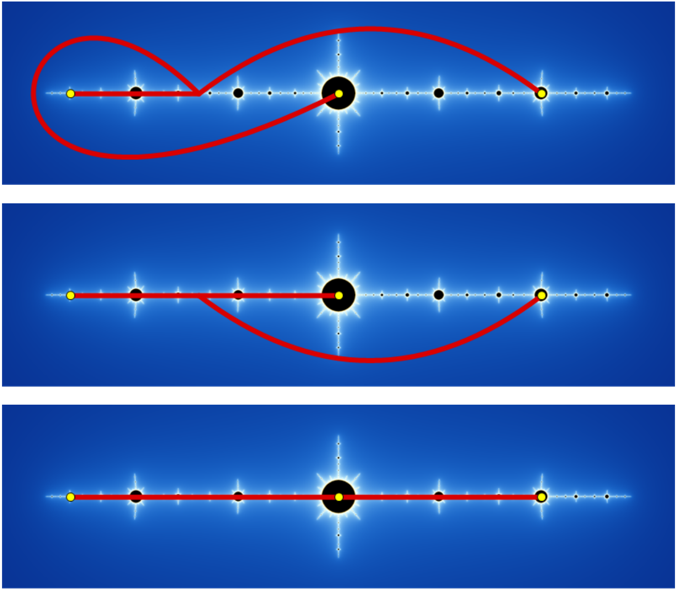
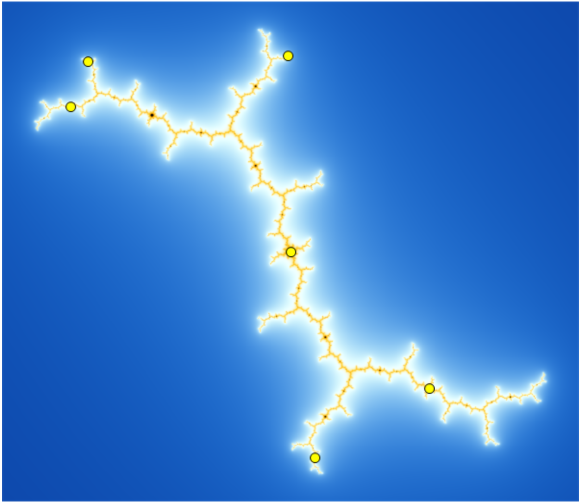


More Marked Points

Things don't get much harder with more marked points.



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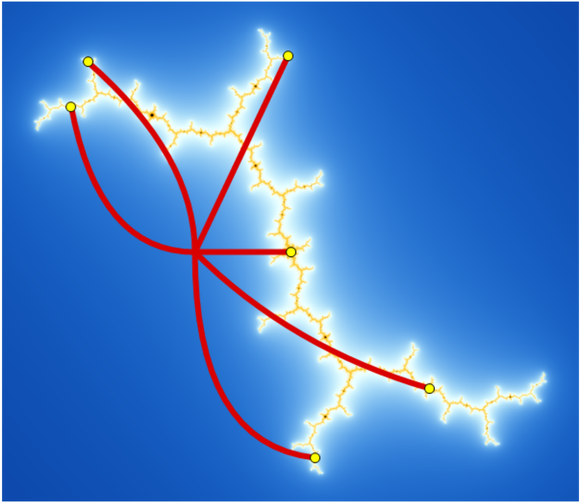

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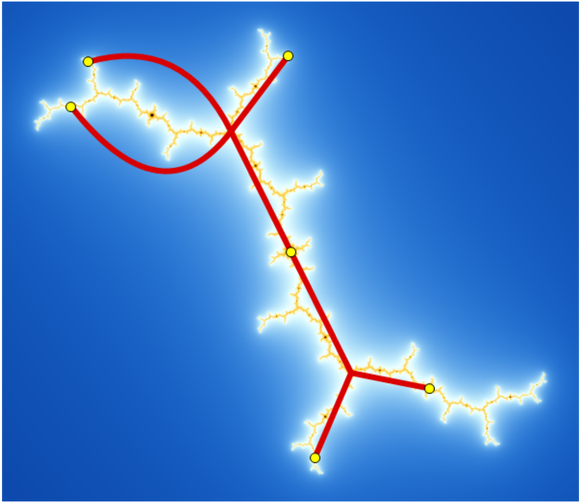

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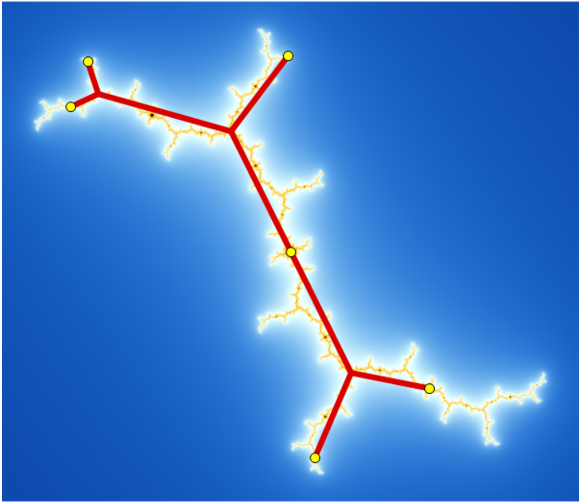

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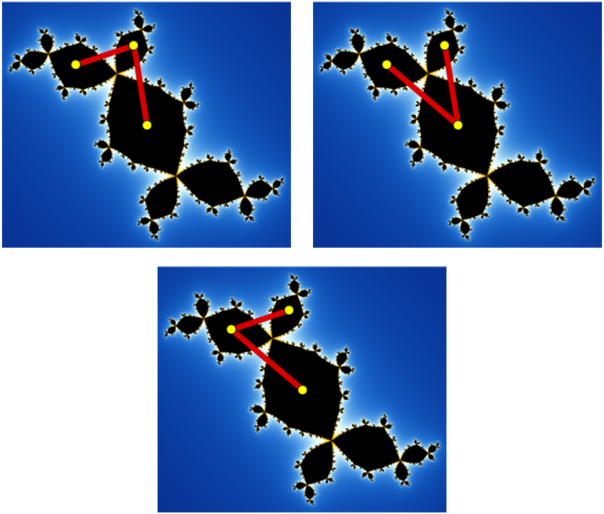

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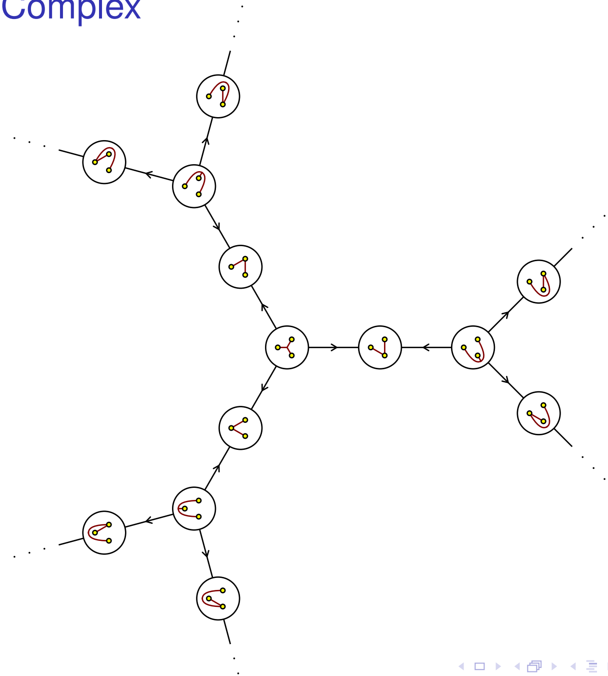
So the algorithm must include a resolution procedure to find the Hubbard tree once we land in the nucleus.

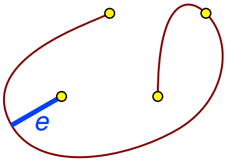
Dynamics of λ_f

The Tree Complex



Collapsing Subforests

Let T be an allowed tree, and let e be an edge of T whose endpoints do not both lie in M.



tree T

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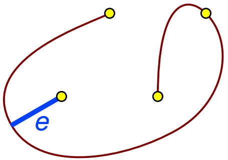

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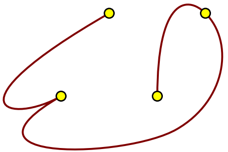

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Then collapsing e to a point yields another allowed tree T/e.

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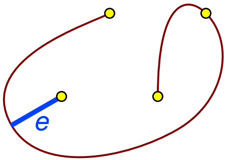

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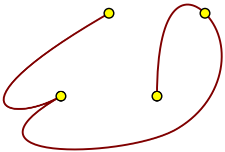

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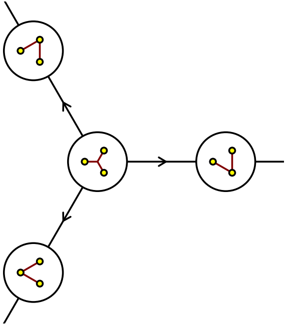
More generally, we can collapse any forest in T as long as no pair of marked points are identified.

The Tree Complex

The ***tree complex*** for M has:

- One vertex for each allowed tree, and
- A directed edge $T \longrightarrow T'$ for each forest collapse.



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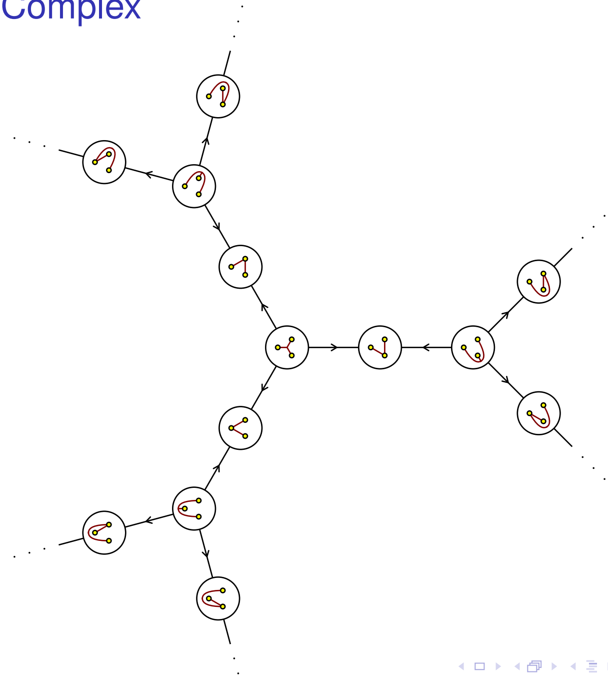
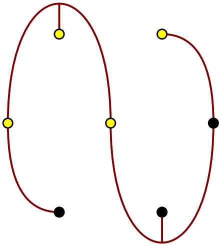


Lifting Forest Collapses

Any forest collapse $T \longrightarrow T'$ lifts to $f^{-1}(T)$.



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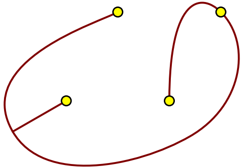

preimage $f^{-1}(T)$ tree T

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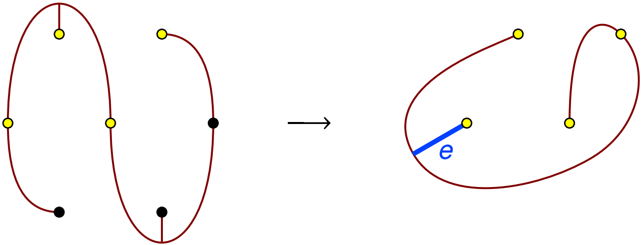

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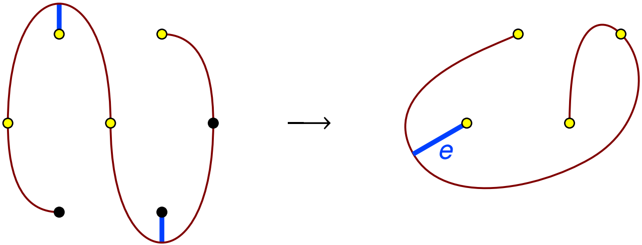

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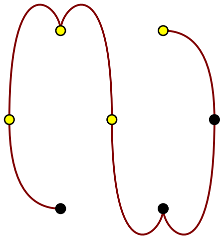

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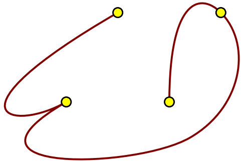

preimage $f^{-1}(T')$

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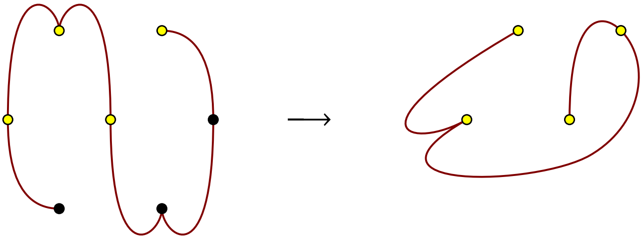

preimage $f^{-1}(T')$ collapsed tree T'

It follows that either

$$\lambda_f(T) \to \lambda_f(T') \qquad \text{or} \qquad \lambda_f(T) = \lambda_f(T').$$

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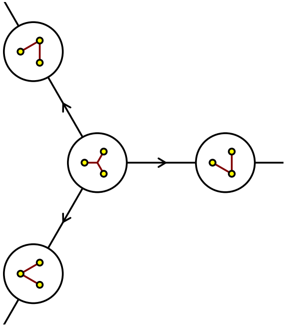
So λ_f induces a non-expanding map on the tree complex. This is the ***lifting map***.



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Proof.

Since the Hubbard tree H is fixed and λ_f is non-expanding, each ball in the complex centered at H maps into itself. Such a ball has finitely many trees. $\square$

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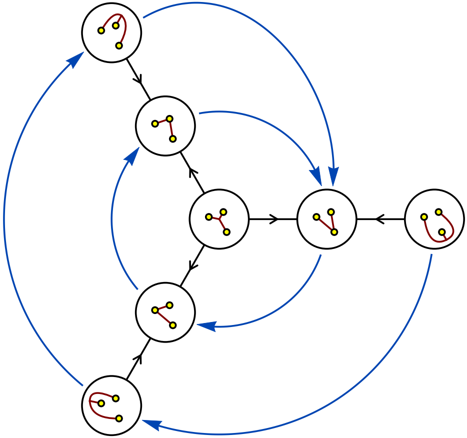
Theorem (BLMW 2019)

The nucleus for λ_f lies within the 2-ball centered at H.

Example: The Rabbit Nucleus

The nucleus for the rabbit is the 1-neighborhood of the Hubbard tree.



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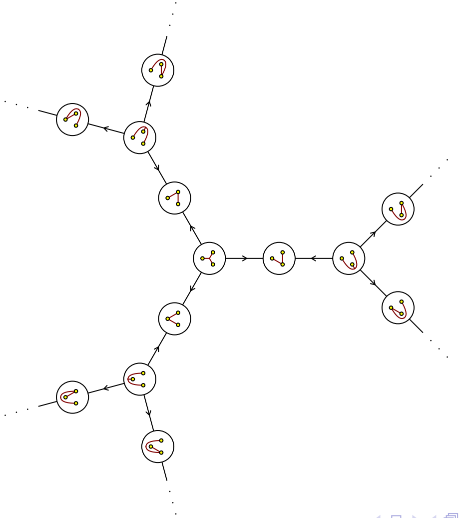
The tree complex is actually the spine of a certain simplicial subdivision of Teichmüller space (discovered by Penner).



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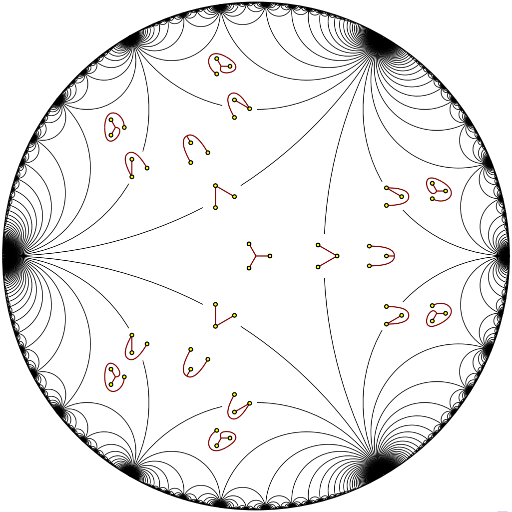

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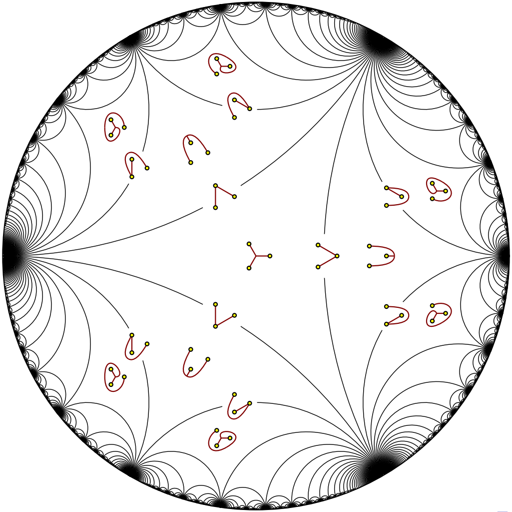
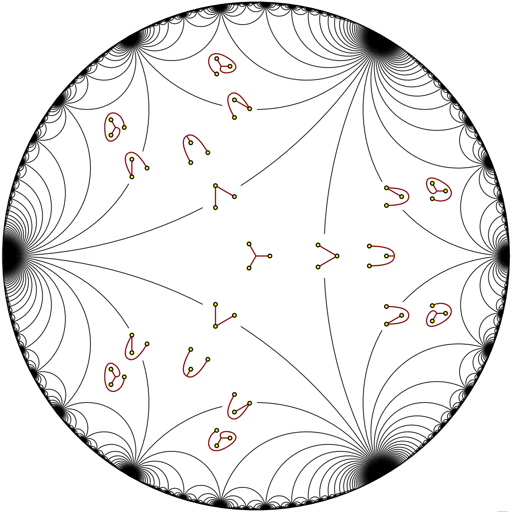
Each tree corresponds to an open simplex. Different points in the simplex correspond to different metrics on the tree.



What's Going On?

The lifting map λ_f seems to be a combinatorial version of Thurston's pullback map.



Finding the Hubbard Tree

The Story So Far

So far: We can iterate lifting until we find a periodic tree.

This gets us within 2 steps of the Hubbard tree.

Questions

1. How do we get to the Hubbard tree itself?
2. How would we even recognize the Hubbard tree if we found it?

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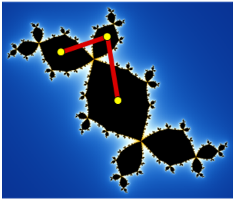 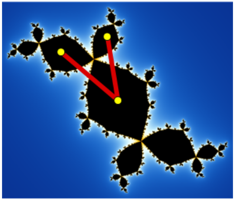 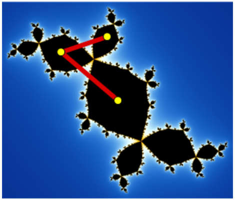

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How do we tell whether an invariant tree T is the Hubbard tree?

Answer

It suffices for there to exist *any* polynomial with Hubbard tree T that induces the same mapping $f^{-1}(T) \to T$.

Poirier's Conditions

Alfredo Poirier completely classified possible Hubbard trees in 1993.

Theorem (Poirier's Conditions)

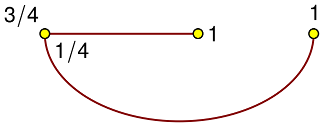
An invariant tree T for (f, M) is a topological Hubbard tree if and only if the following conditions are satisfied:

1. **(Angle Condition)** T has an invariant angle assignment.
2. **(Expanding Condition)** Every forward-invariant subforest of T contains a critical point.

The Angle Condition

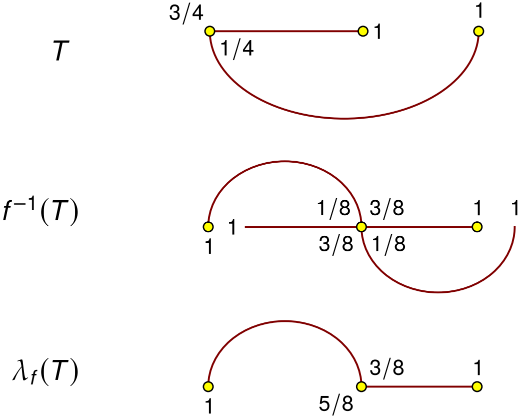
Here is an ***angle assignment*** for a tree T.

T



The Angle Condition

We can ***lift*** the angle assignment to $\lambda_f(T)$.



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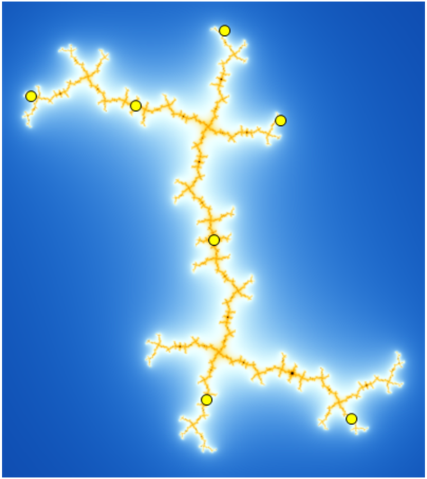
An invariant tree satisfies the ***angle condition*** if there exists an angle assignment that lifts to itself.



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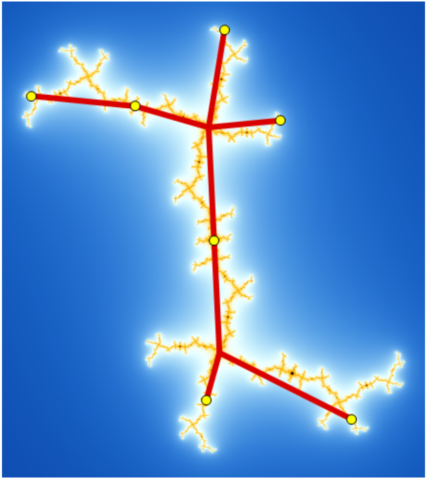

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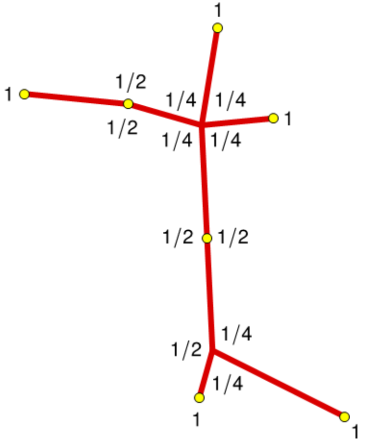

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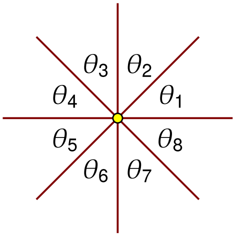

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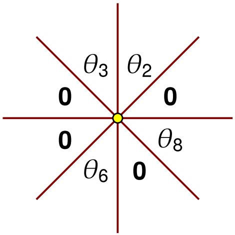

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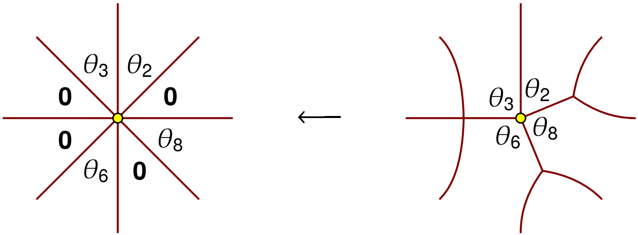


The Expanding Condition

Let T be an invariant tree for (f, M).

A subforest $S \subset T$ is ***forward invariant*** if $f(S) \subset S$. The tree T satisfies the ***expanding condition*** if every such S contains a critical point.

Theorem (BLMW 2019)

Every invariant tree that satisfies the angle condition is adjacent to the Hubbard tree.

Proof.

Collapse the unique maximal invariant forest that contains no critical points. $\square$

The Algorithm

So given an (f, M), the algorithm is as follows:

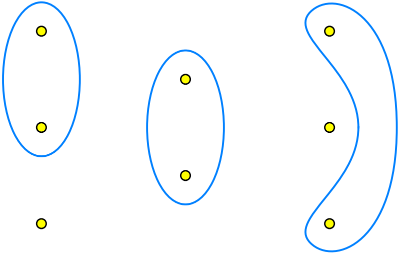
1. Start with any allowed tree and iterate lifting until you find a periodic tree T.
2. Check if T satisfies the angle condition. If it doesn't, move to an adjacent tree T' that does.
3. Check if T' satisfies the expanding condition. If it doesn't, move to an adjacent tree T'' that does.

Then T'' is the topological Hubbard tree.

The Obstructed Case

The Canonical Obstruction

Every obstructed (f, M) has a special collection of curves called the ***canonical obstruction***.



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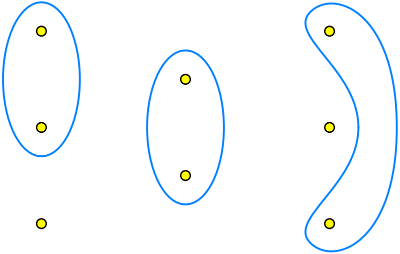

These are the curves whose hyperbolic lengths go to zero.

Pilgrim (2001) proved that the canonical obstruction is fully invariant under f, and is a Thurston obstruction.

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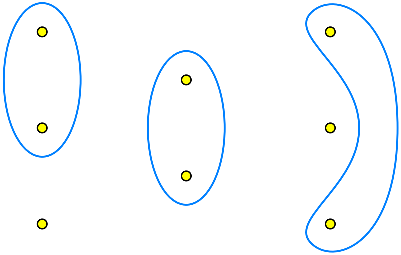

The curves of the canonical obstruction bound disjoint disks.

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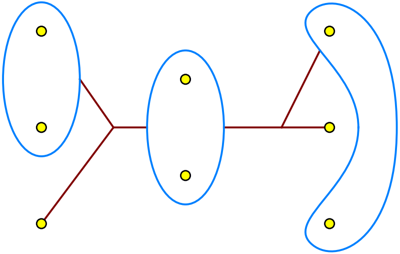

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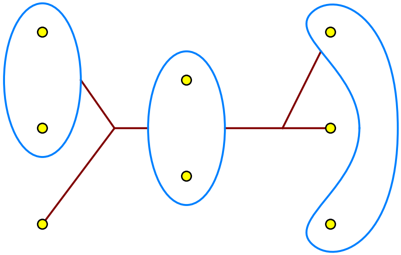


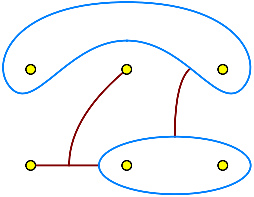
We call this the ***Hubbard bubble tree*** for the obstructed map.

When (f, M) is obstructed, we can use the tree lifting algorithm to find the Hubbard bubble tree.

Finding the Hubbard Bubble Tree

In general, a ***bubble tree*** consists of:

1. Finitely many essential curves in $(\mathbb{C}, M)$ with disjoint interiors.
2. A tree on the exterior of these curves.



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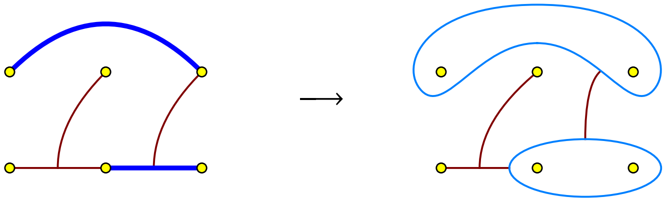

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Theorem (BLMW 2019)

For an obstructed (f, M), the sequence of lifts eventually lands in the 2-neighborhood of the Hubbard bubble tree in the augmented complex.

The End