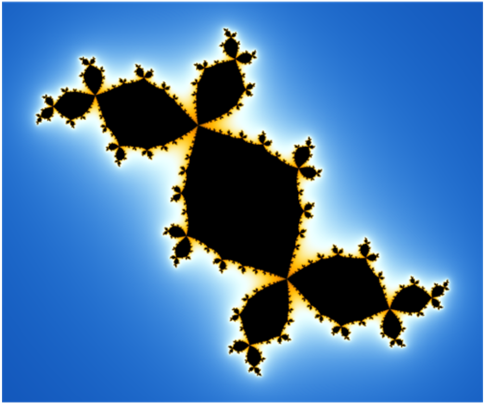
A Geometric Solution to the Twisted Rabbit Problem



Jim Belk, University of St Andrews

Collaborators for this Project

Justin Lanier

Dan Margalit

Becca Winarski

Note: This talk represents work in progress.

Postcritically Finite Polynomials

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial function of degree ≥ 2.

Then f has a finite set C_f of critical points.

We say that f is ***postcritically finite*** if every critical point of f is either periodic or pre-periodic under iteration.

Postcritically Finite Polynomials

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial function of degree ≥ 2.

Then f has a finite set C_f of critical points.

We say that f is ***postcritically finite*** if every critical point of f is either periodic or pre-periodic under iteration.

That is, f is postcritically finite if the ***postcritical set***

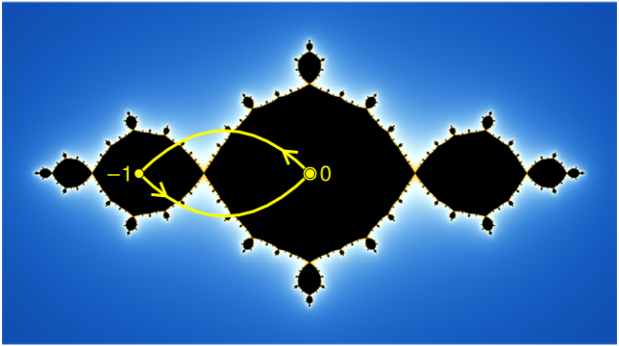
$$P_f = \bigcup_{n \geq 1} f^n(C_f)$$

is a finite set.

Postcritically Finite Polynomials

Example

The polynomial $f(z) = z^2 - 1$ is postcritically finite.

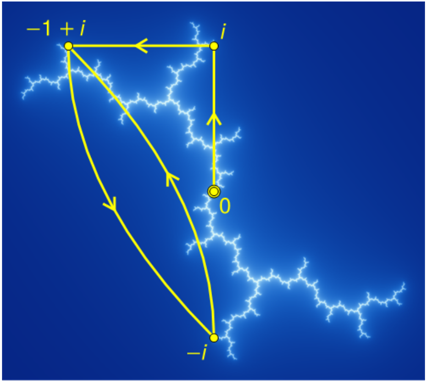


Here $P_f = \{0, -1\}$.

Postcritically Finite Polynomials

Example

The polynomial $f(z) = z^2 + i$ is postcritically finite.



Here $P_f = \{i, -1+i, -i\}$.

Ramification Portrait

Every postcritically finite polynomial has a ***ramification portrait*** that describes the forward orbits of the critical points.

Here's the portrait for $z^2 - 1$:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "2"] & \bullet \arrow[l, bend left] \end{tikzcd}$$

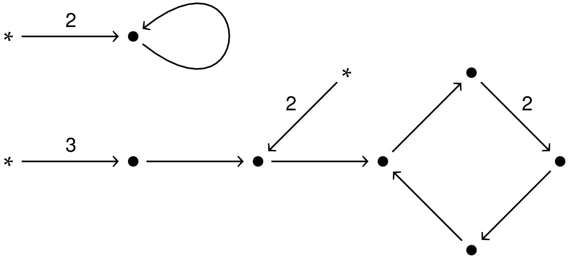
And here's the portrait for $z^2 + i$:

$$\begin{tikzcd} * \arrow[r, "2"] & \bullet \arrow[r] & \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

Ramification Portrait

Every postcritically finite polynomial has a ***ramification portrait*** that describes the forward orbits of the critical points.

Here's a portrait for a polynomial of degree six:



Affine Conjugacy

If $f(z)$ is a postcritically finite polynomial and

$$A(z) = cz + d \qquad (c \neq 0)$$

is an affine function, then

$$g = A \circ f \circ A^{-1}$$

is an ***affine conjugate*** of f. Affine conjugate polynomials have essentially the same dynamics.

Question

What can we say about the affine conjugacy classes of postcritically finite polynomials with a given portrait?

Period Two Quadratics

Example

Up to affine conjugacy, $z^2 - 1$ is the only polynomial realizing

$$\begin{tikzcd} \bullet \arrow[r, bend left, "2"] & \bullet \arrow[l, bend left] \end{tikzcd}$$

To see this, observe that any quadratic polynomial is affine conjugate to one of the form

$$f(z) = z^2 + c.$$

For such a polynomial to realize the above scheme, we need

$$(0^2 + c)^2 + c = 0$$

so $c = 0$ or $c = -1$.

Period Three Quadratics

Example

Up to affine conjugacy, there are *three* polynomials that realize

$$\begin{tikzcd} \bullet \arrow[r, "2"] & \bullet \arrow[r] & \bullet \arrow[ll, bend left] \end{tikzcd}$$

These are the quadratics $f(z) = z^2 + c$, where c is a nonzero solution to

$$((0^2 + c)^2 + c)^2 + c = 0.$$

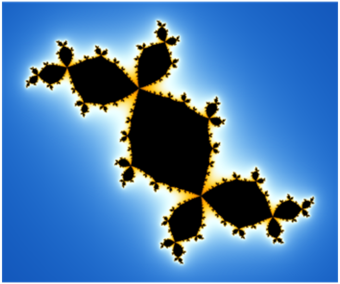
The solutions are:

The Rabbit: $c \approx -0.1226 + 0.7449i$

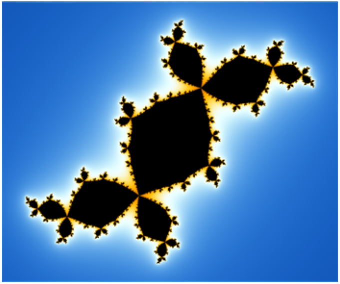
The Corabbit: $c \approx -0.1226 - 0.7449i$

The Airplane: $c \approx -1.754878$

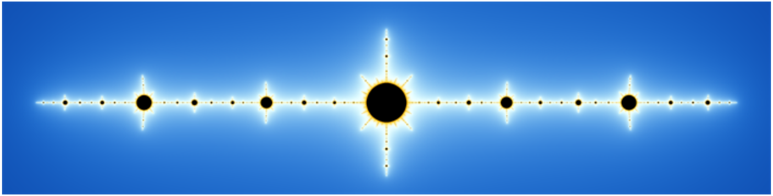
Period Three Quadratics



rabbit

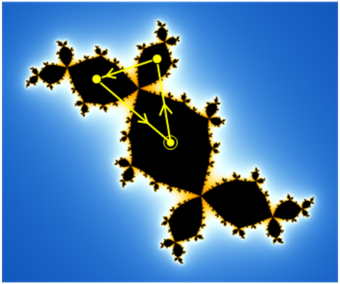


corabbit

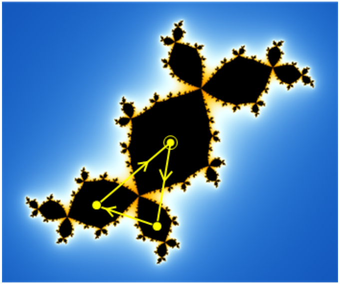


airplane

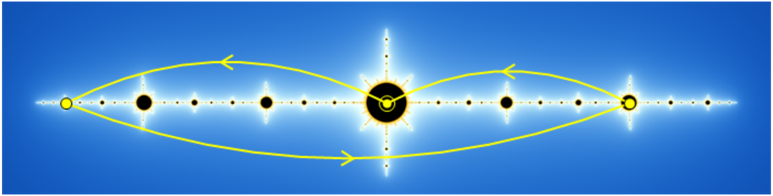
Period Three Quadratics



rabbit



corabbit



airplane

Finiteness Theorem

There are at most finitely many affine conjugacy classes of polynomials realizing any given ramification portrait.

For quadratics with periodic critical point, the number of possibilities grows exponentially as the period increases.

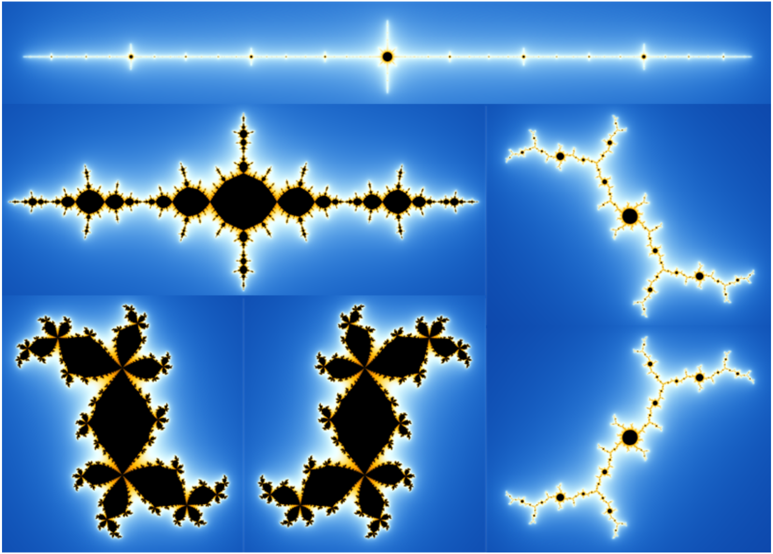
period	1	2	3	4	5	6	7	$\cdots$
# of c-values	1	1	3	6	15	27	63	$\cdots$

Period 3: Roots of $c^3 + 2c^2 + c + 1$.

Period 4: Roots of $c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1$.

$\vdots$

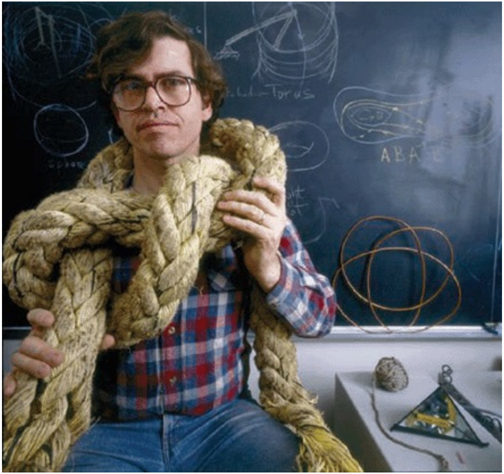
Period Four Quadratics



Thurston's Theorem

Thurston's Theorem

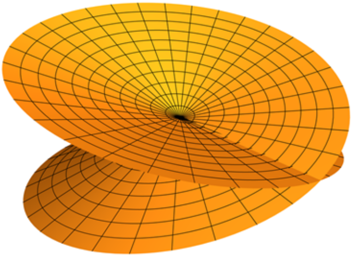
Bill Thurston introduced a purely topological viewpoint towards affine conjugacy classes of postcritically finite polynomials.



Thurston's Theorem

Bill Thurston introduced a purely topological viewpoint towards affine conjugacy classes of postcritically finite polynomials.

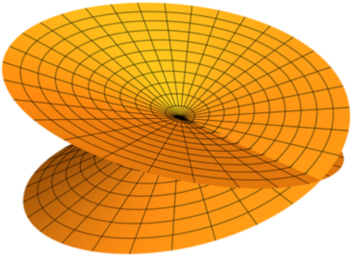
Thurston defined a ***topological polynomial*** to be any orientation-preserving branched cover $F : \mathbb{C} \to \mathbb{C}$ of finite degree.



Thurston's Theorem

Bill Thurston introduced a purely topological viewpoint towards affine conjugacy classes of postcritically finite polynomials.

Thurston defined a ***topological polynomial*** to be any orientation-preserving branched cover $F\colon \mathbb{C} \to \mathbb{C}$ of finite degree.



The points at which F is not locally a homeomorphism are called ***critical points***.

F is ***postcritically finite*** if every critical point is periodic or pre-periodic.

Thurston's Theorem

Two postcritically finite topological polynomials are ***Thurston equivalent*** if they are isotopic.

Thurston's Theorem

Two postcritically finite topological polynomials are ***Thurston equivalent*** if they are isotopic.

Precise Definition

That is, two postcritically finite topological polynomials

$$F_0 : \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad F_1 : \mathbb{C} \longrightarrow \mathbb{C}$$

with the same ramification portrait are ***Thurston equivalent*** if there exists a homotopy

$$F_t : \mathbb{C} \longrightarrow \mathbb{C} \qquad (0 \leq t \leq 1)$$

such that each F_t is a postcritically finite topological polynomial with the same portrait as F_0 and F_1.

Thurston's Theorem

Two postcritically finite topological polynomials are ***Thurston equivalent*** if they are isotopic.

Thurston's Theorem

Two postcritically finite topological polynomials are ***Thurston equivalent*** if they are isotopic.

Thurston Rigidity

Two postcritically finite polynomials are Thurston equivalent if and only if they are conjugate by an affine function.

Thurston's Theorem

Let $F : \mathbb{C} \to \mathbb{C}$ *be a topological polynomial. If every critical point of* F *is periodic, then* F *is Thurston equivalent to a polynomial.*

Thurston's Theorem

Two postcritically finite topological polynomials are ***Thurston equivalent*** if they are isotopic.

Thurston Rigidity

Two postcritically finite polynomials are Thurston equivalent if and only if they are conjugate by an affine function.

Thurston's Theorem

Let $F : \mathbb{C} \to \mathbb{C}$ be a topological polynomial. If every critical point of F is periodic, then F is Thurston equivalent to a polynomial.

Note 1: If F has pre-periodic critical points then it may be ***obstructed*** (i.e. not Thurston equivalent to a polynomial).

Thurston's Theorem

Two postcritically finite topological polynomials are ***Thurston equivalent*** if they are isotopic.

Thurston Rigidity

Two postcritically finite polynomials are Thurston equivalent if and only if they are conjugate by an affine function.

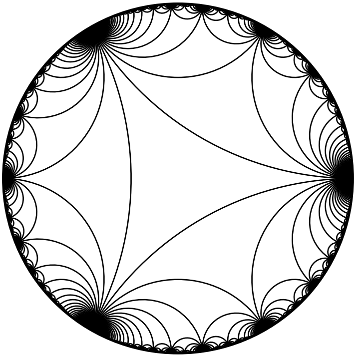
Thurston's Theorem

Let $F \colon \mathbb{C} \to \mathbb{C}$ be a topological polynomial. If every critical point of F is periodic, then F is Thurston equivalent to a polynomial.

Note 2: These are special cases of Thurston's general theorem, which involves postcritically finite branched covers of a sphere.

Thurston's Theorem

The proof involves the ***Teichmüller space*** $\mathcal{T}$ of marked complex structures on a plane with n punctures.



Thurston finds conditions under which the ***pullback map*** $\sigma_F : \mathcal{T} \to \mathcal{T}$ induced by F has a unique fixed point in $\mathcal{T}$.

Twisted Rabbits

Twisted Rabbits

John Hubbard observed that Thurston's proof isn't algorithmic.

Question (Hubbard 1983)

Given a topological polynomial $F : \mathbb{C} \to \mathbb{C}$ with periodic critical points, how do we determine which polynomial f is Thurston equivalent to F?

Twisted Rabbits

For example, suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a topological polynomial with portrait

$$\begin{tikzcd} \bullet \arrow[r, "2"] & \bullet \arrow[r] & \bullet \arrow[ll, bend left] \end{tikzcd}$$

By Thurston's theorem, F is Thurston equivalent to exactly one of

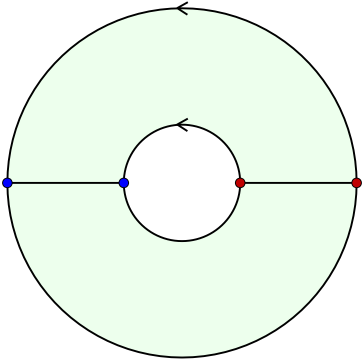
- The rabbit,
- The corabbit, or
- The airplane.

How do we tell which one?

Hubbard gave a very specific example of this question.

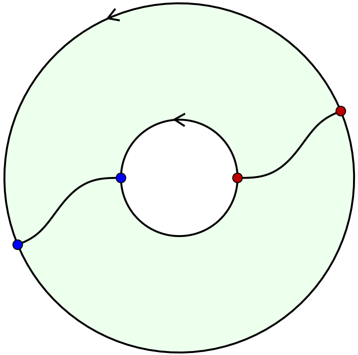
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



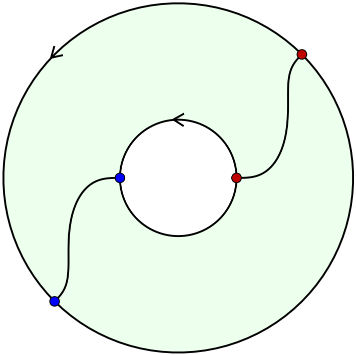
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



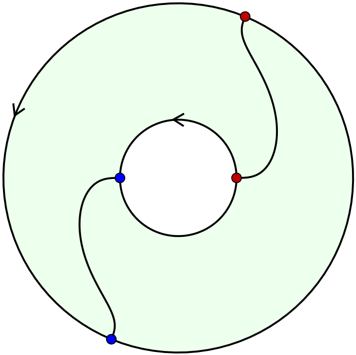
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



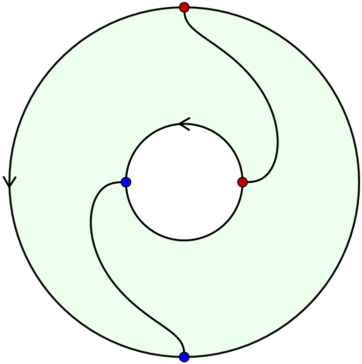
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



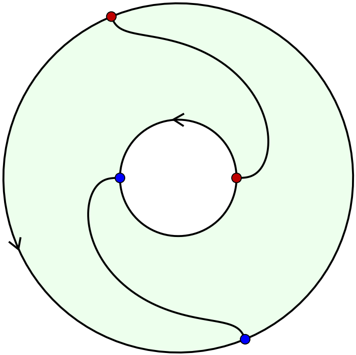
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



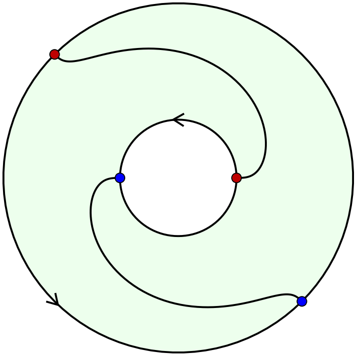
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



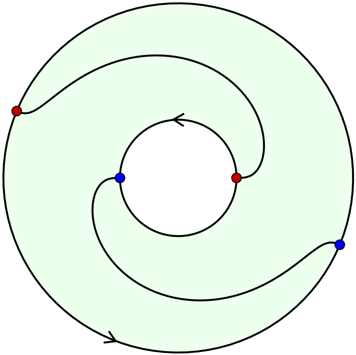
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



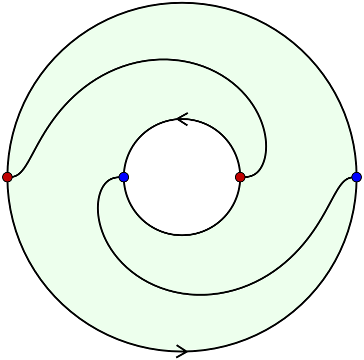
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



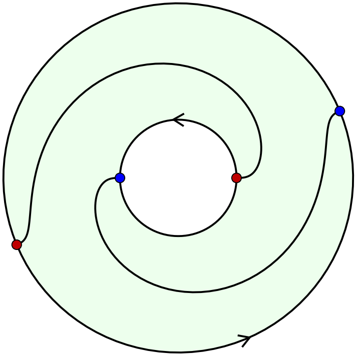
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



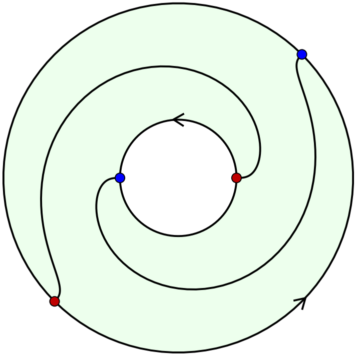
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



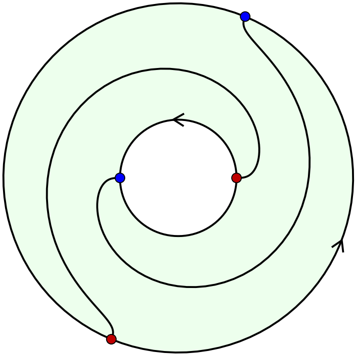
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



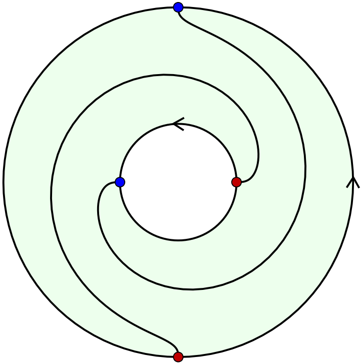
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



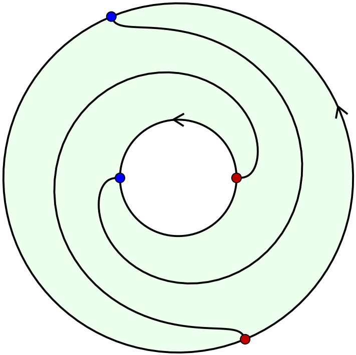
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



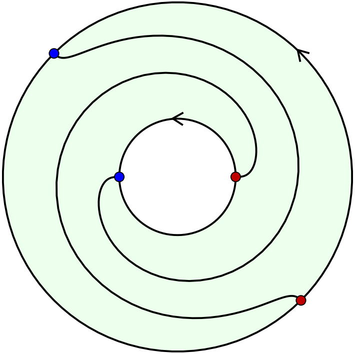
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



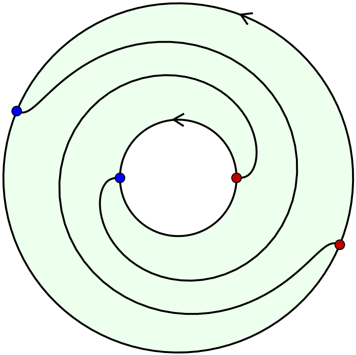
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



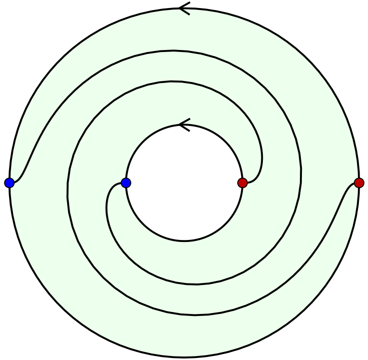
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



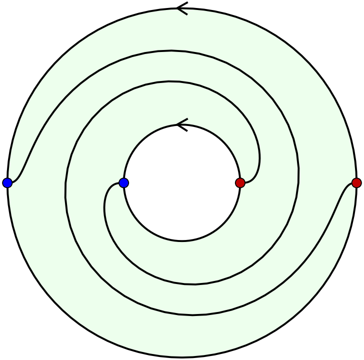
Twisted Rabbits

Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



Twisted Rabbits

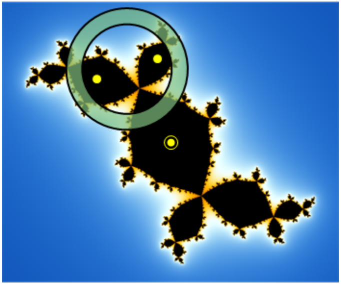
Recall that a ***Dehn twist*** is the following homeomorphism of an annulus.



Note that this homeomorphism is the identity on both boundary circles (and outside the annulus).

Twisted Rabbits

Let $f(z) = z^2 + c$ be the rabbit polynomial, and let $h\colon \mathbb{C} \to \mathbb{C}$ be a Dehn twist around an annulus that surrounds the points in the ears:

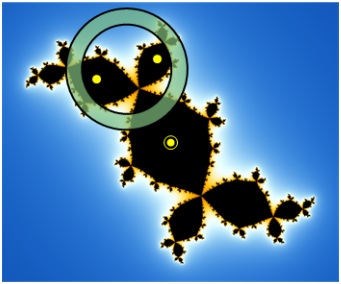


Then $F = h \circ f$ is a topological quadratic whose critical point has period 3.

Twisted Rabbit Problem: Is F Thurston equivalent to the rabbit, the corabbit, or the airplane?

Twisted Rabbits

Let $f(z) = z^2 + c$ be the rabbit polynomial, and let $h\colon \mathbb{C} \to \mathbb{C}$ be a Dehn twist around an annulus that surrounds the points in the ears:



Then $F = h \circ f$ is a topological quadratic whose critical point has period 3.

Twisted Rabbit Problem: Is F Thurston equivalent to the rabbit, the corabbit, or the airplane? What about $F_k = h^k \circ f$?

Twisted Rabbits

The twisted rabbit problem was solved by Laurent Bartholdi and Volodomyr Nekrashevych in 2006 using methods from group theory.

Specifically, their solution uses ***iterated monodromy groups***, which are certain synchronous automata groups that can be associated to critically periodic topological quadratics.

Twisted Rabbits

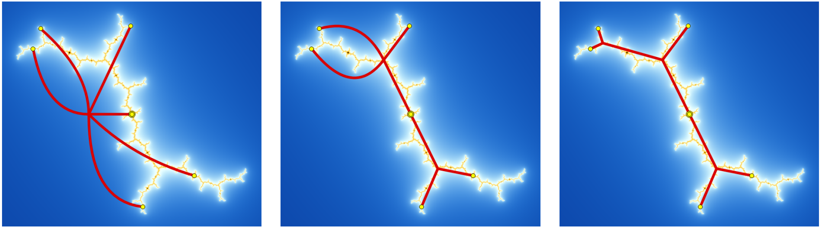
The twisted rabbit problem was solved by Laurent Bartholdi and Volodomyr Nekrashevych in 2006 using methods from group theory.

Unfortunately, the algebra gets quite complicated, so their methods can be difficult to apply when the portrait involves 4 or more points.

Twisted Rabbits

We give an entirely geometric solution to the twisted rabbit problem using a lifting procedure on trees.



The procedure is quite fast and does not appear to increase in complexity significantly when the portrait gets more complicated.

Lifting Trees

Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.

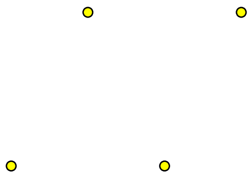
Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.



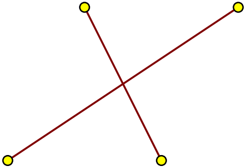
Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.



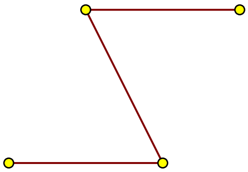
Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.



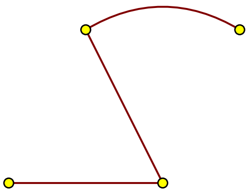
Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.



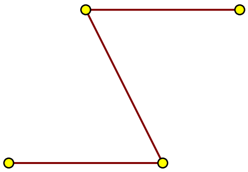
Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.



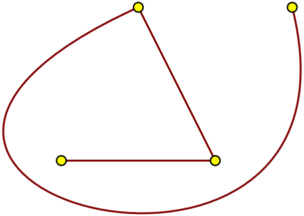
Lifting Trees

Let $F\colon \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.



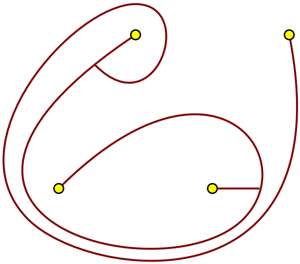
Lifting Trees

Let $F : \mathbb{C} \to \mathbb{C}$ be a postcritically finite topological polynomial.

An ***allowed tree*** for F is a finite tree $T \subset \mathbb{C}$ such that:

1. T contains the postcritical set P_f, and
2. Every leaf of T lies in P_f.

Isotopic trees are considered the same.

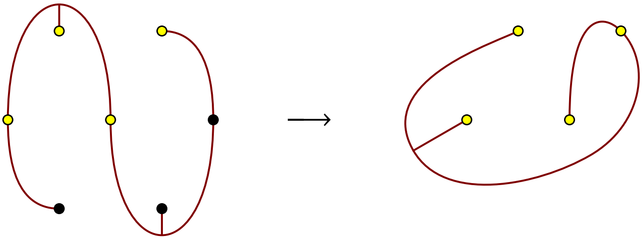


Lifting Trees

If T is an allowed tree for F, then $F^{-1}(T)$ is a tree, and the map

$$F\colon F^{-1}(T) \longrightarrow T$$

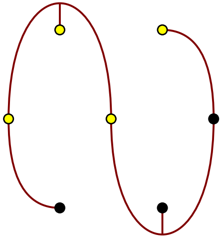
determines F up to homotopy.



preimage $F^{-1}(T)$ allowed tree T

Lifting Trees

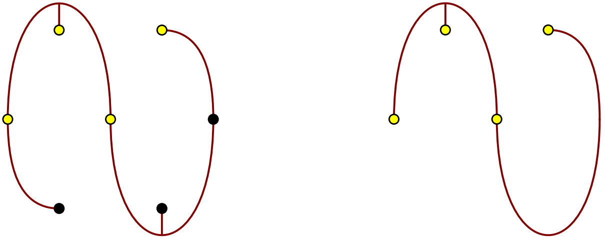
Note: The preimage $F^{-1}(T)$ of an allowed tree is not usually an allowed tree, since not all of its leaves are in P.



preimage $F^{-1}(T)$

Lifting Trees

Note: The preimage $F^{-1}(T)$ of an allowed tree is not usually an allowed tree, since not all of its leaves are in P.

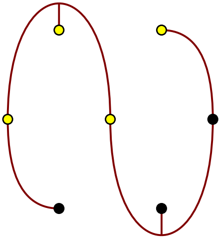


preimage $F^{-1}(T)$ Lift of T

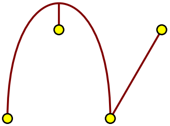
The ***lift*** of an allowed tree T is the (unique) allowed subtree of $F^{-1}(T)$.

Lifting Trees

Note: The preimage $F^{-1}(T)$ of an allowed tree is not usually an allowed tree, since not all of its leaves are in P.



preimage $F^{-1}(T)$

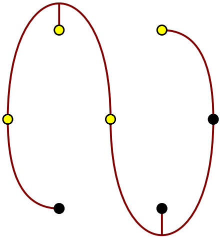


Lift of T

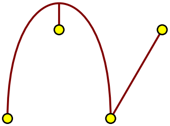
The ***lift*** of an allowed tree T is the (unique) allowed subtree of $F^{-1}(T)$.

Lifting Trees

Note: The preimage $F^{-1}(T)$ of an allowed tree is not usually an allowed tree, since not all of its leaves are in P.



preimage $F^{-1}(T)$



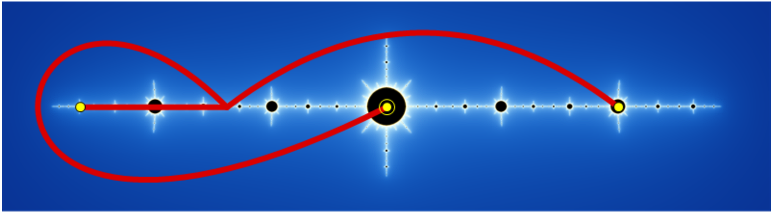
Lift of T

The ***lift*** of an allowed tree T is the (unique) allowed subtree of $F^{-1}(T)$.

Idea: Iterate this lifting procedure to obtain information about F.

Iterated Lifting for the Airplane

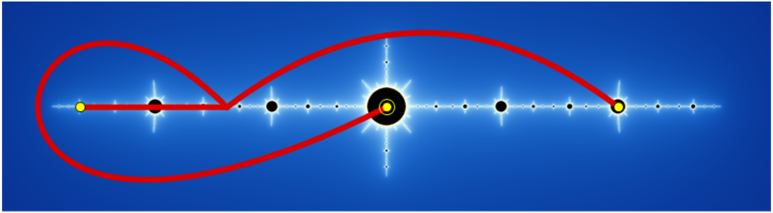
Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



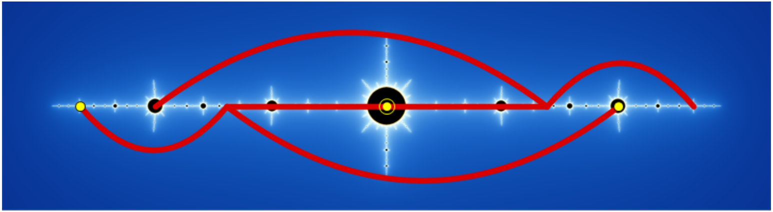
original tree T_0

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



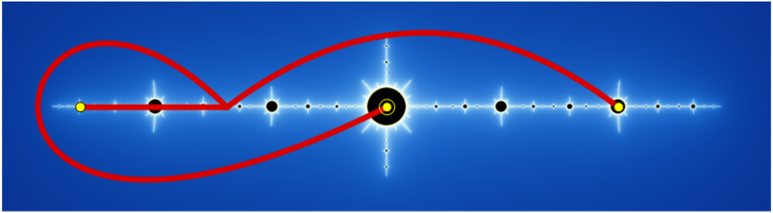
original tree T_0



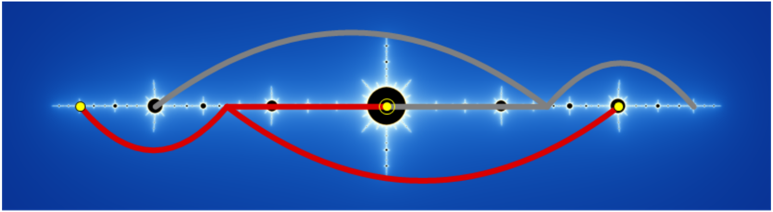
preimage $f^{-1}(T_0)$

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



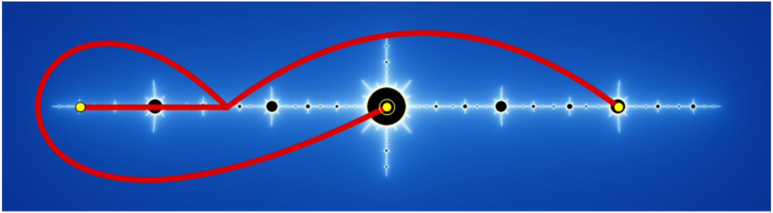
original tree T_0



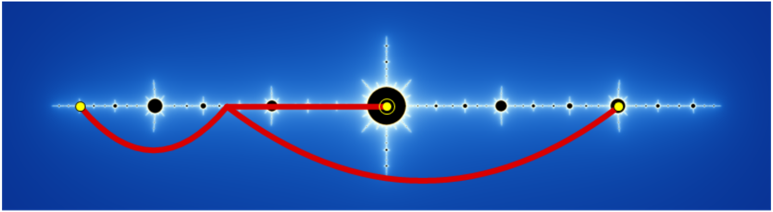
preimage $f^{-1}(T_0)$

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



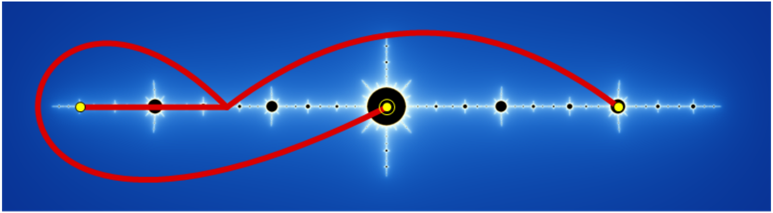
original tree T_0



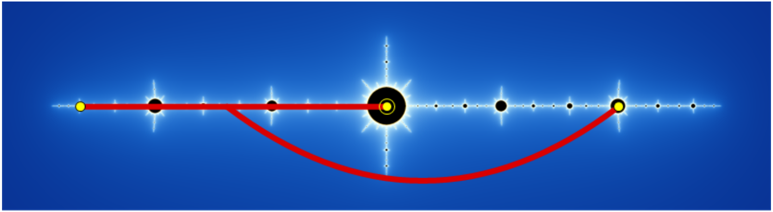
lift of T_0

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



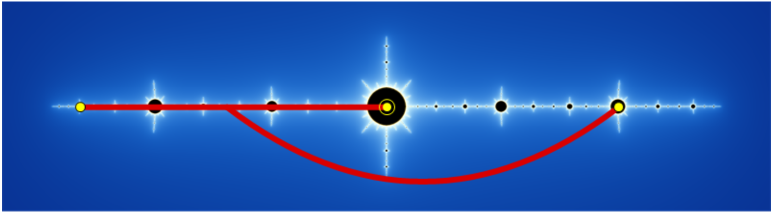
original tree T_0



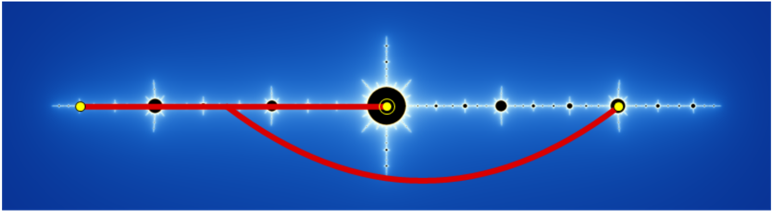
lift of T_0

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



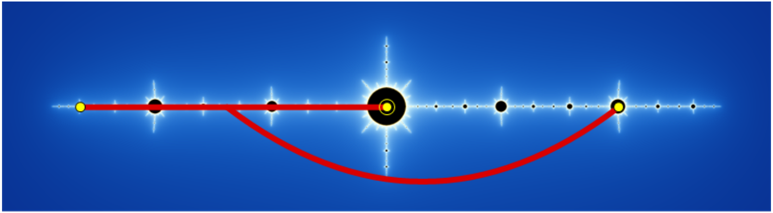
first lift T_1



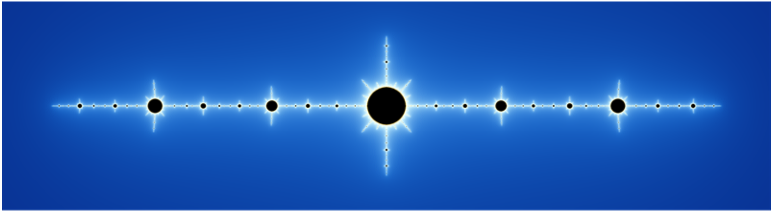
lift of T_0

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.

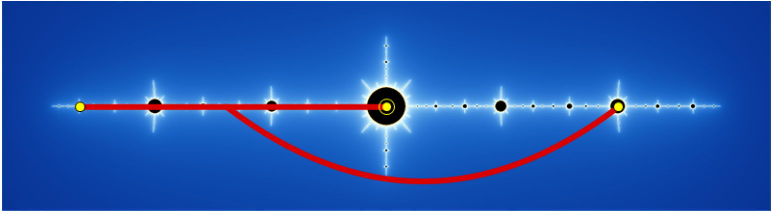


first lift T_1

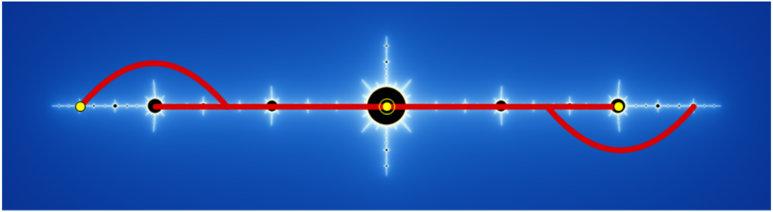


Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



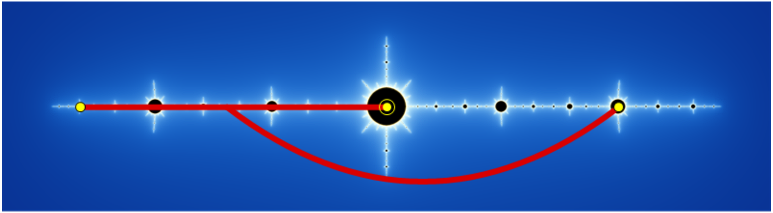
first lift T_1



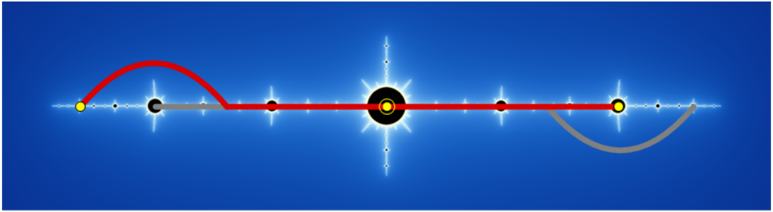
preimage $f^{-1}(T_1)$

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \to \mathbb{C}$ be the airplane polynomial.



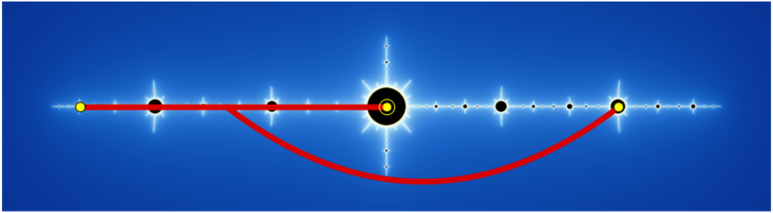
first lift T_1



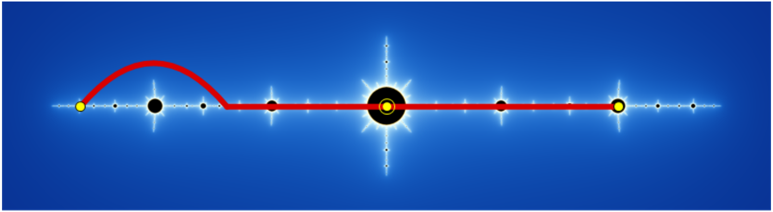
preimage $f^{-1}(T_1)$

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



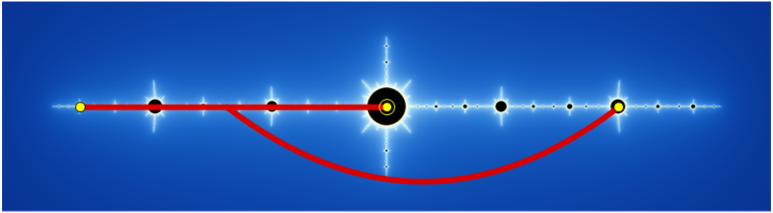
first lift T_1



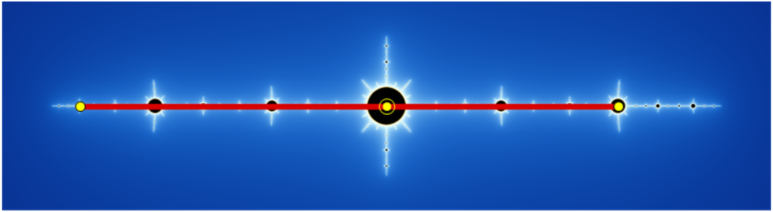
second lift T_2

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



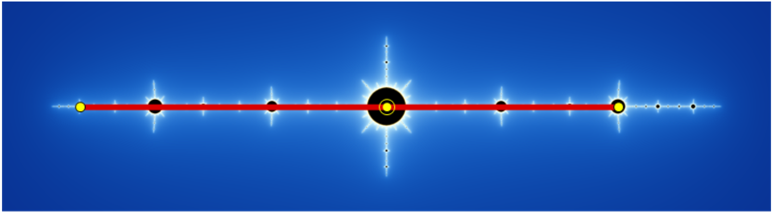
first lift T_1



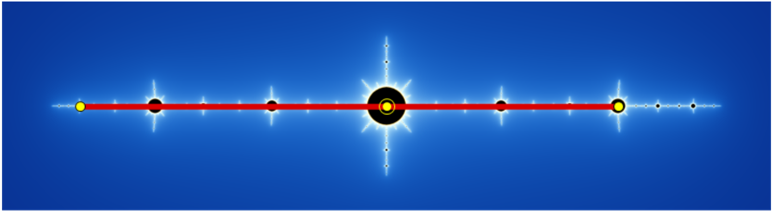
second lift T_2

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.



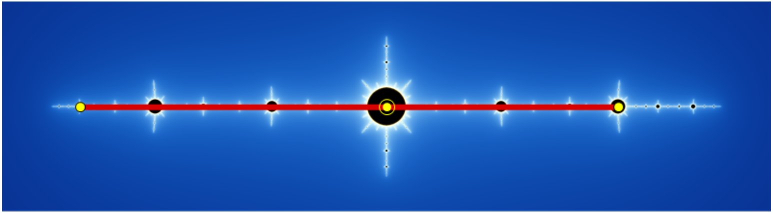
second lift T_2



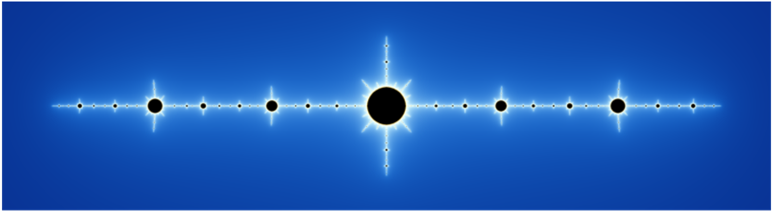
second lift T_2

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.

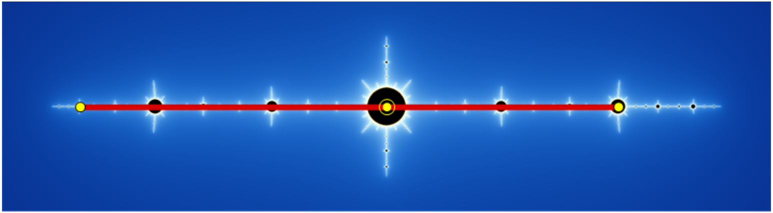


second lift T_2

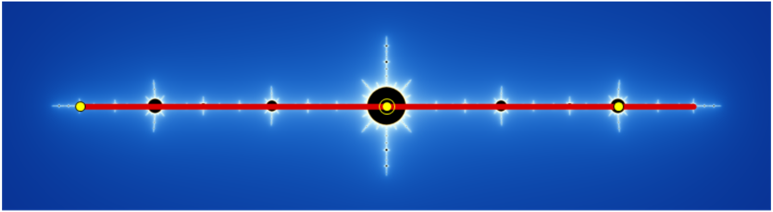


Iterated Lifting for the Airplane

Let $f : \mathbb{C} \to \mathbb{C}$ be the airplane polynomial.



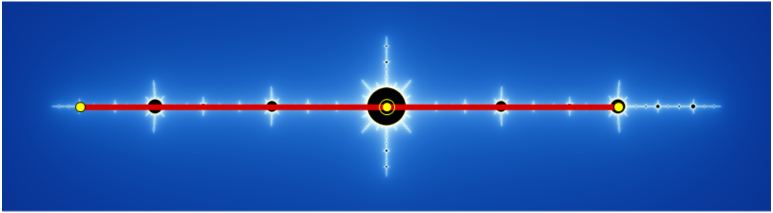
second lift T_2



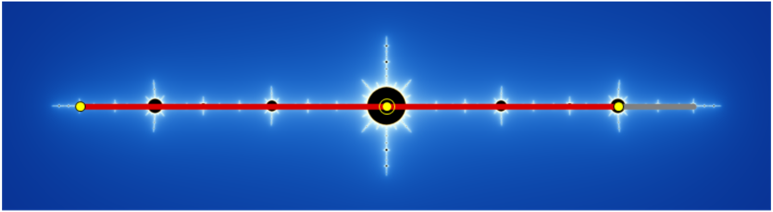
preimage $f^{-1}(T_2)$

Iterated Lifting for the Airplane

Let $f\colon \mathbb{C} \to \mathbb{C}$ be the airplane polynomial.



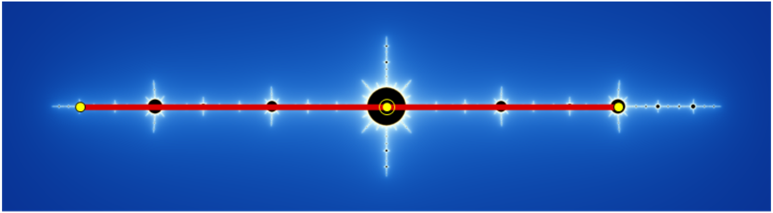
second lift T_2



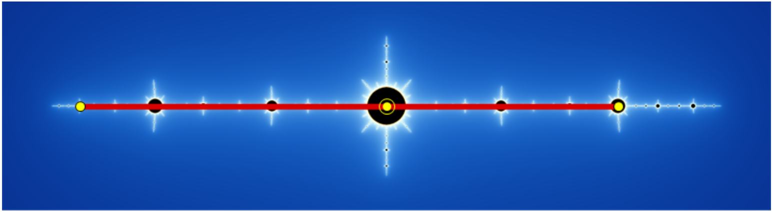
preimage $f^{-1}(T_2)$

Iterated Lifting for the Airplane

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the airplane polynomial.

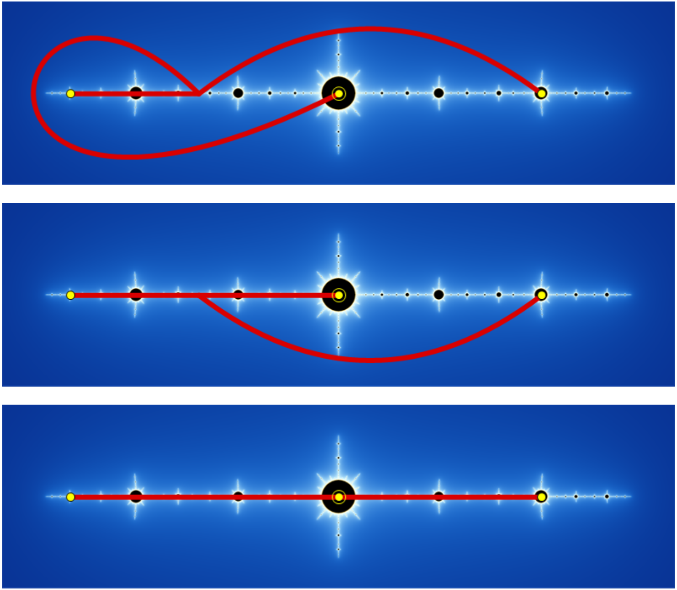


second lift T_2

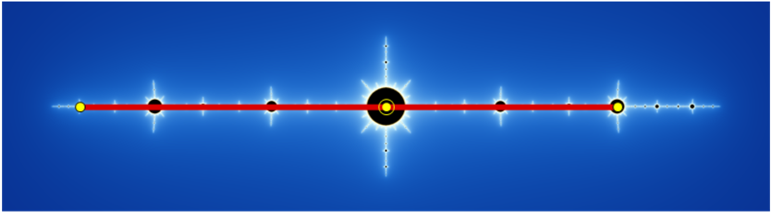


third lift T_3

Iterated Lifting for the Airplane



Iterated Lifting for the Airplane



The tree above is known as the ***Hubbard Tree*** T_{Hub} for the airplane polynomial.

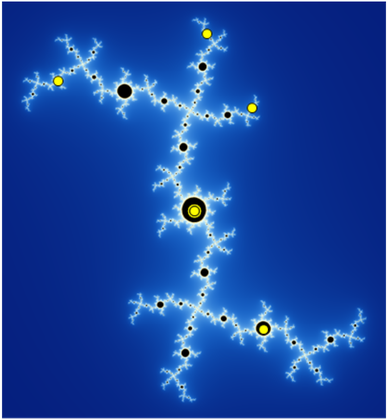
Airplane Theorem (BLMW 2018)

For any allowed tree T_0, the sequence $\{T_n\}$ of lifts under the airplane polynomial arrives at T_{Hub} after finitely many steps.

Hubbard Trees

Theorem (Hubbard and Douady, 1981)

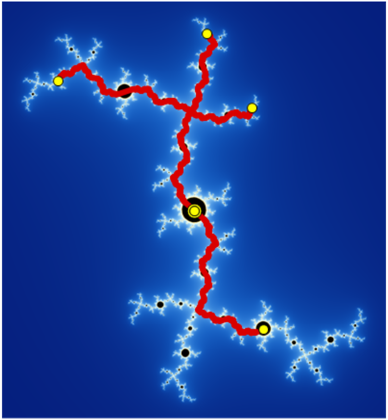
For any postcritically finite polynomial $f(z)$, there exists an allowed tree T for which $f(T) \subseteq T$.



Hubbard Trees

Theorem (Hubbard and Douady, 1981)

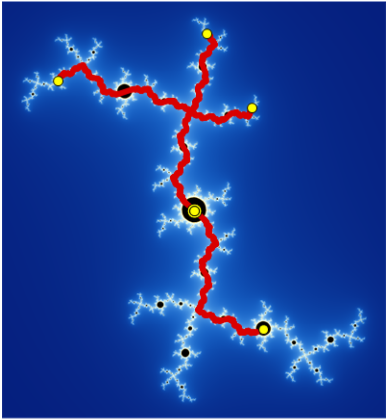
For any postcritically finite polynomial $f(z)$, there exists an allowed tree T for which $f(T) \subseteq T$.



Hubbard Trees

Theorem (Hubbard and Douady, 1981)

For any postcritically finite polynomial $f(z)$, there exists an allowed tree T for which $f(T) \subseteq T$.



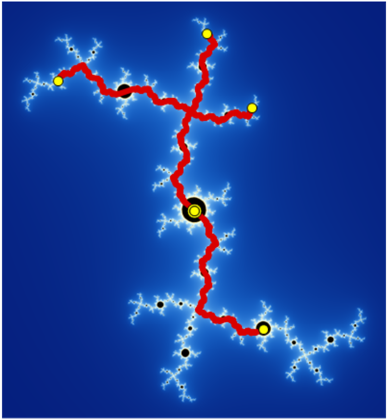
This is the ***Hubbard tree*** for $f(z)$.

Note that $T \subseteq f^{-1}(T)$, so the Hubbard tree is a fixed point of the lifting operation.

Hubbard Trees

Theorem (Hubbard and Douady, 1981)

For any postcritically finite polynomial $f(z)$, there exists an allowed tree T for which $f(T) \subseteq T$.



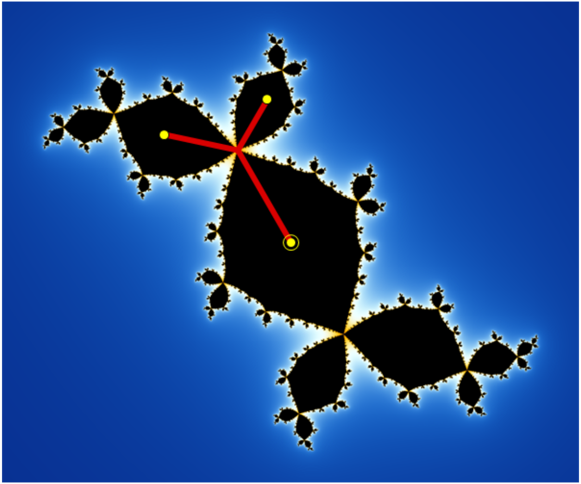
This is the ***Hubbard tree*** for $f(z)$.

Note that $T \subseteq f^{-1}(T)$, so the Hubbard tree is a fixed point of the lifting operation.

However, iterated lifting does not always find the Hubbard tree.

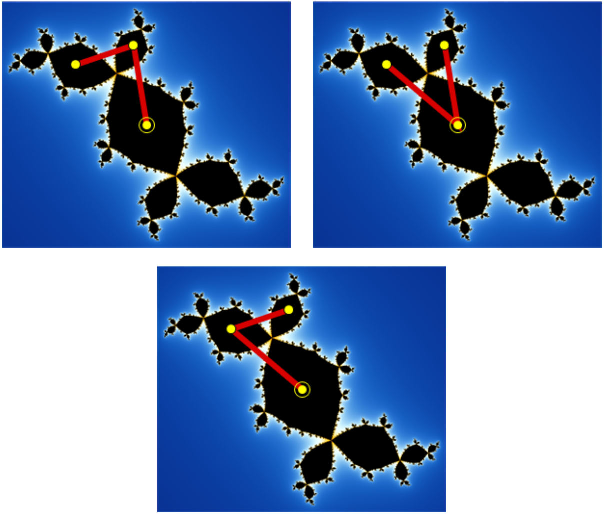
Iterated Lifting for the Rabbit

The Hubbard tree for the rabbit polynomial f_{rabbit} is a tripod.



Iterated Lifting for the Rabbit

But there is also a 3-cycle of allowed trees.



Iterated Lifting for the Rabbit

Let T_{Hub} be the Hubbard tree for f_{rabbit} and let T_{c1}, T_{c2}, T_{c3} be the trees in the 3-cycle.

We call the set $\{T_{\text{Hub}}, T_{c1}, T_{c2}, T_{c3}\}$ the ***nucleus*** for the rabbit.

Rabbit Theorem (BLMW 2018)

For any allowed tree T_0, the sequence $\{T_n\}$ of lifts under f_{rabbit} reaches the nucleus after finitely many steps.

Note: The situation for the corabbit is similar.

Recognition Algorithm

Given: A topological quadratic $F\colon \mathbb{C} \to \mathbb{C}$ whose critical point has period 3.

Procedure: Start with any T_0, and compute the iterated lifts $\{T_n\}$ until the sequence begins to repeat.

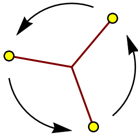
1. If we find a fixed path of length two, then F is Thurston equivalent to the airplane.
2. If we find a fixed tripod or a 3-cycle of paths, then F is Thurston equivalent to the rabbit or corabbit.

Recognition Algorithm

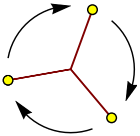
Given: A topological quadratic $F : \mathbb{C} \to \mathbb{C}$ whose critical point has period 3.

Procedure: Start with any T_0, and compute the iterated lifts $\{T_n\}$ until the sequence begins to repeat.

1. If we find a fixed path of length two, then F is Thurston equivalent to the airplane.
2. If we find a fixed tripod or a 3-cycle of paths, then F is Thurston equivalent to the rabbit or corabbit.



rabbit



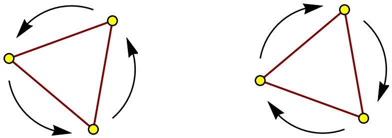
corabbit

Recognition Algorithm

Given: A topological quadratic $F : \mathbb{C} \to \mathbb{C}$ whose critical point has period 3.

Procedure: Start with any T_0, and compute the iterated lifts $\{T_n\}$ until the sequence begins to repeat.

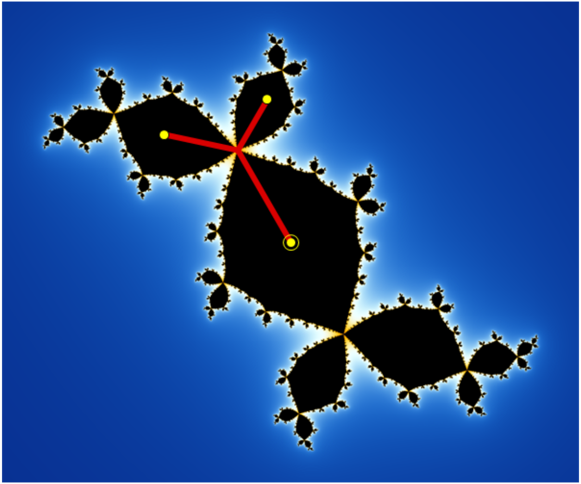
1. If we find a fixed path of length two, then F is Thurston equivalent to the airplane.
2. If we find a fixed tripod or a 3-cycle of paths, then F is Thurston equivalent to the rabbit or corabbit.



rabbit corabbit

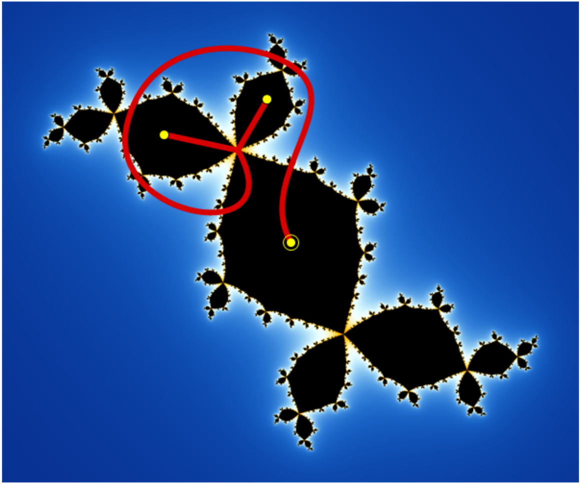
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



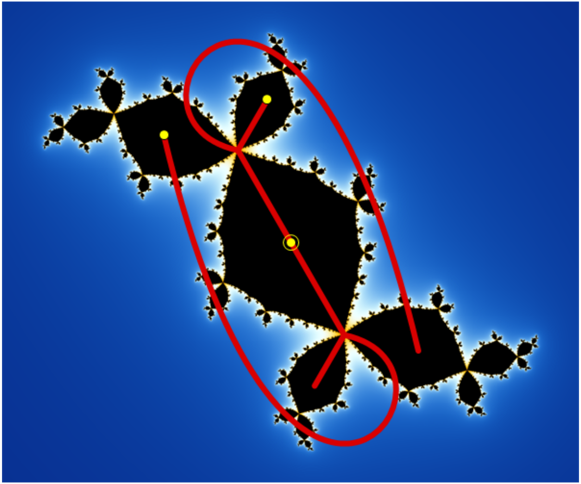
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



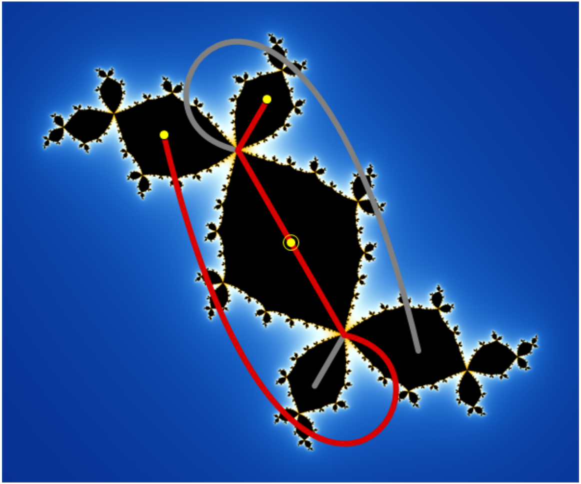
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



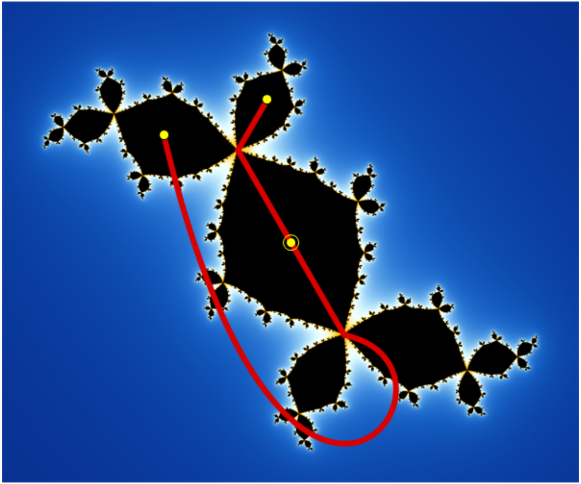
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



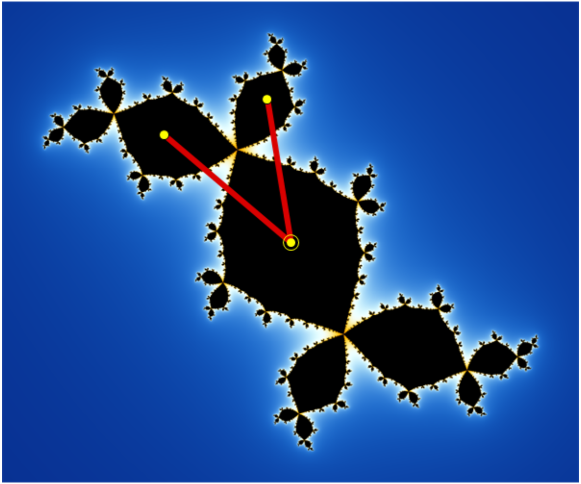
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



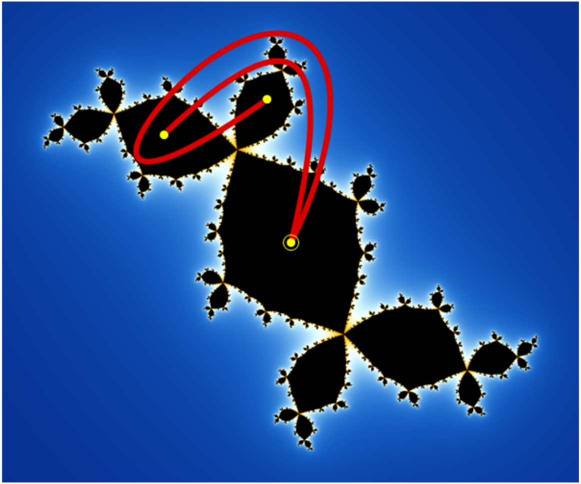
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



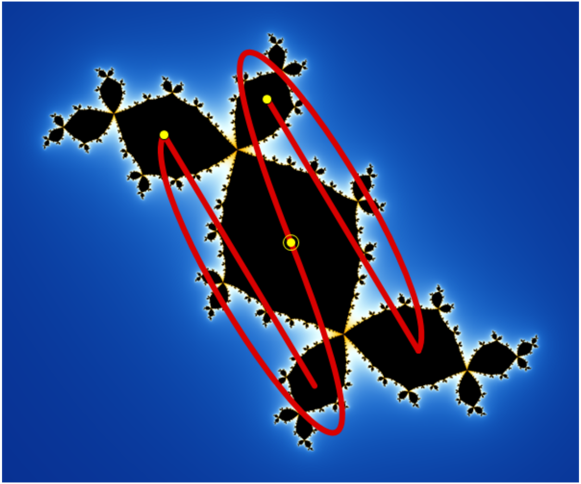
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



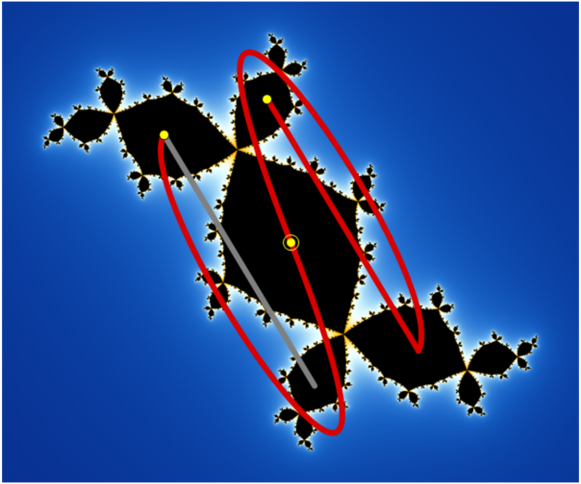
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



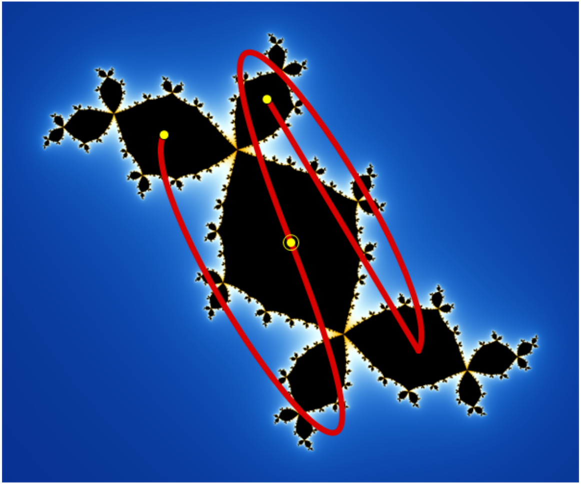
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



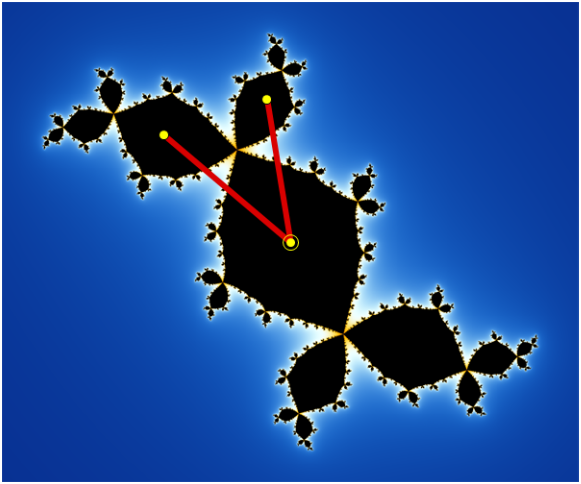
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



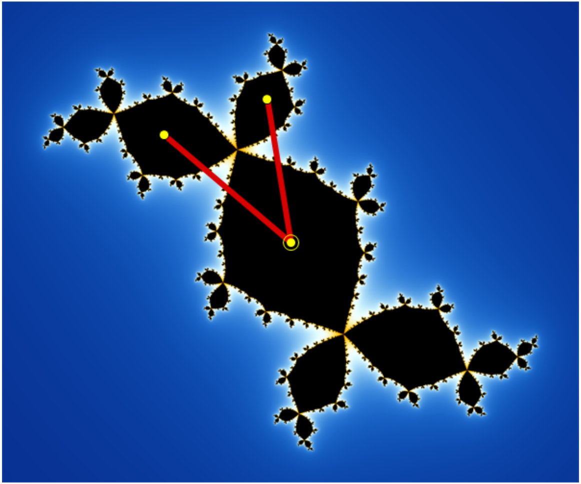
Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



Example: A Twisted Rabbit

Let $F = h \circ f_{\text{rabbit}}$, where h is the full twist around the ears.



It's an airplane!

Methods of Proof

Sketch of Proof

Airplane Theorem (BLMW 2018)

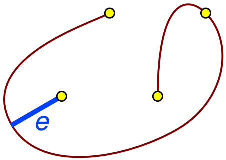
For any allowed tree T_0, the sequence $\{T_n\}$ of lifts under f_{airplane} arrives at T_{Hub} after finitely many steps.

Rabbit Theorem (BLMW 2018)

For any allowed tree T_0, the sequence $\{T_n\}$ of lifts under f_{rabbit} reaches the nucleus after finitely many steps.

Contracting Trees

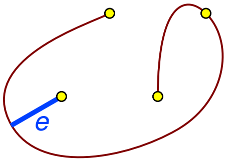
Let T be an allowed tree, and let e be an edge of T whose endpoints do not both lie in P.



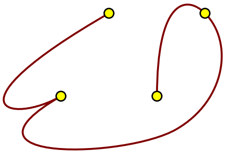
allowed tree T

Contracting Trees

Let T be an allowed tree, and let e be an edge of T whose endpoints do not both lie in P.



allowed tree T

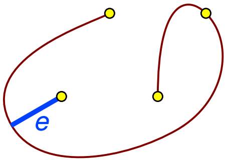


contraction T/e

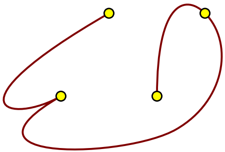
Then the ***contraction*** T/e obtained by contracting e is again an allowed tree.

Contracting Trees

Let T be an allowed tree, and let e be an edge of T whose endpoints do not both lie in P.



allowed tree T



contraction T/e

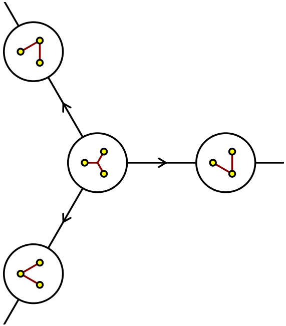
Then the ***contraction*** T/e obtained by contracting e is again an allowed tree.

More generally, a ***contraction*** of an allowed tree T is obtained by contracting any suitable collection of edges in T.

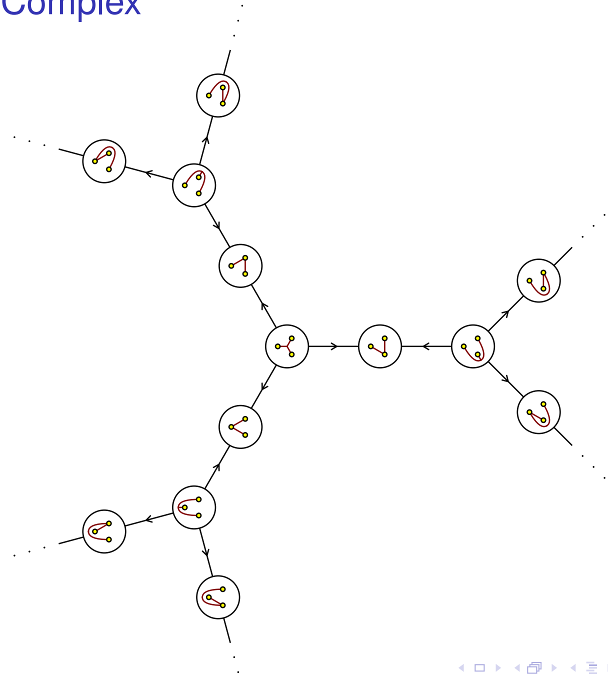
The Tree Complex

Given a critically periodic F, the associated ***tree complex*** has:

- One vertex for each allowed tree T, and
- A directed edge $T \to T'$ whenever T' is a contraction of T.

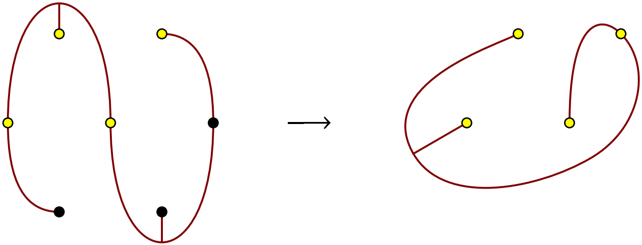


The Tree Complex



Lifting Contractions

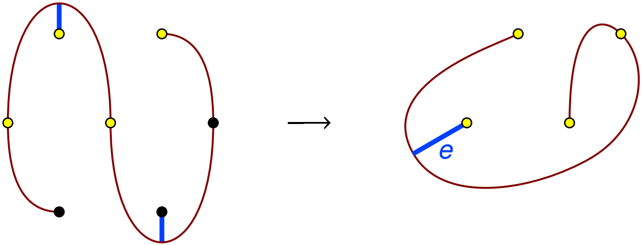
If T' is a contraction of T, then $F^{-1}(T')$ is a contraction of $F^{-1}(T)$.



preimage $F^{-1}(T)$ allowed tree T

Lifting Contractions

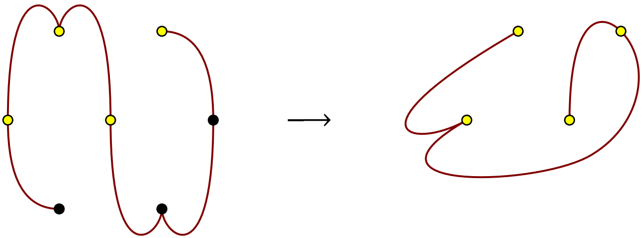
If T' is a contraction of T, then $F^{-1}(T')$ is a contraction of $F^{-1}(T)$.



preimage $F^{-1}(T)$ allowed tree T

Lifting Contractions

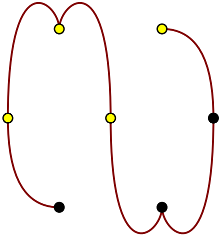
If T' is a contraction of T, then $F^{-1}(T')$ is a contraction of $F^{-1}(T)$.



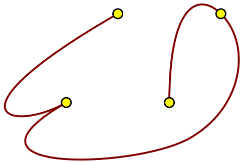
preimage $F^{-1}(T')$ $\qquad$ contraction T'

Lifting Contractions

If T' is a contraction of T, then $F^{-1}(T')$ is a contraction of $F^{-1}(T)$.



$\longrightarrow$



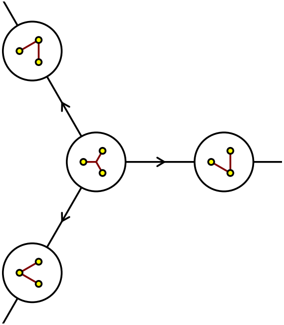
preimage $F^{-1}(T')$ contraction T'

It follows that the lift of T' is either:

- A contraction of the lift of T, or
- The same as the lift of T.

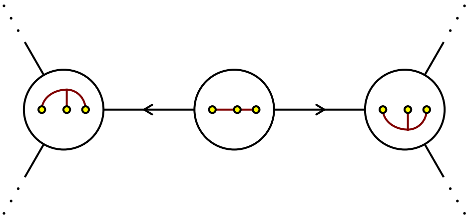
The Tree Complex

So lifting of trees defines a non-expanding map on the tree complex. This is the ***lifting map***.



Proof of the Airplane Theorem

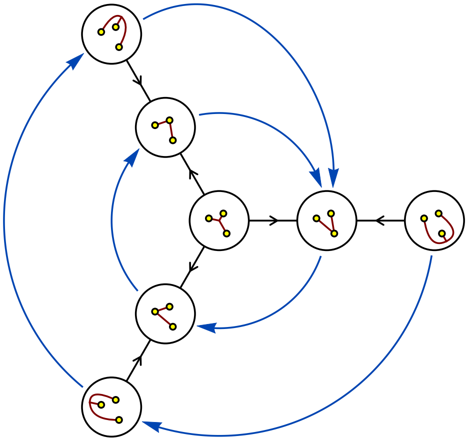
Both allowed trees adjacent to T_{Hub} in the tree complex lift to T_{Hub}.



It follows that the lift of an allowed tree is always at least one step closer to T_{Hub} in the tree complex.

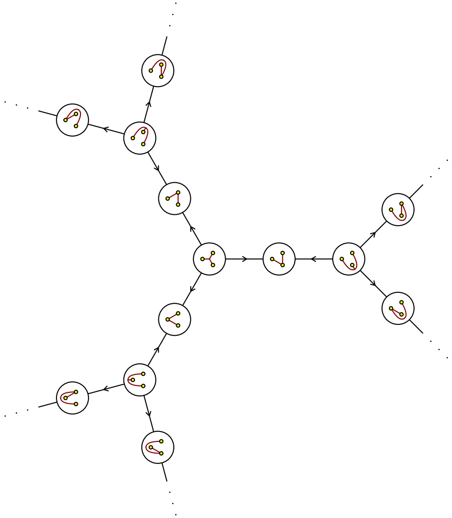
Proof of the Rabbit Theorem

For the rabbit, the 2-neighborhood of T_{Hub} maps into the 1-neighborhood after three iterations.



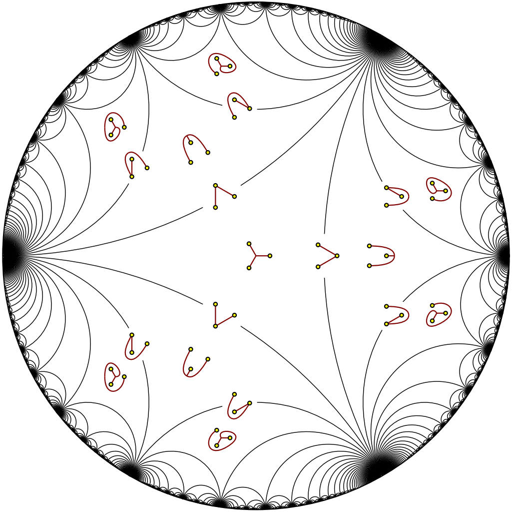
What's Going On?

The tree complex is actually the spine of a certain simplicial subdivision of Teichmüller space.



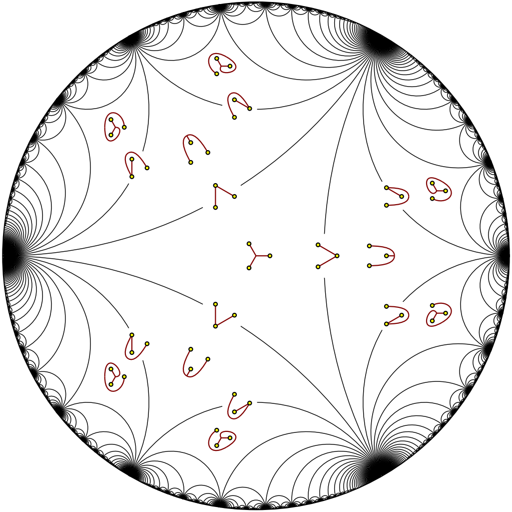
What's Going On?

The tree complex is actually the spine of a certain simplicial subdivision of Teichmüller space.



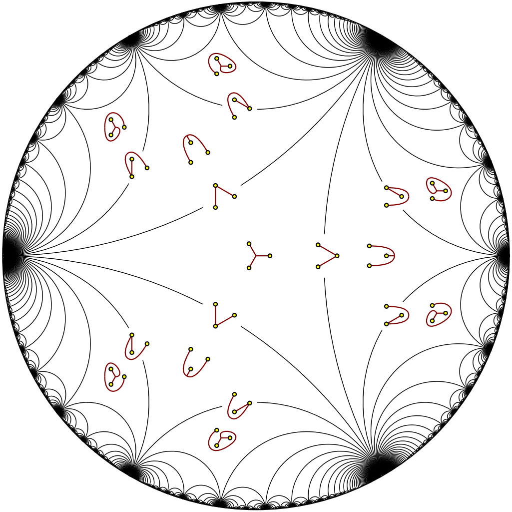
What's Going On?

Each allowed tree corresponds to an open simplex. Different points in the simplex correspond to different metrics on the tree.



What's Going On?

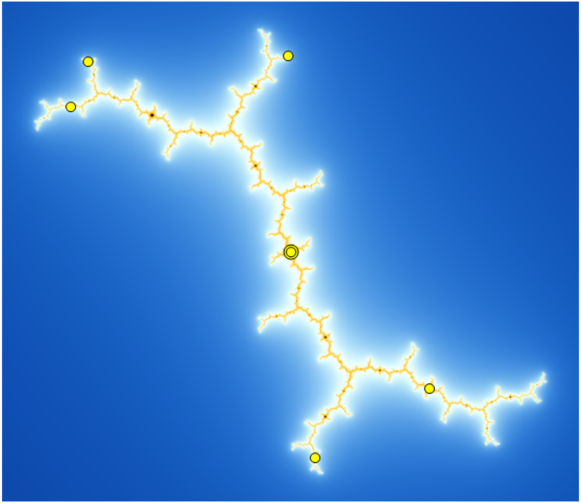
The lifting map seems to be a combinatorial version of Thurston's pullback map $\sigma_F : \mathcal{T} \to \mathcal{T}$.



Generalization

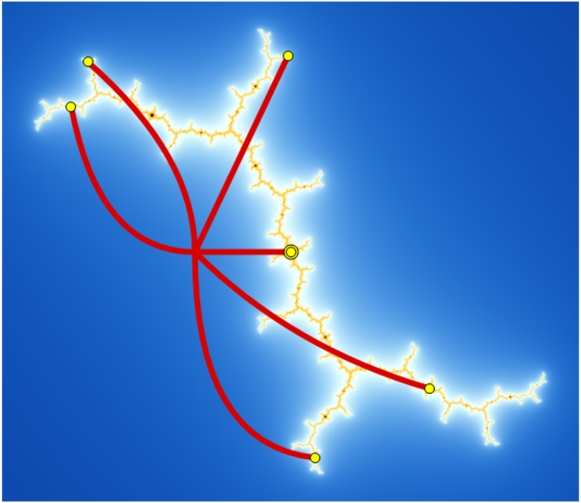
Generalization

The lifting procedure makes sense for any postcritically finite polynomial.



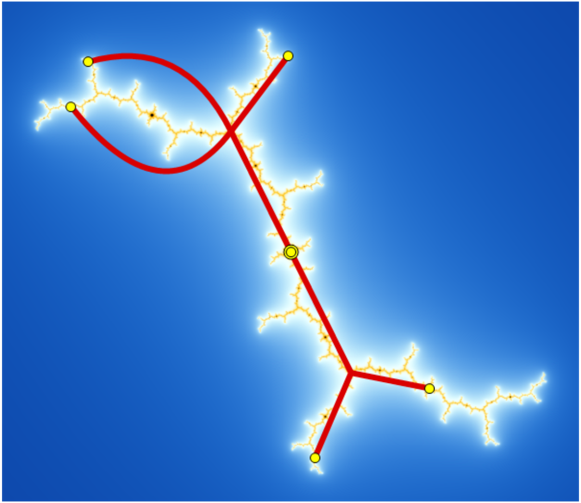
Generalization

The lifting procedure makes sense for any postcritically finite polynomial.



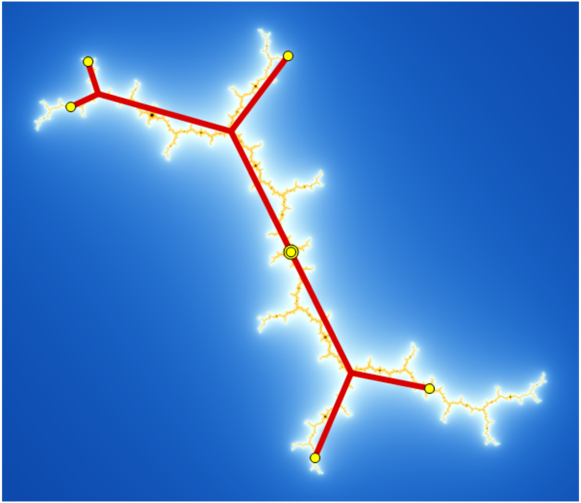
Generalization

The lifting procedure makes sense for any postcritically finite polynomial.



Generalization

The lifting procedure makes sense for any postcritically finite polynomial.



The Nucleus

Let $f : \mathbb{C} \to \mathbb{C}$ be a postcritically finite polynomial.

Theorem (BLMW 2018)

Every allowed tree for f is periodic or pre-periodic under lifting.

The ***nucleus*** $\mathcal{N}_f$ is the set of all periodic trees. This includes the Hubbard tree, and is a connected subset of the tree complex.

Conjecture

The nucleus $\mathcal{N}_f$ is always finite set.

Note: Whenever $\mathcal{N}_f$ is finite, we get an algorithmic solution to the corresponding twisted rabbit problem.

Progress So Far

For the following theorem, a polynomial $f(z)$ is ***unicritical*** if it has only one critical point. Up to affine conjugacy, such a polynomial has the form

$$f(z) = z^d + c.$$

Theorem (BLMW 2018)

Let $f : \mathbb{C} \to \mathbb{C}$ be a unicritical polynomial whose critical point is periodic. Then the nucleus $\mathcal{N}_f$ is finite.

Indeed, $\mathcal{N}_f$ is contained in the 2-neighborhood of T_{Hub}.

Theorem (BLMW 2018)

If c is real, then $\mathcal{N}_f = \{T_{\mathrm{Hub}}\}$.

Questions

Question

Is the nucleus $\mathcal{N}_f$ always finite?

Question

How does iterated tree lifting behave for obstructed topological polynomials? Can we use it to recover a Levy cycle?

Question

What exactly is the relationship between the tree lifting map and Thurston's pullback map σ_F?

Question

How can we extend our methods to rational maps? The preimage of a tree is not always a tree!

The End