Function Spaces

A **function space** is a topological space whose points are functions. There are many different kinds of function spaces, and there are usually several different topologies that can be placed on a given set of functions. These notes describe three topologies that can be placed on the set of all functions from a set X to a space Y: the product topology, the box topology, and the uniform topology.

Sets of Functions

We will be using the following notation for sets of functions:

Notation: Sets of Functions
If X and Y are sets, let Y^X denote the set of all functions from X to Y.

This notation may seem a bit confusing: in what sense is the set Y^X a power of Y? The idea is that Y^X is a generalization of the finite powers Y^n. The following example should explain this connection.

EXAMPLE 1 Let Y be a set, and let $X = \{x_1, \ldots, x_n\}$ be a finite set with n elements. Then the set Y^X consists of all functions $\{x_1, \ldots, x_n\} \to Y$. Any such function can be thought of as an n tuple of points in Y:

$$f \;=\; \big(f(x_1), f(x_2), \ldots, f(x_n)\big).$$

Thus we can identify Y^X with the Cartesian power $Y^n = Y \times \cdots \times Y$.

In fact, the nth Cartesian power Y^n is sometimes *defined* as the set $Y^{\{1,\ldots,n\}}$ of all functions $\{1, \ldots, n\} \to Y$. Using this definition, every ordered n-tuple $(y_1, \ldots, y_n)$ is actually a function, with y_k being an alternative notation for $y(k)$. ■

In general, the set Y^X can be viewed as a product of copies of Y:

$$Y^X \;=\; \prod_{x\in X} Y$$

EXAMPLE 2 Let $\mathbb{N}$ be the natural numbers. If Y is a set, then $Y^{\mathbb{N}}$ (denoted Y^ω in the book) is the set of all functions $\mathbb{N} \to Y$. This can be thought of as an infinite product:

$$Y^{\mathbb{N}} \;=\; \prod_{n\in\mathbb{N}} Y \;=\; Y \times Y \times Y \times \cdots .$$

Every element of $Y^{\mathbb{N}}$ can be viewed as an infinite tuple (or sequence) of elements of Y:

$$(y_1,\; y_2,\; y_3,\; \ldots)$$ ■

EXAMPLE 3 Consider the set $\mathbb{R}^{\mathbb{R}}$ of all functions $\mathbb{R} \to \mathbb{R}$. This set can be viewed as a product of copies of $\mathbb{R}$:

$$\mathbb{R}^{\mathbb{R}} \;=\; \prod_{x\in\mathbb{R}} \mathbb{R}.$$

The idea here is that a function $f\colon \mathbb{R} \to \mathbb{R}$ can be thought of as a vector with one coordinate for each $x \in \mathbb{R}$. ■

Of course, we have yet to define a topology on the function space Y^X. Among other things, such a topology would give us a notion of convergence for functions—given a sequence of functions $f_n \in Y^X$, we would be able to say whether it converges to a function $f \in Y^X$.

Pointwise Convergence

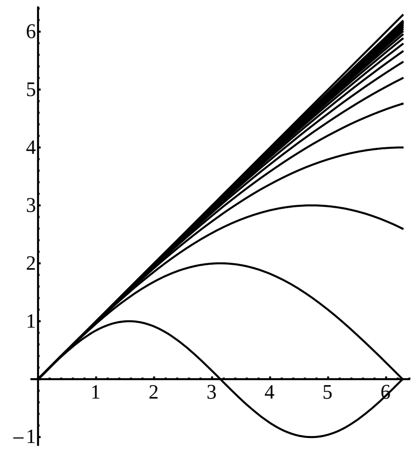
We are used to the idea of a sequence x_n of real numbers converging to some real number x. More generally, we know what it means for a sequence x_n of points in a topological space to converge to a point x. But what does it mean for a sequence of functions to converge to a function?

The following example should be illuminating:

EXAMPLE 4 Consider the sequence of functions $f_n\colon [0, 2\pi] \to \mathbb{R}$ defined as follows:

$$f_1(x) = \sin(x), \qquad f_2(x) = 2\sin\!\left(\frac{x}{2}\right), \qquad f_3(x) = 3\sin\!\left(\frac{x}{3}\right), \qquad \ldots$$

The graphs of the first 20 functions in this sequence is shown below, along with the line $y = x$:



As you can see, the graphs of successive functions in this sequence become closer and closer to the graph of the function $f(x) = x$. Thus, it is reasonable to say that the sequence f_n converges to the function f. ■

Definition: Pointwise Convergence
Let X be a set, let Y be a topological space, and let $f_n\colon X \to Y$ be a sequence of functions. We say that f_n **converges pointwise** to a function $f\colon X \to Y$ if for every $x \in X$ the sequence $f_n(x)$ converges to $f(x)$ in Y.

That is, the sequence of functions f_n converges pointwise to f if

$$\lim_{n\to\infty} f_n(x) \;=\; f(x)$$

for each individual value of x.

EXAMPLE 5 The functions

$$f_n(x) = n \sin\left(\frac{x}{n}\right)$$

from the previous example converges pointwise to the function $f(x) = x$. In particular,

$$\lim_{n\to\infty} n \sin\left(\frac{x}{n}\right) \;=\; x$$

for every $x \in [0, 2\pi]$. ■

EXAMPLE 6 Here is an example involving functions $\mathbb{N} \to \mathbb{R}$, which we write as infinite tuples. Consider the following sequence of functions:

$$\begin{aligned} f_1 &= (1,\ 2,\ 3,\ 4,\ 5,\ \ldots) \\ f_2 &= \left(\tfrac{1}{2},\ \tfrac{2}{2},\ \tfrac{3}{2},\ \tfrac{4}{2},\ \tfrac{5}{2},\ \ldots\right) \\ f_3 &= \left(\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{3}{3},\ \tfrac{4}{3},\ \tfrac{5}{3},\ \ldots\right) \\ &\vdots \end{aligned}$$

For any fixed $k \in \mathbb{N}$, the sequence $f_n(k)$ consists of the numbers k/n, and thus converges to 0. (Each of these sequences corresponds to a column of numbers above.) Therefore, the functions f_n converge pointwise to the zero function:

$$f = (0,\ 0,\ 0,\ 0,\ 0,\ \ldots). \qquad \blacksquare$$

The Product Topology

Our next task is to define a topology on Y^X under which convergence of sequences corresponds to pointwise convergence of functions.

> **Definition: The Product Topology**
> Let X be a set, and let Y be a topological space. Given any $x \in X$ and any open set $U \subset Y$, define
> $$S(x, U) = \{f \in Y^X \mid f(x) \in U\}.$$
> Then the sets $S(x, U)$ form a subbasis for a topology on Y^X, known as the **product topology**.

As the following example illustrates, this product topology agrees with the product topology for the Cartesian product of two sets defined in §15.

EXAMPLE 7 If Y is a topological space, then the product $Y \times Y$ can be viewed as a function space Y^X, where $X = \{1, 2\}$. If $U \subset Y$ is open, then

$$S(1, U) = U \times Y \qquad \text{and} \qquad S(2, U) = Y \times U.$$

It is easy to see that sets of this form are a subbasis for the product topology on $Y \times Y$ as defined in §15. Thus the definition above agrees with our existing definition of the product topology. $\blacksquare$

Be aware that the sets $S(x,U)$ are a subbasis for the product topology, not a basis. A basic open set would be a finite intersection of subbasic open sets:

$$S(x_1,U_1) \cap \cdots \cap S(x_n,U_n).$$

Because this intersection is finite, a basic open set can include restrictions on only finitely many different function values.

Although our definition of a subbasic $S(x,U)$ involves an arbitrary open set U, it is often helpful to restrict to the case where U is a basic open set:

Theorem 1 Subbasis for the Product Topology

Let X be a set, let Y be a topological space, and let $\mathcal{B}$ be a basis for the topology on Y. Then the collection

$$\{S(x,B) \mid x \in X,\ B \in \mathcal{B}\}$$

is a subbasis for the product topology on Y^X.

PROOF Consider an element $S(x,U)$ of the standard subbasis for the product topology. Then U is an open subset of Y, so U can be expressed as the union of some family $\{B_\alpha\}_{\alpha\in J}$ of elements of $\mathcal{B}$. Therefore

$$S(x,U) \;=\; \bigcup_{\alpha\in J} S(x,B_\alpha),$$

which proves $S(x,U)$ lies in the topology generated by the sets $S(x,B)$. ■

EXAMPLE 8 Consider the space $\mathbb{R}^{\mathbb{N}}$ (or $\mathbb{R}^\omega$) of infinite sequences in $\mathbb{R}$. This space can be thought of as an infinite product:

$$\mathbb{R}^{\mathbb{N}} \;=\; \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots.$$

If (c,d) is an open interval in $\mathbb{R}$, then

$$S(3,(c,d)) \;=\; \mathbb{R}\times\mathbb{R}\times(c,d)\times\mathbb{R}\times\mathbb{R}\times\cdots$$

is an example of a subbasic open set in $\mathbb{R}^{\mathbb{N}}$. If (a,b) is another open interval in $\mathbb{R}$, then

$$S(1,(a,b))\cap S(3,(c,d)) \;=\; (a,b)\times\mathbb{R}\times(c,d)\times\mathbb{R}\times\mathbb{R}\times\cdots$$

is an example of a basic open set in $\mathbb{R}^{\mathbb{N}}$. In general, a basic open set in $\mathbb{R}^{\mathbb{N}}$ may involve restrictions on any finite number of coordinates of a tuple. ■

As we have indicated, convergence in the product topology is the same as pointwise convergence of functions:

Theorem 2 Convergence in the Product Topology

Let X be a set, let Y be a topological space, let f_n be a sequence in Y^X, and let $f \in Y^X$. Then $f_n \to f$ under the product topology if and only if the functions f_n converge pointwise to f.

PROOF Suppose first that f_n converges to f under the product topology, and let $x \in X$. If U is a neighborhood of $f(x)$ in Y, then $S(x, U)$ is a neighborhood of f in Y^X, so $f_n \in S(x, U)$ for all but finitely many n. It follows that $f_n(x) \in U$ for all but finitely many n, which proves that $f_n(x) \to f(x)$.

For the converse, suppose that f_n converges pointwise to f, and let $S(x, U)$ be a neighborhood of f in Y^X. Then U is a neighborhood of $f(x)$ in Y. Since $f_n(x) \to f(x)$, it follows that $f_n(x) \in U$ for all but finitely many n. Then $f_n \in S(x, U)$ for all but finitely many n, which proves that $f_n \to f$ under the product topology. ■

Because of this theorem, the product topology on a function space is sometimes referred to as the **topology of pointwise convergence**.

The product topology has several other nice properties. Here is one of the most important:

Theorem 3 Continuous Functions in the Product Topology

Let X be a set, and let Y be a topological space. For each $x \in X$, let $\pi_x : Y^X \to Y$ be the projection function $\pi_x(f) = f(x)$. Then:

1. *Each function π_x is continuous under the product topology.*
2. *The product topology is the smallest topology on Y^X for which all of the functions π_x are continuous.*
3. *If A is a topological space and $g : A \to Y^X$ is a function, then g is continuous under the product topology if and only if every function $\pi_x \circ g : A \to Y$ is continuous.*

PROOF Observe that, if $x \in X$ and $U \subset Y$ is open, then

$$\pi_x^{-1}(U) \;=\; S(x, U).$$

Thus the subbasic open sets $S(x, U)$ for the product topology are precisely the preimages of open sets under the projections π_x. This proves assertions (1) and (2).

For the last assertion, let $g \colon A \to Y^X$ be a function, and suppose that each composition $\pi_x \circ g$ is continuous. Then $g^{-1}(S(x, U)) = (\pi_x \circ g)^{-1}(U)$ is open for each subbasic open set $S(x, U)$ in Y^X, which proves that g is continuous. The converse follows from (1) and the fact that the composition of continuous functions is continuous. ■

The Box Topology

We now discuss a second possible topology on Y^X.

Definition: The Box Topology
Let X be a set and let Y be a topological space. Given a family $\{U_x\}_{x \in X}$ of open sets in Y, the product

$$\prod_{x \in X} U_x \;=\; \{f \in Y^X \mid f(x) \in U_x \text{ for every } x \in X\}$$

is called an **open box** in Y^X. The collection of all open boxes forms a basis for a topology on Y^X, known as the **box topology**.

As with the product topology, it is not necessary to use arbitrary open subsets of Y to form the basis for Y^X:

Theorem 4 Basis for the Box Topology

Let X be a set, let Y be a topological space, and let $\mathcal{B}$ be a basis for the topology on Y. Then the collection of sets

$$\{\textstyle\prod B_x \mid B_x \in \mathcal{B} \text{ for each } x \in X\}$$

is a basis for the box topology on Y^X.

PROOF Let $U = \prod U_x$ be an arbitrary open box in Y^X, and let $f \in U$. Then $f(x) \in U_x$ for each $x \in X$, so there exists a $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U_x$. Then $f \in \prod B_x$, and $\prod B_x \subset U$. ■

EXAMPLE 9 Consider again the function space $\mathbb{R}^{\mathbb{N}}$. For any sequence of open intervals $(a_1, b_1), (a_2, b_2), \ldots$ in $\mathbb{R}$, the set

$$(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \cdots$$

is an example of a basic open set in the box topology. Note that such a set is not open in the product topology. ■

Though the box topology may seem more natural than the product topology, it not actually very useful. In particular, very few sequences of functions converge in the box topology:

EXAMPLE 10 Consider the following sequence in $\mathbb{R}^{\mathbb{N}}$:

$$\begin{aligned} f_1 &= (1,\ 1,\ 1,\ 1,\ 1,\ \ldots) \\ f_2 &= \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \ldots\right) \\ f_3 &= \left(\tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ \ldots\right) \\ &\ \ \vdots \end{aligned}$$

This sequence converges to the point $f = (0, 0, 0, \ldots)$ in the product topology, and seems that f_n should converge to f under any "reasonable" notion of convergence.

However, the sequence f_n does not converge to f in the box topology. In particular, the open box

$$(-1, 1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \left(-\tfrac{1}{4}, \tfrac{1}{4}\right) \times \cdots$$

contains f, but does not contain f_n for any value of n. ■

Because the box topology does not correspond to a useful notion of convergence of functions, it is hardly ever used for applications in functional analysis. Its primary purpose is to serve as a counterexample for statements about arbitrary topological spaces. For example, one might ask whether every topological space is homeomorphic to a metric space. The answer is no, with the box topology providing an easy counterexample:

Theorem 5 A Non-Metrizable Space

There does not exist a metric for the box topology on $\mathbb{R}^{\mathbb{N}}$.

PROOF We give an argument involving neighborhoods of the origin. See pg. 132 of Munkres for a proof involving sequences.

Suppose that d were a metric on $\mathbb{R}^{\mathbb{N}}$ whose corresponding metric topology were the same as the box topology. Let $\mathbf{0}$ denote the zero function $(0, 0, 0, \ldots)$ in $\mathbb{R}^{\mathbb{N}}$, and consider the following sequence of open balls:

$$B_d(\mathbf{0}, 1) \quad \supset \quad B_d(\mathbf{0}, 1/2) \quad \supset \quad B_d(\mathbf{0}, 1/3) \quad \supset \quad \cdots$$

By assumption, each of these balls is open in the box topology, so each ball $B_d(\mathbf{0}, 1/n)$ must contain a basic open box around $\mathbf{0}$. Thus, there exist positive real numbers a_{ij} such that:

$$\begin{aligned}
B_d(\mathbf{0}, 1) &\supset (-a_{11}, a_{11}) \times (-a_{12}, a_{12}) \times (-a_{13}, a_{13}) \times \cdots \\
B_d(\mathbf{0}, 1/2) &\supset (-a_{21}, a_{21}) \times (-a_{22}, a_{22}) \times (-a_{23}, a_{23}) \times \cdots \\
B_d(\mathbf{0}, 1/3) &\supset (-a_{31}, a_{31}) \times (-a_{32}, a_{32}) \times (-a_{33}, a_{33}) \times \cdots \\
&\vdots
\end{aligned}$$

Without loss of generality, we may assume that the intervals in each column are shrinking, i.e. that $a_{1k} \geq a_{2k} \geq a_{3k} \geq \cdots$ for each k. Now consider the open box formed by the intervals along the diagonal:

$$(-a_{11}, a_{11}) \times (-a_{22}, a_{22}) \times (-a_{33}, a_{33}) \times \cdots.$$

This set is a neighborhood of $\mathbf{0}$ in the box topology, but it cannot contain any of the open balls $B_d(\mathbf{0}, 1/n)$, a contradiction. Thus no such metric d exists, and $\mathbb{R}^{\mathbb{N}}$ under the box topology is not metrizable. ■

Uniform Convergence

There are a few problems with pointwise convergence that make it less than useful for many applications. To illustrate the problem, we present two examples of sequences of functions that converge pointwise in a counterintuitive way.

EXAMPLE 11 Consider the following sequence of functions in $\mathbb{R}^{\mathbb{N}}$:

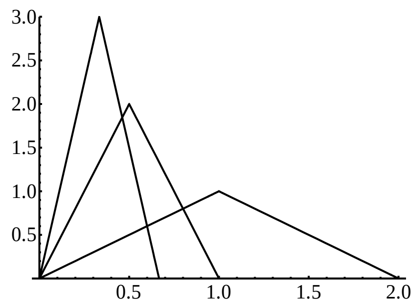
$$\begin{aligned}
f_1 &= (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ldots) \\
f_2 &= (0,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots) \\
f_3 &= (0,\ 0,\ 3,\ 3,\ 3,\ 3,\ 3,\ \ldots) \\
f_4 &= (0,\ 0,\ 0,\ 4,\ 4,\ 4,\ 4,\ \ldots) \\
&\vdots
\end{aligned}$$

Since each column is eventually zero, these functions converge pointwise to the zero function $(0, 0, 0, \ldots)$, despite the fact that the average value diverges to infinity. ■

EXAMPLE 12 Consider the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n \\ 2n - n^2 x & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{otherwise.} \end{cases}$$

The graph of f_n is a triangular spike with height n and total area 1. For example, the graphs of f_1, f_2, and f_3 are shown below:



Because the spikes are becoming thinner as n increases, each individual value of x lies in only finitely many spikes; it follows that each sequence $f_n(x)$ is eventually 0, so the functions f_n converge pointwise to the constant zero function. Again, this does not really agree with our intuitive notion of convergence. ■

Uniform convergence is an alternative to pointwise convergence which is a bit more strict. As a result, it has nicer theoretical properties, and conforms more closely with our intuitive notion of convergence. It is based on a measure of distance between functions:

Definition: Uniform Distance
Let X be a set, let Y be a metric space with metric d, and let $f, g\colon X \to Y$ be functions. The **uniform distance** $\rho(f, g)$ from f to g is defined as follows:

$$\rho(f, g) \;=\; \sup\{d(f(x), g(x)) \mid x \in X\}.$$

If the set $\{d(f(x), g(x)) \mid x \in X\}$ is unbounded, then $\rho(f, g)$ is infinite.

The uniform distance $\rho(f, g)$ should be thought of as the maximum distance between $f(x)$ and $g(x)$. In some cases, such as when one function has an asymptote, this maximum may not be realized, making it necessary to define $\rho(f, g)$ as a supremum.

Definition: Uniform Convergence
Let X be a set, let Y be a metric space, and let $f_n \colon X \to Y$ be a sequence of functions. We say that f_n **converges uniformly** to a function $f \colon X \to Y$ if $\rho(f_n, f) \to 0$ as $n \to \infty$.

For example, the functions in examples 11 and 12 have $\rho(f_n, f) = n$ for all n, and therefore do not converge uniformly. On the other hand, the functions in example 10 have $\rho(f_n, f) = 1/n$ for each n, and therefore do converge uniformly to the zero function.

Note that uniform convergence can only be defined when Y is a metric space, since it depends on being able to measure distances between y-values. Section 46 of Munkres discusses some related notions of convergence that work for any topological space Y.

Assuming Y is a metric space, there is an obvious topology on Y^X under which convergence of sequences is the same thing as uniform convergence:

Definition: Uniform Topology
Let X be a set, and let Y be a metric space. For each $f \in Y^X$ and $\epsilon > 0$, define

$$B_\rho(f, \epsilon) \;=\; \{g \in Y^X \mid \rho(f, g) < \epsilon\}.$$

Then the sets $B_\rho(f, \epsilon)$ form a basis for a topology on Y^X, known as the **uniform topology**.

Theorem 6 Convergence in the Uniform Topology

Let X be a set, let Y be a metric space, let f_n be a sequence in Y^X, and let $f \in Y^X$. Then $f_n \to f$ under the uniform topology if and only if the functions f_n converge uniformly to f.

Though it may appear from the definition that the uniform topology is a metric topology with metric ρ, this is not actually the case. The problem is that $\rho(f, g)$ is often infinite, which is not allowed by the definition of a metric. This is less of a problem than it seems: it works perfectly well to simply allow metrics to take infinite values. Alternatively, we can define the **bounded uniform metric** $\overline{\rho}$ by

$$\overline{\rho}(f, g) \;=\; \min\{\rho(f, g), 1\}.$$

Then $\overline{\rho}$ is a legitimate metric, and the corresponding metric topology is the same as the uniform topology.

EXAMPLE 13 Consider the space $\mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}$ is given the standard metric. Let $f \in \mathbb{R}^{\mathbb{N}}$ be the constant zero function. Then the basic open set $B_\rho(f,1)$ consists of all functions $g\colon \mathbb{N} \to \mathbb{R}$ such that

$$\sup\bigl\{|g(k)| \bigm| k \in \mathbb{N}\bigr\} < 1.$$

Note that $B_\rho(f,1)$ is not simply an open box:

$$B_\rho(f,1) \;\neq\; (-1,1) \times (-1,1) \times (-1,1) \times \cdots .$$

The reason is that a function may take values in the interval $(-1,1)$, but still have supremum equal to 1. For example, the function

$$g \;=\; \bigl(\tfrac{1}{2},\, \tfrac{2}{3},\, \tfrac{3}{4},\, \tfrac{4}{5},\, \ldots\bigr)$$

lies in the box $(-1,1)^{\mathbb{N}}$, but $\rho(f,g) = 1$, and therefore $g \notin B_\rho(f,1)$.

Incidentally, it can be shown that the box $(-1,1)^{\mathbb{N}}$ is not even open in the uniform topology on $\mathbb{R}^{\mathbb{N}}$, and hence uniform and box topologies are different on $\mathbb{R}^{\mathbb{N}}$. ■

One of the nicest theoretical properties of the uniform topology is the following:

Theorem 7 $\mathcal{C}(X,Y)$ is Closed

Let X be a topological space, let Y be a metric space, and let

$$\mathcal{C}(X,Y) \;=\; \{f\colon X \to Y \mid f \text{ is continuous}\}.$$

Then $\mathcal{C}(X,Y)$ is a closed subset of Y^X under the uniform topology.

Since Y^X is a metric space under the uniform topology, this theorem is equivalent to the statement that the limit of any convergent sequence of points in $\mathcal{C}(X,Y)$ is an element of $\mathcal{C}(X,Y)$. That is, the uniform limit of a sequence of continuous functions is again continuous. This result is known as the **uniform limit theorem**, and appears as theorem 21.6 of Munkres.

PROOF Let f be an element of the closure of $\mathcal{C}(X,Y)$, and let $x_0 \in X$. We claim that f is continuous at x_0.

Let $\epsilon > 0$. We must show that there exists a neighborhood U of x_0 so that $f(U) \subset B_d(f(x_0),\epsilon)$, where d is the metric on Y. Since f is in the closure of $\mathcal{C}(X,Y)$, there exists a $g \in \mathcal{C}(X, Y)$ so that $\rho(f, g) < \epsilon/3$. Then g is continuous, so there exists a neighborhood U of x_0 so that $g(U) \subset B_d(g(x_0), \epsilon/3)$. If $x \in U$, we know that

$$d\big(f(x), g(x)\big) < \frac{\epsilon}{3}, \qquad d\big(g(x), g(x_0)\big) < \frac{\epsilon}{3}, \qquad \text{and} \qquad d\big(g(x_0), f(x_0)\big) < \frac{\epsilon}{3},$$

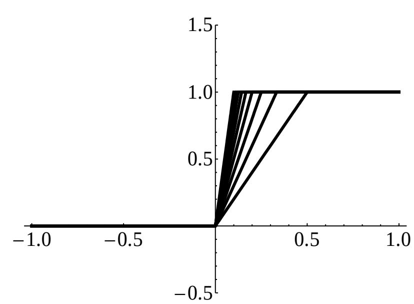
and therefore $d\big(f(x), f(x_0)\big) < \epsilon$ by the triangle inequality. Thus $f(U) \subset B_d(f(x_0), \epsilon)$, which proves that f is continuous. ■

The theorem above does not hold if the uniform topology is replaced by the product topology. Indeed, as the following example shows, it is perfectly possible for a sequence of continuous functions to converge pointwise to a discontinuous function.

EXAMPLE 14 Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ nx & \text{if } 0 \leq x \leq 1/n \\ 1 & \text{if } x \geq 1/n. \end{cases}$$

The functions $f_2, \ldots, f_{10}$ are graphed below:



These functions are all continuous, but they move from $y = 0$ to $y = 1$ over shorter and shorter periods of time as n increases. The result is that the sequence f_n converges pointwise to a function f that has a jump discontinuity at $x = 0$:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$ ■

The following theorem describes the relationship between the three topologies we have discussed. It is the same as theorem 20.4 in Munkres:

Theorem 8 Comparing the Three Topologies

Let X be a set, and let Y be a metric space. Let $\mathcal{T}_{\text{product}}$, $\mathcal{T}_{\text{box}}$, and $\mathcal{T}_{\text{uniform}}$ denote the three topologies on Y^X. Then

$$\mathcal{T}_{\text{product}} \quad \subset \quad \mathcal{T}_{\text{uniform}} \quad \subset \quad \mathcal{T}_{\text{box}}.$$

PROOF We first prove that any subbasic open set in the product topology is open in the uniform topology. Let $S(x,U)$ be such a set, and let $f \in S(x,U)$. Then U is open in Y and $f(x) \in U$, so there exists an $\epsilon > 0$ such that $B(f(x),\epsilon) \subset U$. Then every element of $B_\rho(f,\epsilon)$ must also lie in $S(x,U)$. This proves that $S(x,U)$ is open in the uniform topology, and therefore $\mathcal{T}_{\text{product}} \subset \mathcal{T}_{\text{uniform}}$.

Next we must show that any basic open set in the uniform topology is open in the box topology. Let $B_\rho(f,\epsilon)$ be such a set, and let $g \in B_\rho(f,\epsilon)$. Then there exists an $\epsilon' > 0$ so that $B_\rho(g,\epsilon') \subset B_\rho(f,\epsilon)$. Then $\prod_{x\in X}\big(g(x)-\epsilon'/2, g(x)+\epsilon'/2\big)$ is an open set in the box topology that contains g and is contained in $B_\rho(g,\epsilon')$, and is hence also contained in $B_\rho(f,\epsilon)$. This proves that $B_\rho(f,\epsilon)$ is open in the box topology, and therefore $\mathcal{T}_{\text{uniform}} \subset \mathcal{T}_{\text{box}}$. ■

Note that the ordering of the three topologies above corresponds to how many sequences converge: lots of sequences converge in the product topology, some sequences converge in the uniform topology, and almost no sequences converge in the box topology.

Finally, we end with a theorem that illustrates the difference between these three topologies. (See exercises 19.7 and 20.5 in Munkres.)

Theorem 9 Closure of $\mathbb{R}^\infty$

Let $\mathbb{R}^\infty$ be the following subset of $\mathbb{R}^{\mathbb{N}}$:

$$\mathbb{R}^\infty \;=\; \{f \in \mathbb{R}^{\mathbb{N}} \mid f(k) = 0 \text{ for all but finitely many } k\}.$$

1. *In the box topology, $\mathbb{R}^\infty$ is a closed set.*

2. *In the uniform topology, the closure of $\mathbb{R}^\infty$ is the set*

$$\{f \in \mathbb{R}^{\mathbb{N}} \mid f(n) \to 0 \text{ as } n \to \infty\}.$$

3. *In the product topology, the closure of $\mathbb{R}^\infty$ is all of $\mathbb{R}^{\mathbb{N}}$.*

PROOF In the box topology, define a sequence U_n of open boxes in $\mathbb{R}^{\mathbb{N}}$ by

$$U_n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \times (\mathbb{R} - \{0\}) \times (\mathbb{R} - \{0\}) \times \cdots.$$

Then $\bigcup_{n=1}^{\infty} U_n$ is the complement of $\mathbb{R}^\infty$, and hence $\mathbb{R}^\infty$ is closed in the box topology.

In the uniform topology, let $f \in \mathbb{R}^{\mathbb{N}}$. If $f(n) \to 0$ as $n \to \infty$, then every ball $B_\rho(f, \epsilon)$ must contain an element of $\mathbb{R}^\infty$, and therefore f is in the closure of $\mathbb{R}^\infty$. Conversely, if $f(n) \not\to 0$ as $n \to \infty$, then there exists an $\epsilon > 0$ such that $f(n) \notin (-\epsilon, \epsilon)$ for infinitely many n. Then $B_\rho(f, \epsilon)$ does not contain any element of $\mathbb{R}^\infty$, and therefore f does not lie in the closure of $\mathbb{R}^\infty$.

Finally, in the product topology observe that every basic open set

$$S(x_1, U_1) \cap \cdots \cap S(x_n, U_n)$$

contains a point from $\mathbb{R}^\infty$. It follows that the closure of $\mathbb{R}^\infty$ is all of $\mathbb{R}^{\mathbb{N}}$. ■

Exercises

1. Let X, Y, and Z be sets.
   (a) Prove that there exists a bijection $(Y \times Z)^X \to Y^X \times Z^X$.
   (b) If $X \cap Y = \emptyset$, prove that there exists a bijection $Z^{X \cup Y} \to Z^X \times Z^Y$.

2. If Y is Hausdorff, prove that Y^X is Hausdorff in both the product and box topologies.

3. Consider the product topology, the uniform topology, and the box topology on the space $\{0,1\}^{\mathbb{N}}$. Are all three topologies different? How do these topologies compare with the discrete topology on $\{0,1\}^{\mathbb{N}}$?

4. Consider $\mathbb{R}^\infty$ as a subspace of $\mathbb{R}^{\mathbb{N}}$ under the product, uniform, and box topologies. Show that the three resulting subspace topologies on $\mathbb{R}^\infty$ are all distinct.

5. Let X be a set. For each $f \in \mathbb{R}^X$ and $\epsilon > 0$, let

   $$W(f, \epsilon) = \{g \in \mathbb{R}^X : |g(x) - f(x)| < \epsilon \text{ for every } x \in X\}.$$

   Prove that the sets $W(f, \epsilon)$ are a subbasis for the box topology on $\mathbb{R}^X$.

6. Let Y be a bounded metric space, and define a metric D on $Y^{\mathbb{N}}$ by

   $$D(f, g) = \sup\{d(f(n), g(n))/n \mid n \in \mathbb{N}\}.$$

   Prove that metric topology on $Y^{\mathbb{N}}$ determined by D is the same as the product topology.