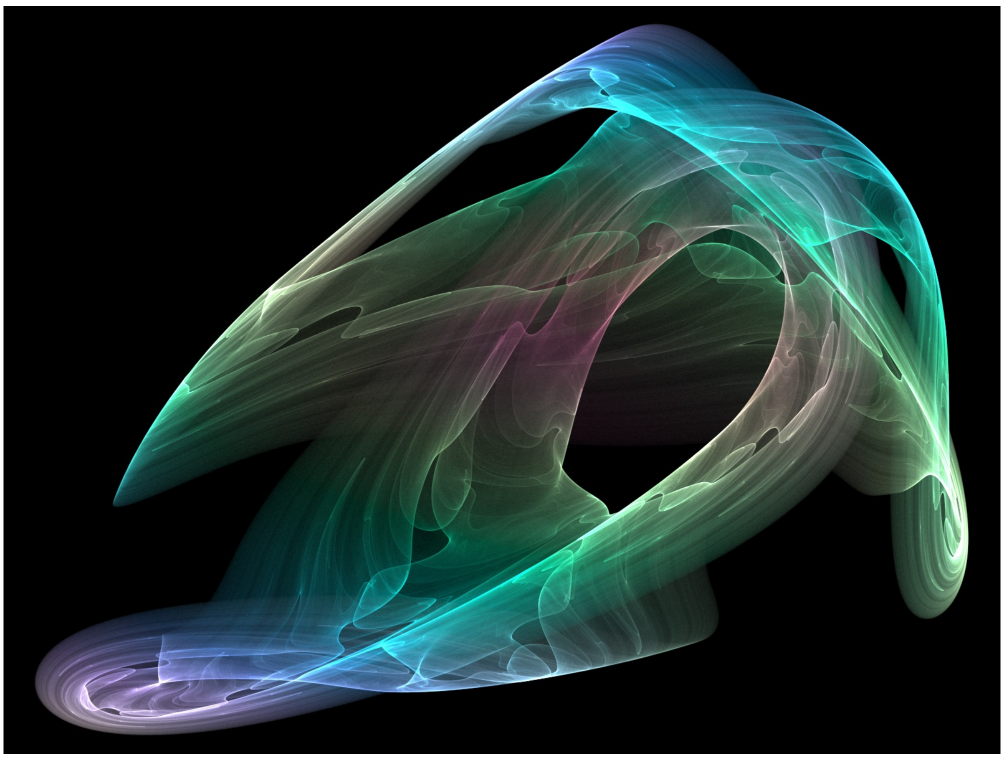
1 Differential Equations



◀ The strange attractor for a Sprott system consisting of three quadratic differential equations.[1]

A DIFFERENTIAL EQUATION is any equation that involves a derivative. For example, Newton's second law

$$F = ma$$

is actually a differential equation, since the acceleration a is the second derivative of position. We can make the differential nature of this equation more apparent by writing

[1]Based on the image Atractor Poisson Saturne by Nicolas Desprez, licensed under CC BY-SA 3.0.

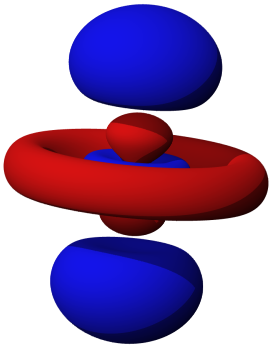
the acceleration explicitly as a second derivative:

$$F = m\,\frac{d^2x}{dt^2}$$

As you can see, differential equations are fundamental to physics, and our current belief is that all of the laws of nature can be expressed as differential equations. For example, Maxwell's equations, which govern the behavior of electromagnetic fields, are also differential equations, as is the **Schrödinger equation**

$$i\hbar\frac{\partial\Psi}{\partial t} = H[\Psi],$$

which governs the evolution of the wavefunction Ψ of a system in quantum mechanics. Among other applications, the Schrödinger equation can be used to predict the behavior of electrons in atoms, making it vital to both physics and chemistry.



▲ The $4d_{z^2}$ electron orbital. The shapes of electron orbitals are governed by the Schrödinger equation.

Applications of differential equations are not limited to physics. In general, a **dynamical system** is any system that changes or evolves over time according to fixed rules. Such systems appear throughout the natural and social sciences, and include mechanical systems, electric circuits, ongoing chemical reactions, biomechanical systems, populations of organisms and ecosystems, business and financial markets, and social networks. Each of these system has its own rules for how it evolves, and typically these rules can be described using one or more differential equations. The process of discovering these rules is known as **mathematical modeling**, and the resulting differential equations are a **mathematical model** of the given dynamical system.

For example, in chemistry the rate at which a chemical reaction occurs is governed by a **rate equation**. For a simple chemical reaction with only one reactant (or only one reactant in short supply), this equation takes the form

$$\frac{dC}{dt} = kC^n.$$

▲ The rate at which a chemical reaction occurs is governed by a rate equation.

Here C denotes the concentration of the reactant, k is a constant called the **rate constant**, and n is an integer called the **order** of the reaction. A more complicated chemical reaction with more than one reactant would have one differential equation for the concentration of each substance in the solution.

In biology, differential equations are often used to model populations of organisms in a given environment. For example, a population of animals growing in an environment with abundant resources might follow the **exponential growth equation**

$$\frac{dP}{dt} = kP.$$

▲ Differential equations are used to predict fish populations in underwater ecosystems and commercial fisheries.[2]

Here P is the size of the population and k is a constant called the **growth constant**. If instead food or space is limited, the population might grow according to the **logistic equation**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{P_{\max}}\right),$$

where $P_{\max}$ represents the maximum stable population that the given resources can support.

Finally, differential equations are often used in economics to model the behavior of economies and markets. For example, the **Solow growth model** describes the growth of economies over time using the differential equation

[2] Moofushi Kandu fish by Bruno de Giusti, licensed under CC BY-SA 2.5, via Wikimedia Commons.

$$\frac{dk}{dt} = s f(k) - (n + g + \delta)k.$$

Here k is the ratio of capital to labor, f is the **production function**, and the constants s, n, g, δ represent respectively the fraction of economic output devoted to investment, the exponential growth rate of labor, the exponential growth rate of technology, and the rate of depreciation of capital. A common choice for the production function $f(k)$ is a **Cobb-Douglas production function** $f(k) = k^\alpha$, where α is the elasticity of output with respect to capital, in which case the differential equation takes the form

$$\frac{dk}{dt} = sk^\alpha - (n + g + \delta)k.$$

1.1 The Study of Differential Equations

From a mathematical point of view, a **differential equation** is any equation that involves the derivative of an unknown function. For example, the equation

$$f'(x) = 3f(x)$$

involves the derivative of the unknown function $f(x)$. A **solution** to a differential equation is any function $f(x)$ that agrees with the given information. For example, the function $f(x) = e^{3x}$ is a solution the equation above, since the derivative of e^{3x} is equal to $3e^{3x}$.

Unlike an algebraic equation, whose solution is an unknown *number*, the solution to a differential equation is an unknown *function*.

In applications, the unknown function usually describes the way in which a particular variable changes with time. For example, if $P(t)$ describes the population of a bacteria colony at time t, then $P(t)$ might satisfy the differential equation

$$P'(t) = 3P(t).$$

This equation could also be written

$$\frac{dP}{dt} = 3P,$$

where we have used the Leibniz notation for derivatives, and we have simply written P for the population instead of $P(t)$. Here a solution to the equation would be an explicit formula for P in terms of t, such as $P(t) = e^{3t}$.

In some disciplines, it is common to write a dot above a variable instead of a prime to indicate the derivative with respect to time. Thus the equation to the left could also be written

$$\dot{P}(t) = 3P(t).$$

or simply

$$\dot{P} = 3P.$$

When discussing differential equations abstractly, we usually use x for the independent variable and y for the dependent variable, i.e. $y = y(x)$. Thus the equation above could be written

$$y'(x) = 3y(x),$$

or simply

$$y' = 3y,$$

with $y = e^{3x}$ being a possible solution.

EXAMPLE 1

Which of the following functions is a solution to the equation $xy' = 3y$?

(a) $y = e^x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = 0$

SOLUTION If $y = e^x$, then $y' = e^x$ as well. Substituting both of these into the equation

$$xy' = 3y$$

gives us

$$xe^x = 3e^x.$$

The two sides are not equal, so e^x is not a solution to this equation.

If $y = x^2$, then $y' = 2x$, and the equation $xy' = 3y$ becomes

$$2x^2 = 3x^2.$$

Again, the two sides are not equal, so $y = x^2$ is not a solution to this equation.

If $y = x^3$, then $y' = 3x^2$, and the equation $xy' = 3y$ becomes

$$3x^3 = 3x^3.$$

This time the two sides of the equation are the same, and therefore $y = x^3$ is a solution to the equation $xy' = 3y$.

Finally, if $y = 0$ (the constant zero function), then $y' = 0$ as well, and both sides of the equation $xy' = 3y$ are zero. Since the two sides of the equation are the same, it follows that $y = 0$ is also a solution to the equation $xy' = 3y$.

Of course, xe^x is the same as $3e^x$ when $x = 3$, but that doesn't mean that $y = e^x$ is a solution to the given equation. To be a solution to a differential equation, a function $y(x)$ must satisfy the equation for *all* values of x. That is, the two sides of the differential equation must be equal *as functions*.

Order of an Equation

▲ Like most systems in Newtonian mechanics, the motion of a spinning top is goverend by second-order differential equations.[3]

The **order** of a differential equation is the highest order of derivative that appears in it. A **first-order equation** involves only the first derivative of the unknown function. Most of the differential equations discussed so far have been first-order equations, and such equations are prevalent in chemistry, biology, and the social sciences.

A **second-order equation** is a differential equation that involves a second derivative. For example, Newton's second law

$$F = m\,\frac{d^2x}{dt^2}$$

is a second-order equation, since it involves the second derivative of position (i.e. the acceleration). As a result, most of the differential equations that arise in classical mechanics are second-order.

It is also possible to have **third-order equations**, **fourth-order equations**, and so forth, but these rarely arise in applications. For the most part, we will concentrate on first and second order equations.

Systems of Equations

▲ A system of two differential equations can be used to model the populations of interacting predator and prey species.[4]

A **system** of differential equations is a set of several such equations that involve the same collection of variables. Typically there is one equation describing the rate of change of each variable. For example, the motion of a satellite moving around the Earth can be modeled by the system of differential equations

$$\begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{MGx}{(x^2+y^2+z^2)^{3/2}},\\
\frac{d^2y}{dt^2} &= -\frac{MGy}{(x^2+y^2+z^2)^{3/2}},\\
\frac{d^2z}{dt^2} &= -\frac{MGz}{(x^2+y^2+z^2)^{3/2}},
\end{aligned}$$

[3] Physics in Sepia by Randen Pederson, licensed under CC BY 2.0, cropped from the original.
[4] Photo by NJR ZA via Wikimedia Commons, licensed under CC BY-SA 3.0

where M is the mass of the Earth, G is Newton's gravitational constant, and (x, y, z) denotes the position of the satellite in three-dimensional space. In general, any situation that involves more than one variable will usually require a system of differential equations to model it.

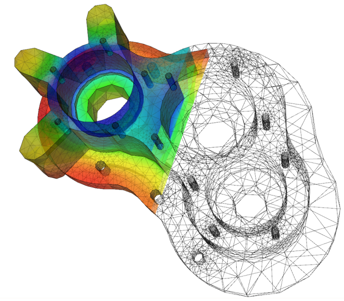
Ordinary vs. Partial Differential Equations

The two main types of differential equations are **ordinary differential equations** (**ODE's**) and **partial differential equations** (**PDE's**). For an ordinary differential equation, the unknown function is a function of a single variable, such as x or t. Most of the equations discussed so far have been ordinary differential equations.

For a partial differential equation, the unknown function is a **multivariable function** that takes several different inputs. For example, the unknown function might be the temperature $T(x, y, z, t)$ inside a solid body, which depends on the three coordinates x, y, z that describe locations inside as well as the time t. A differential equation for such a function involves its **partial derivatives**, e.g.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}.$$

In general, partial differential equations are required to model spatial phenomena such as heat flow or wave propagation, including wave functions in quantum mechanics. Though partial differential equations are quite important in both mathematics and science, their study requires significantly more calculus and analysis than the study of ordinary differential equations, and for this reason we will concentrate exclusively on ODE's.



▲ Partial differential equations are required to model the flow of heat in solid objects, such as this pump casing.[5]

▲ Water waves can be modeled using partial differential equations.[6]

[5] Image via Wikimedia Commons, licensed under CC BY-SA 3.0.

[6] Surface Waves by Roger McLassus via Wikimedia Commons, licensed under CC BY-SA 3.0.

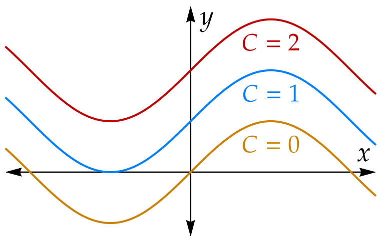
1.2 Integrable Equations

There are a few differential equations that we already know how to solve. For example, consider the equation

$$y' = \cos x.$$

A solution to this equation is any function $y(x)$ whose derivative is $\cos x$. Thus y is given by the indefinite integral

$$y = \int \cos x\, dx.$$



▲ **Figure 1:** Three curves of the form $y = \sin x + C$.

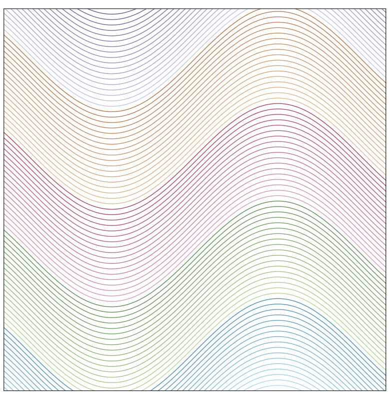
We conclude that

$$y = \sin x + C.$$

Here C is an arbitrary constant, with different values of C corresponding to different solutions. For example, $C = 0$, $C = 1$, and $C = 2$ correspond to the solutions $y = \sin x$, $y = \sin x + 1$, and $y = \sin x + 2$, respectively. Graphs of these solutions are shown in Figure 1.

When studying differential equations, any one solution to a differential equation is called a **particular solution**, while a general formula for *all* possible solutions is called a **general solution**. In this case, the formula

$$y = \sin x + C$$



▲ **Figure 2:** The family of curves $y = \sin x + C$. completely fills the plane.

is the general solution to the differential equation, with specific values of C giving the particular solutions. Note that there are infinitely many possible values of C, and therefore this differential equation has infinitely many different particular solutions. Indeed, the graphs of all of the particular solutions completely fill the plane, as shown in Figure 2

Though integration often plays a role in the solution to a differential equation, most differential equations cannot be solved simply by evaluating an indefinite integral. In fact, this only works for differential equations of the specific form

$$y' = f(x),$$

where $f(x)$ can be any function of x.

Directly Integrable Equations

A differential equation is **directly integrable** if it has the form

$$y' = f(x),$$

where $f(x)$ is a function of x. In this case, the solutions are given by the indefinite integral

$$y = \int f(x)\, dx.$$

We will assume that the reader is familiar with basic techniques for evaluating indefinite integrals, including substitution and integration by parts. Table 1.1 shows several common integrals that we will be using in examples and exercises.

Sometimes a differential equation is not directly integrable, but can be put into a directly integrable form using a little algebra. The following examples illustrate this procedure.

Common Integrals

$\int k\,dx = kx + C$	$\int e^x\,dx = e^x + C$
$\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C \;\; (p \neq -1)$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int \cos x\,dx = \sin x + C$	$\int \sin x\,dx = -\cos x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \dfrac{1}{1+x^2}\,dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

◀ **Table 1.1:** Some common integrals that arise when solving differential equations.

EXAMPLE 2

Find the general solution to the differential equation $x^2 y' = x - y'$.

SOLUTION First we must solve for y'. Adding y' to both sides gives

$$x^2 y' + y' = x.$$

We can now factor out a y'

$$(x^2+1)\,y' = x$$

and divide through by x^2+1 to get

$$y' = \frac{x}{x^2+1}.$$

This equation is now directly integrable. The solutions are given by

$$y = \int \frac{x}{x^2+1}\,dx.$$

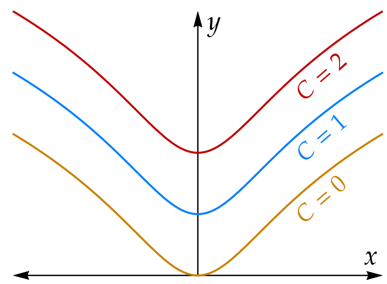
We can evaluate this integral using substitution. If we let $u = x^2+1$, then $du = 2x\,dx$ and

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

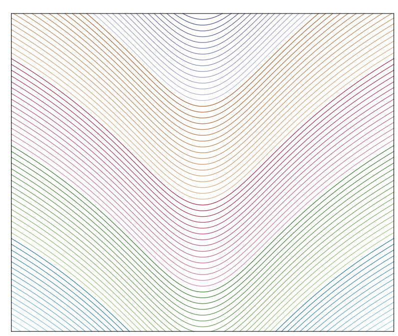
Thus the general solution is

$$y = \ln(x^2+1) + C,$$

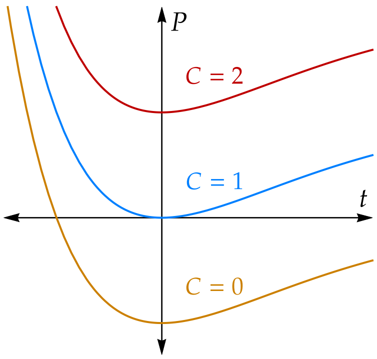
where we have dropped the absolute value around the x^2+1 since x^2+1 is always positive. Figures 3 and 4 show the graphs of these solutions.



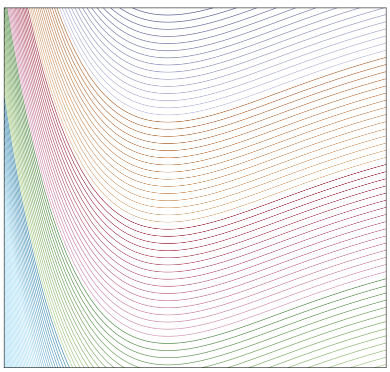
▲ **Figure 3:** Three curves of the form $y = \ln(x^2+1) + C$.



▲ **Figure 4:** The family of curves $y = \ln(x^2+1) + C$ completely fills the plane.



▲ **Figure 5:** Three functions of the form $P(t) = -(t+1)e^{-t} + C$.



▲ **Figure 6:** The graphs of the functions $P(t) = -(t+1)e^{-t} + C$ completely fill the plane.

EXAMPLE 3

Find the general solution to the differential equation $e^t \dfrac{dP}{dt} = t$.

SOLUTION We can put this equation into a directly integrable form by solving for $\dfrac{dP}{dt}$:

$$\frac{dP}{dt} = te^{-t}.$$

Then the solutions are given by

$$P = \int te^{-t}\,dt.$$

This integral requires integration by parts, which is based on the formula

$$\int u\,dv = uv - \int v\,du.$$

In this case we let $u = t$ and $dv = e^{-t}\,dt$. Then $du = dt$ and $v = -e^{-t}$, so

$$\int te^{-t}\,dt = -te^{-t} - \int(-e^{-t})\,dt = -te^{-t} - e^{-t} + C.$$

Thus the general solution is

$$P(t) = -(t+1)e^{-t} + C.$$

Figure 5 shows the particular solutions corresponding to $C = 0$, $C = 1$, and $C = 2$, and Figure 6 shows how the solutions to this differential equation completely fill the plane.

EXERCISES

1–6 ■ Use integration to find the general solution to the given differential equation.

1. $y' = x\sqrt{x^2+1}$

2. $\dfrac{dr}{dt} = t\cos t$

3. $y' + \cos(3x) = 0$

4. $e^t \dfrac{dM}{dt} = 1$

5. $xy' + 4x^3 = 1$

6. $y' = 1 - x^2y'$

1.3 Finding Solutions

As we have seen, some differential equations can be solved by directly integrating. For example, to solve

$$y' = x \cos x$$

we need only compute the integral of $x \cos x$. However, method doesn't work for an equation such as

$$y' = x^2 y.$$

The trouble here is that the right side of the equation has a y in it. Instead of just giving us a formula for y' in terms of x, this differential equation expresses a *relationship* between the derivative y' and the original function y. This makes the equation much more difficult to solve.

In general, only differential equations of the form

$$y' = f(x)$$

can be integrated directly, where $f(x)$ is any formula involving just x.

We begin with a simple example of such an equation. Pay careful attention to the solution here, for we will be returning to this example again and again.

EXAMPLE 4

Consider the following differential equation:

$$y' = y.$$

In words, this equation says that *the function y is equal to its own derivative*. What, then, are the possibilities for y?

There are two possibilities that immediately present themselves, namely

$$y = 0 \qquad \text{and} \qquad y = e^x.$$

The constant function $y(x) = 0$ is a solution the given equation, since the derivative of 0 is again just 0. Similarly, the function $y(x) = e^x$ is a solution, since the derivative of e^x is again just e^x.

It takes a little thought to come up with any other solutions. Based on our experience with integrals, we might guess that

$$y = e^x + C$$

would be a solution for any constant C, but this is not correct. For if we take the derivative of $e^x + C$, we just get e^x, which is only the same as $e^x + C$ in the case where C is 0.

However, there is a general family of solutions to this equation. If C is any constant, then

$$y = Ce^x$$

is a solution to the given equation, since the derivative of Ce^x is again just Ce^x. The graphs of these solutions are shown in Figure 7.

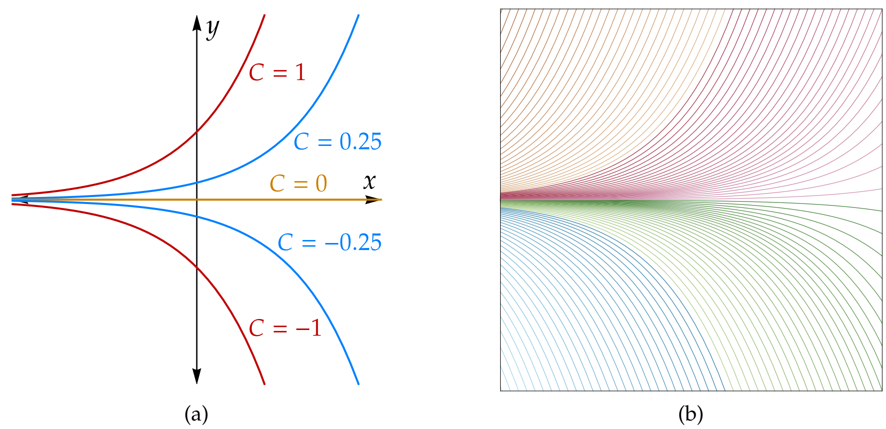
In the previous example, the solutions

$$y = 0 \qquad \text{and} \qquad y = e^x$$

were **particular solutions** to the given differential equation. These were both special cases of the more general formula

$$y = Ce^x.$$

Specifically, $y = 0$ corresponds to $C = 0$, and $y = e^x$ corresponds to $C = 1$. Indeed, though it may not be obvious, it turns out that *every* solution to the equation $y' = y$ has the form $y = Ce^x$ for some constant C. For this reason, we refer to the formula $y = Ce^x$ as the **general solution** to the differential equation.



▶ **Figure 7:** (a) Five curves of the form $y = Ce^x$. (b) The family of all such curves completely fills the plane.

This behavior is all fairly typical. Most differential equations have an infinite family of solutions, which can be written in the form of a single general solution. For a first-order equation, this general solution is often a formula that involves a single arbitrary constant C.

This suggests a procedure for solving first-order equations: first we try to guess a particular solution to the differential equation, and then we try to guess how to include a constant C in a way that makes the solution more general.

EXAMPLE 5

Find a general solution to the following differential equation.

$$y' = -y^2$$

SOLUTION To solve this equation, we must start by guessing a particular solution. Where should we begin? Well, we have no idea what formula might work here, so it probably makes sense to start with some very simple formulas:

$$y = \sin x, \qquad y = e^x, \qquad y = \sqrt{x}, \qquad y = \frac{1}{x}, \qquad y = x^2, \qquad y = \ln x.$$

Do any of these work? Check them for yourself before continuing.

It turns out that $y = 1/x$ is the right guess. If $y = 1/x$, then $y' = -1/x^2$, and the equation becomes

$$-\frac{1}{x^2} = -\left(\frac{1}{x}\right)^2.$$

So $y = 1/x$ is a particular solution to this differential equation.

Later on, we will learn a method called *separation of variables* that allows us to solve this equation without any guessing.

Now, what about the general solution? We need to figure out how to include an arbitrary constant C. Again, we just have to guess where in the formula the C might go:

$$y = \frac{1}{x} + C, \qquad y = \frac{C}{x}, \qquad y = \frac{1}{Cx}, \qquad y = \frac{1}{x+C}, \qquad y = \frac{1}{x^C}.$$

Do any of these work? Yes—it is easy to check that

$$\boxed{y = \frac{1}{x+C}}$$

is always a solution.

But is it the general solution? Presumably it is, since it includes an arbitrary constant, but it's hard to be sure about such things. We would need to somehow know that *every* solution to the given equation has this form.

A Closer Look — Proving a Solution is General

Though we have discussed how to *find* general solutions, we have mostly been ignoring the question of how to tell whether a given solution is actually general. For example, consider the differential equation

$$y' = y.$$

We know that $y = Ce^x$ is a solution for every constant C, but how do we know that these are the only possible solutions? That is, how do we know that every solution to the given equation has this form?

This is actually not too difficult to prove. Suppose that $y(x)$ is any solution to the differential equation, i.e. any function satisfying $y'(x) = y(x)$. Then we can define a new function $C(x)$ by the formula

$$C(x) = y(x)\,e^{-x}.$$

Note then that

$$y(x) = C(x)e^x.$$

We wish to prove that $C(x)$ is actually a constant. To do so, we simply take the derivative of $C(x)$ using the product rule:

$$C'(x) = y'(x)e^{-x} - y(x)e^{-x}.$$

Since $y'(x) = y(x)$, the two terms on the right cancel, and thus $C'(x) = 0$. We conclude that $C(x)$ is actually a constant function, and thus $y(x)$ has the desired form.

A *proof* is a mathematical argument that establishes the truth of a given statement.

In fact, this is not quite the general solution. It turns out that every solution has the above form, with the exception of the constant function $y = 0$. This is also a solution, but it does not correspond to any value for C.

Intuitively, $y = 0$ is the solution that you get when $C = \infty$, i.e. when you take the limit as $C \to \infty$.

Sometimes it helps to make a sequence of educated guesses.

EXAMPLE 6

Find a particular solution to the following differential equation.

$$xy' + 2y = 14x^5$$

SOLUTION Again we should just start by guessing formulas for y. However, the right side of this equation gives us a clue: maybe we should try something involving x^5. If we just try $y = x^5$, we get

$$x(5x^4) + 2(x^5) = 14x^5,$$

which isn't quit right, since the left side is just $7x^5$.

Our guess of x^5 came very close, but our left side was off by a factor of 2. Perhaps it would work to insert a 2 somewhere in our guess? Indeed, if $y = 2x^5$, then the left side will work out correctly:

$$x(10x^4) + 2(2x^5) = 14x^5.$$

Thus $\boxed{y = 2x^5}$ is one solution to this differential equation.

The general solution here is more complicated and would be hard to guess. It turns out to be $y = 2x^5 + Cx^{-2}$, where C is an arbitrary constant. The solution we found corresponds to $C = 0$.

A Closer Look Making Up Differential Equations

Although our goal is to understand how to solve differential equations, you can learn a lot by trying to make up differential equations that have a certain solution. For example, suppose we want a differential equation that has

$$y = x^3$$

as a solution. The simplest possibility is

$$y' = 3x^2.$$

However, any differential equation that holds when you plug in $y = x^3$ and $y' = 3x^2$ will work. For example, since $x(3x^2) = 3x^3$, the equation

$$xy' = 3y$$

has $y = x^3$ as a solution. Some other differential equations with $y = x^3$ as a solution include

$$(y')^3 = 27y^2, \qquad xy' + 4y = 7x^3, \qquad \text{and} \qquad yy' = 3x^5.$$

On your own, you could try making some differential equations that have $y = x^2$ as a solution, or perhaps $y = \sin x$.

EXERCISES

1–2 ■ Use guess and check to find the general solution to the given differential equation.

1. $y' + y \tan x = 0$

2. $(y')^2 = 4y$

3–5 ■ Use guess and check to find a particular solution to the given differential equation.

3. $y' + y = 9e^{2x}$

4. $yy' = 4e^{8x}$

5. $x^2y' + e^y = 2x$

1.4 Initial Value Problems

As we have seen, most differential equations have more than one solution. For a first-order equation, the general solution usually involves an arbitrary constant C, with one particular solution corresponding to each value of C.

What this means is that knowing a differential equation that a function $y(x)$ satisfies is not enough information to determine $y(x)$. To find the formula for $y(x)$ precisely, we need one more piece of information, usually called an **initial condition**.

For example, suppose we know that a function $y(x)$ satisfies the differential equation

$$y' = y.$$

It follows that

$$y(x) = Ce^x$$

for some constant C. If we want to determine C, we need at least one more piece of information about the function $y(x)$. For example, if we also know that

$$y(0) = 3,$$

the the value of C must be 3, and hence $y(x) = 3e^x$.

Initial Value Problems

An **initial value problem** consists of

1. A first-order differential equation $y' = f(x, y)$, and
2. An **initial condition** of the form $y(a) = b$.

For example,

$$y' = y, \qquad y(0) = 3$$

is an initial value problem, whose solution is

$$y = 3e^x.$$

In general, **we expect that every initial value problem has exactly one solution**. We can find this solution using the following procedure.

Solving Initial Value Problems

Given an initial value problem

$$y' = f(x, y), \qquad y(a) = b,$$

we can solve it using the following procedure:

1. Find the general solution to the given differential equation, involving an arbitrary constant C.
2. Substitute $x = a$ and $y = b$ to get an equation for C.
3. Solve for C and then substitute the answer back into the formula for y.

EXAMPLE 7

Find the solution to the following initial value problem:

$$y' = -y^2, \qquad y(0) = 5.$$

SOLUTION We previously found the general solution to this differential equation:

$$y = \frac{1}{x+C},$$

Plugging in $x = 0$ and $y = 5$ gives the equation

$$5 = \frac{1}{0+C}.$$

Solving for C gives $C = 1/5$, so

$$y = \frac{1}{x+(1/5)}.$$

This simplifies to

$$\boxed{y = \frac{5}{5x+1}}$$

In this last step we multiplied the numerator and denominator by 5 to simplify the fraction of fractions.

EXAMPLE 8

Find the solution to the following initial value problem:

$$y' = 2y, \qquad y(0) = 5.$$

SOLUTION The given differential equation isn't very different from the equation

$$y' = y.$$

In that case, the general solution was $y = Ce^x$. How can we modify this solution to account for the extra 2?

A few moments of thought reveals the answer:

$$y = Ce^{2x}$$

So this is the general solution to the given equation. Plugging in $x = 0$ and $y = 5$ gives the equation

$$5 = Ce^0,$$

so $C = 5$ and the solution is

$$\boxed{y = 5e^{2x}}$$

More generally, the solution to any equation of the form $y' = ky$ (where k is a constant) is $y = Ce^{kx}$.

The Fundamental Theorem of ODE's (Optional)

As a general rule, we expect any initial value problem of the form

$$y' = f(x,y), \qquad y(a) = b$$

to have a unique solution. The following theorem gives specific conditions which guarantee that this holds.

Fundamental Theorem of ODE's

Consider an initial value problem of the form

$$y' = f(x, y), \qquad y(a) = b.$$

If the function $f(x, y)$ is continuously differentiable for all values of x and y, then this initial value problem has a unique solution.

Here **continuously differentiable** means that both partial derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

exist and are continuous.

This theorem is also known as the **existence and uniqueness theorem for first-order ODE's**, since it guarantees both that the solution exists and that it is unique.

The hypothesis that the function $f(x, y)$ is continuously differentiable is important for the theorem. In fact, there are initial value problems that do not satisfy this hypothesis that have more than one solution. For example, the initial value problem

$$y' = \frac{y}{x}, \qquad y(0) = 0$$

has infinitely many different solutions, namely the lines $y = Cx$ for all possible values of C. The function $f(x, y)$ in this case is y/x, which is not defined (and hence not continuously differentiable) when $x = 0$.

There is a nice geometric interpretation of the fundamental theorem. As we have seen, the solutions to a differential equation can be viewed as a family of solution curves in the xy-plane. For example, Figure 8 shows the curves $y = \ln(x + C)$, which are the solutions to the differential equation

$$y' = e^{-y}.$$

From a geometric point of view, an initial condition $y(a) = b$ is the same as a point (a, b) that the solution curve must pass through. Thus, saying that the initial value problem

$$y' = f(x, y), \qquad y(a) = b$$

has a unique solution is the same as saying that the point (a, b) has exactly one solution curve passing through it. This leads us to the following restatement of the fundamental theorem of ODE's.

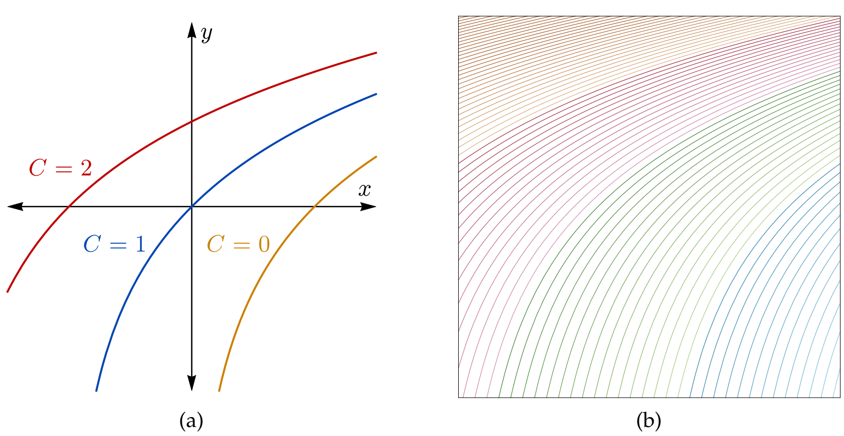


Figure 8: (a) Three curves of the form $y = \ln(x + C)$. (b) The family of all such curves completely fills the plane.

Fundamental Theorem of ODE's (Geometric Version)

Consider a first-order differential equation of the form

$$y' = f(x, y),$$

where the function $f(x, y)$ is continuously differentiable. Then:

1. The solution curves for this differential equation completely fill the plane, and
2. Solution curves for different solutions do not intersect.

Here statement (1) is the same as saying that every point (a, b) lies on at least one solution curve, i.e. every initial condition gives at least one solution. Statement (2) is the same as saying that no point (a, b) lies on more than one solution curve, i.e. every initial condition has at most one solution.

EXERCISES

1–2 ■ Solve the given initial value problem.

1. $y' = xe^x$, $y(0) = 3$

2. $y' = 3y$, $y(2) = 4$

1.5 Second-Order Equations

Recall that the **second derivative** of a function y is the derivative of the derivative. This can be written

$$y'' \qquad \text{or} \qquad \frac{d^2y}{dx^2}.$$

When using t (for time) instead of x, the second derivative is sometimes written with two dots, i.e. $\ddot{y}$.

A **second-order equation** is a differential equation that involves y'', as well as perhaps y', y, and x.

Second-order equations are quite important in physics, since acceleration is the second derivative of position. Indeed, virtually all important differential equations in physics are second-order, whereas most important differential equations in biology, chemistry, and economics are first-order. As we will see, second-order equations and first-order equations behave quite differently.

EXAMPLE 9

Find the general solution to the following second-order equation.

$$y'' = 12x^2.$$

SOLUTION Integrating once gives a formula for y':

$$y' = \int 12x^2\,dx = 4x^3 + C.$$

We can now integrate again to get a formula for y.

$$y = \int (4x^3 + C)\,dx = x^4 + Cx + C_2.$$

Here C_2 represents a *new* constant of integration, which may be different from the original C. Actually, it would make more sense to refer to the original C as C_1:

$$\boxed{y = x^4 + C_1x + C_2}$$

This is the general solution to the given second-order equation.

In general, any second-order equation of the form

$$y'' = f(x)$$

can be solved by integrating twice.

The general solution we found in the last example involved *two* arbitrary constants C_1 and C_2. This is typical for a second-order equation.

1. The general solution to a **first-order** equation usually involves **one** arbitrary constant.
2. The general solution to a **second-order** equation usually involves **two** arbitrary constants.

Incidentally, there are also third-order equations (involving the third derivative), fourth-order equations, and so forth. As you would expect, the general solution to an nth-order differential equation usually involves n arbitrary constants. However, we will mostly restrict our attention to first and second order equations, since equations of third order and higher are rare in both science and mathematics.

EXAMPLE 10

Find the general solution to the following second-order equation.

$$y'' = y.$$

SOLUTION Obviously $y = e^x$ is a solution, and more generally $y = C_1 e^x$ is a solution for any constant C_1. However, this is not the general solution—we are expecting one more arbitrary constant.

So how can we find another solution to this differential equation? Think about this for a minute—we want a function other than a multiple of e^x that is equal to its own second derivative.

The answer is quite clever: what about $y = e^{-x}$? Though the derivative of e^{-x} has an extra minus sign, the second derivative is again e^{-x}, so e^{-x} is a solution to the above equation. Indeed, anything of the form $y = C_2 e^{-x}$ is a solution, where C_2 can be any constant.

But how can we combine the two solutions into a single formula? In this case, it turns out that it works to just add them together:

$$\boxed{y = C_1 e^x + C_2 e^{-x}}$$

(The reader may want to check this by plugging this formula into the original equation.) This formula includes two arbitrary constants, so it ought to be the general solution to the given second-order equation.

Because the general solution to a second-order equation involves two arbitrary constants, you need two additional pieces of information to determine a single solution. One option is to give two different values for y, e.g. $y(0)$ and $y(1)$. This is called a **boundary value problem**, and you can solve it using the following procedure.

It is common in applications that the two known values of y are at the boundary points of the interval of possible x-values. Hence the terminology "boundary value problem".

Solving Boundary-Value Problems

1. Find the general solution to the given second-order equation, involving constants C_1 and C_2.
2. Plug in the first value for y to get an equation involving C_1 and C_2.
3. Plug in the second value for y' to get another equation involving C_1 and C_2.
4. Solve the two equations for the unknown constants C_1 and C_2.

EXAMPLE 11

Find the solution to the following boundary-value problem

$$y'' = 12x, \qquad y(-1) = 3, \qquad y(1) = 5.$$

SOLUTION We can integrate to get a formula for y':

$$y' = \int 12x\,dx = 6x^2 + C_1,$$

and then integrate again to get a formula for y:

$$y = \int (6x^2 + C_1)\,dx = 2x^3 + C_1 x + C_2,$$

All that remains is to find the values of C_1 and C_2.

Plugging in $x = -1$ and $y = 3$ gives the equation

$$3 = -2 - C_1 + C_2,$$

and plugging in $x = 1$ and $y = 5$ gives the equation

$$5 = 2 + C_1 + C_2,$$

We can solve these two equations to get $C_1 = -1$ and $C_2 = 4$, so

$$\boxed{y = 2x^3 - x + 4}$$

Instead of giving two pieces of information about y, another way of specifying a single solution to a second-order differential equation is to give one piece of information about y and one piece of information about y'. In particular, a **second-order initial value problem** consists of the following information:

1. A second-order differential equation involving an unknown function y.
2. An initial condition for y, such as the value of $y(0)$.
3. An initial condition for y', such as the value of $y'(0)$.

Such conditions are common in physics, where $y(0)$ would represent the initial *position* of an object, and $y'(0)$ would represent the initial *velocity* of an object. We can solve such a problem using the following procedure.

Solving Second-Order Initial Value Problems

1. Find the general solution to the given second-order equation, involving constants C_1 and C_2.

2. Plug in the initial value for y to get an equation involving C_1 and C_2.

3. Take the derivative of the general formula for y to get a general formula for y'.

4. Plug in the initial value for y' to get another equation involving C_1 and C_2.

5. Solve the two equations for the unknown constants C_1 and C_2.

EXAMPLE 12

Find the solution to the following initial value problem.

$$y'' = y, \qquad y(0) = 7, \qquad y'(0) = 3.$$

SOLUTION As we saw in Example 10, the general solution to the given equation is

$$y = C_1 e^x + C_2 e^{-x}.$$

Therefore, we need only figure out the values of C_1 and C_2.

Plugging in $y(0) = 7$ gives the equation

$$7 = C_1 + C_2.$$

To plug in $y'(0) = 3$, we must start by taking the derivative of our general formula for y:

$$y' = C_1 e^x - C_2 e^{-x}.$$

We can now plug in $y'(0) = 3$ to get the equation

$$3 = C_1 - C_2.$$

We now have two equations for C_1 and C_2:

$$C_1 + C_2 = 7 \qquad \text{and} \qquad C_1 - C_2 = 3.$$

Solving yields $C_1 = 5$ and $C_2 = 2$, so

$$\boxed{y = 5e^x + 2e^{-x}}$$

EXERCISES

1–2 ■ Use integration to find the general solution to the given differential equation.

1. $y'' = \sqrt[3]{x}$

2. $x^3 y'' = x + 2$

3. Use guess and check to find a particular solution to the equation $y'y'' = 14y + 4x^3$.

4–5 ■ Solve the given boundary value problem.

4. $y'' = \sin x$, $y(0) = 4$, $y(\pi) = 6$

5. $y'' = y$, $y(0) = 7$, $y(\ln 2) = 8$

6–7 ■ Solve the given initial value problem.

6. $y'' = x^2$, $y(1) = 1/2$, $y'(1) = 1/2$

7. $y'' = 4y$, $y(0) = 5$, $y'(0) = 2$