10.1 Subspaces

The word "subspace" is used for several different concepts in mathematics. For clarity, the subpaces we discuss here are sometimes called **linear subspaces** or **vector subspaces**.

A **subspace** is simply a flat that goes through the origin. For example, a one-dimensional subspace is a line that goes through the origin, a two-dimensional subspace is a plane that goes through the origin, and so forth. Subspaces are by far the most important flats in linear algebra. As we shall see, they have several nice properties that distinguish them from other flats.

Span of Vectors

Any subspace of $\mathbb{R}^n$ can be defined by a parametric equation of the form

$$(x_1, x_2, \ldots, x_n) = \mathbf{0} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k.$$

where $\mathbf{0} = (0, 0, \ldots, 0)$ is the origin. Since adding $\mathbf{0}$ has no effect, we can omit it from the equation:

$$(x_1, x_2, \ldots, x_n) = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k.$$

What this says is that the subspace consists of all possible linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. This has a special name.

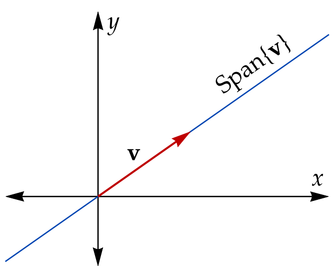
Span of Vectors

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^n$, the **span** of these vectors, written

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

is the subspace of $\mathbb{R}^n$ consisting of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Geometrically, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the flat through the origin in $\mathbb{R}^n$ that contains the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.



▲ **Figure 1:** The span of a single vector $\mathbf{v}$ in $\mathbb{R}^2$.

EXAMPLE 1

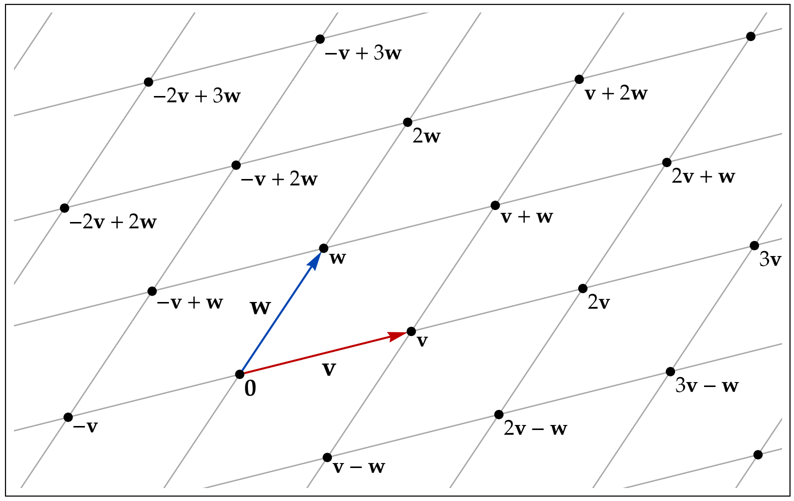
If $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^n$, the span of $\mathbf{v}$ is simply the set of all scalar multiples of $\mathbf{v}$. This is the line defined by the parametric equation

$$(x_1, x_2, \ldots, x_n) = t\,\mathbf{v}.$$

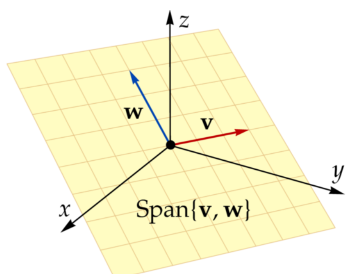
This line is illustrated in Figure 1. Some specific examples include:

- In two dimensions, $\text{Span}\{(1,1)\}$ is the line $y = x$, and $\text{Span}\{(1,2)\}$ is the line $y = 2x$.
- In three dimensions, $\text{Span}\{\mathbf{i}\}$ is the x-axis, $\text{Span}\{\mathbf{j}\}$ is the y-axis, and $\text{Span}\{\mathbf{k}\}$ is the z-axis.
- Also in three dimensions, $\text{Span}\{(1,1,1)\}$ is the line $(x, y, z) = (t, t, t)$, which lies evenly between the three axes.
- In four dimensions, $\text{Span}\{(0,1,0,1)\}$ is a line through the origin on the x_2x_4-plane that makes a $45°$ angle with both the x_2 and x_4 axes.

Incidentally, there is one exception to the rule that the span of a single vector is a line. Specifically, the only scalar multiple of the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ is $\mathbf{0}$ itself, and therefore $\text{Span}\{\mathbf{0}\}$ is just the origin.



▶ **Figure 2:** The span of two vectors $\mathbf{v}$ and $\mathbf{w}$ is a plane. Points inside the parallelograms can be obtained using linear combinations with non-integer coefficients.



▲ **Figure 3:** The span of a two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

EXAMPLE 2

If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ that point in two different directions, the span of $\mathbf{v}$ and $\mathbf{w}$ is the plane through the origin that contains them both, as shown in Figure 3. This subspace is defined by the parametric equation

$$(x_1, x_2, \ldots, x_n) = s\,\mathbf{v} + t\,\mathbf{w}.$$

Figure 2 shows how the different linear combinations of $\mathbf{v}$ and $\mathbf{w}$ fill a plane.

Some specific examples of planes spanned by two vectors include:

- In three dimensions, $\mathrm{Span}\{\mathbf{i},\mathbf{j}\}$ is the xy-plane, $\mathrm{Span}\{\mathbf{i},\mathbf{k}\}$ is the xz-plane, and $\mathrm{Span}\{\mathbf{j},\mathbf{k}\}$ is the yz-plane.
- Also in three dimensions, $\mathrm{Span}\{(1,0,0),(0,1,1)\}$ is the plane $y = z$, which is given by the parametric equation
  $$(x, y, z) = (s, t, t).$$
  Geometrically, this plane contains the x-axis and makes 45° angles with both the y and z axes.
- The x_2x_4-plane in $\mathbb{R}^4$ can be described as $\mathrm{Span}\{\mathbf{e}_2,\mathbf{e}_4\}$, where $\mathbf{e}_2 = (0,1,0,0)$ and $\mathbf{e}_4 = (0,0,0,1)$ are the standard basis vectors.
- In two dimensions, $\mathrm{Span}\{\mathbf{i},\mathbf{j}\}$ is the entire plane $\mathbb{R}^2$. More generally, if $\mathbf{v}$ and $\mathbf{w}$ are any two vectors in $\mathbb{R}^2$ that are not scalar multiples of one another, then $\mathrm{Span}\{\mathbf{v},\mathbf{w}\} = \mathbb{R}^2$.

Incidentally, it is not always true that the span of two vectors is a plane. If $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples of one another, then $\mathrm{Span}\{\mathbf{v},\mathbf{w}\}$ will just be a line, namely the line of all scalar multiples of $\mathbf{v}$ (or $\mathbf{w}$).

EXAMPLE 3

Determine whether the vector $(4,-1,7)$ lies in the span of $(1,2,1)$ and $(2,1,3)$.

SOLUTION The question is whether $(4,-1,7)$ is a linear combination of these two vectors. That is, we want to find values of s and t so that

$$s\begin{bmatrix}1\\2\\1\end{bmatrix} + t\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}4\\-1\\7\end{bmatrix}.$$

This is equivalent to three equations involving s and t:

$$s + 2t = 4, \qquad 2s + t = -1, \qquad s + 3t = 7.$$

Solving the first two equations for s and t gives $s = -2$ and $t = 3$. This works in the third equation also, and therefore $(4, -1, 7)$ does lie in the span of these two vectors:

$$-2\begin{bmatrix}1\\2\\1\end{bmatrix} + 3\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}4\\-1\\7\end{bmatrix}.$$

Points and Parallel Vectors

As we have seen, every subspace of $\mathbb{R}^n$ can be described as the span of a set of vectors. That is, every subspace can be defined by a parametric equation of the form

$$(x_1, x_2, \ldots, x_n) = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_n\mathbf{v}_n$$

Where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors parallel to the flat.

However, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ can also be viewed as *points* on the subspace. For example, in the case of a two-dimensional subspace

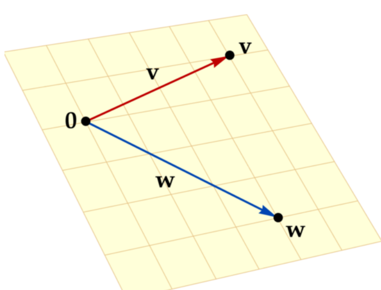
$$(x_1, x_2, \ldots, x_n) = s\mathbf{v} + t\mathbf{w}$$

the point $\mathbf{v}$ corresponds to $s = 1$ and $t = 0$, while the point $\mathbf{w}$ corresponds to $s = 0$ and $t = 1$. For a k-dimensional subspace

$$(x_1, x_2, \ldots, x_n) = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_n\mathbf{v}_n$$

each point $\mathbf{v}_i$ can be obtained by substituting 1 for t_i and 0 for all of the other parameters.

This identification between points and parallel vectors is always true for subspaces.



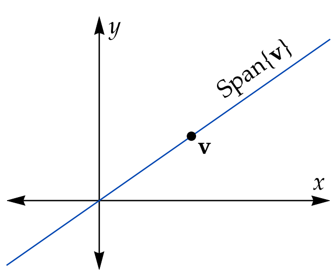
▲ **Figure 4:** Vectors $\mathbf{v}$ and $\mathbf{w}$ parallel to a subspace and the associated points.

The points on a subspace are the same as the vectors parallel to the subspace.

That is, a point $\mathbf{v}$ lies on a given subspace if and only if the vector $\mathbf{v}$ is parallel to the subspace.

This principle is illustrated in Figure 4. In this figure, $\mathbf{v}$ and $\mathbf{w}$ are two vectors parallel to a subspace, which we have drawn as emanating from the origin $\mathbf{0}$. Then the other endpoints of these arrows are the points $\mathbf{v}$ and $\mathbf{w}$, so these points must lie on the subspace. Conversely, if $\mathbf{v}$ and $\mathbf{w}$ are points on a subspace, then the corresponding radial vectors $\mathbf{v}$ and $\mathbf{w}$ must be parallel to the subspace.

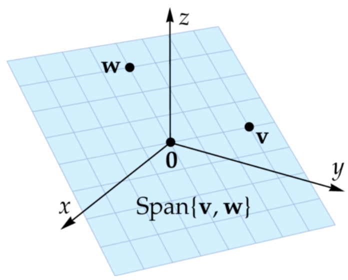
The identification between points and parallel vectors gives us a new interpretation of the span:



▲ **Figure 5:** The span of a single point $\mathbf{v}$ in $\mathbb{R}^2$.

The span of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is the smallest flat that goes through the points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and the origin.

For example, if $\mathbf{v}$ is a nonzero vector, then $\mathrm{Span}\{\mathbf{v}\}$ can be thought of as the line that goes through the point $\mathbf{v}$ and the origin, as shown in Figure 5. Similarly, if $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors, then $\mathrm{Span}\{\mathbf{v}, \mathbf{w}\}$ can be thought of as the plane that goes through the points $\mathbf{v}$ and $\mathbf{w}$ as well as the origin, as shown in Figure 6.



▲ **Figure 6:** The span of two points $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$.

EXAMPLE 4

Find two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ whose span is the plane $3x - 2y + z = 0$.

SOLUTION All we need are two points $\mathbf{v}$ and $\mathbf{w}$ that lie on this plane. For example,

$$\mathbf{v} = (1, 0, -3) \qquad \text{and} \qquad \mathbf{w} = (0, 1, 2)$$

suffice. Note that these vectors are not scalar multiples of one another, so the span is indeed the whole plane $3x - 2y + z = 0$ as opposed to a line.

Properties of Subspaces

Because of the identification between points and parallel vectors, the points on a subspace have certain fundamental properties.

Though it may not be obvious, these two properties actually characterize subspaces. That is, if S is a (nonempty) set of points in $\mathbb{R}^n$ that have these two properties, then S must be a subspace of $\mathbb{R}^n$.

Properties of Subspaces

Let S be a subspace of $\mathbb{R}^n$.

1. For any points $\mathbf{v}, \mathbf{w}$ on S, the sum $\mathbf{v} + \mathbf{w}$ also lies on S.
2. For any point $\mathbf{v}$ on S and any scalar k, the point $k\mathbf{v}$ also lies on S.

It follows from these properties than any linear combination of points on a subspace again lies on the subspace.

Algebraically, the reason that this plane goes through the origin is that constant term of the equation is zero. A linear equation with this property is said to be **homogeneous**.

EXAMPLE 5

The equation

$$x + y + z = 0$$

defines a two-dimensional subspace of $\mathbb{R}^3$, i.e. a plane through the origin.

Because this is a subspace, any scalar multiple of a point on this plane again lies on the plane. For example, $(2, -1, -1)$ lies on this plane, and

$$3(2, -1, -1) = (6, -3, -3),$$

so $(6, -3, -3)$ also lies on this plane.

Similarly, the sum of any two points on this plane will always be another point on this plane. For example, $(2, -1, -1)$ and $(2, 3, -5)$ both lie on this plane, and

$$(2, -1, -1) + (2, 3, -5) = (4, 2, -6)$$

so $(4, 2, -6)$ also lies on this plane.

Indeed, any linear combination of points on this plane will again lie on this plane. For example, the point

$$5(2, -1, -1) + 7(2, 3, -5) = (24, 16, -40)$$

also lies on this plane.

EXERCISES

1. Which of the planes

$$x + y - z = 3, \qquad x + 3z = 1, \qquad \text{and} \qquad x - 2y + 7z = 0$$

is a subspace of $\mathbb{R}^3$?

2. Let L be the line in $\mathbb{R}^2$ that goes through the points $(1,3)$ and $(2,6)$. Is L is a subspace of $\mathbb{R}^2$? Explain.

3. Find the point at which the line $\text{Span}\{(3,2,1)\}$ intersects the plane $x + y + z = 24$.

4. Does the point $(2,1,4)$ lie on the plane $\text{Span}\{(6,3,8),(4,2,5)\}$? Explain.

5. The span of the vectors $(3,1,2)$ and $(2,-1,0)$ is a plane in $\mathbb{R}^3$. Find a Cartesian equation for this plane.

6. (a) Is the span of the vectors $(4,-6,2,-4)$ and $(-6,9,-3,6)$ a line or a plane in $\mathbb{R}^4$? Explain.

(b) Is the span of the vectors $(3,1,4,2,3)$ and $(6,2,8,4,2)$ a line or a plane in $\mathbb{R}^5$? Explain.

7. Suppose that S is a three-dimensional subspace of $\mathbb{R}^5$ that contains the points $(1,2,0,0,0)$ and $(0,0,0,1,1)$. Must S contain the point $(3,6,0,4,4)$? Explain.

8–13 ■ Determine whether the vector $\mathbf{w}$ is in $\text{Span}\{\mathbf{u},\mathbf{v}\}$. Try to solve these problems without writing anything down.

8. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}$

9. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

10. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

11. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$

12. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix}$

13. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$

14–17 ■ Try to solve the given geometry problem without writing anything down.

14. Find the point on the plane $\text{Span}\{(1,0,0),(0,1,0)\}$ closest to the point $(5,7,2)$.

15. Find the distance from the point $(3,0,8)$ to the line $\text{Span}\{(0,0,1)\}$.

16. Find the point at which $\text{Span}\{(1,1,1)\}$ intersects the line $(x,y,z) = (3,6-t,t)$.

17. Find the point at which $\text{Span}\{(2,1,7)\}$ intersects the plane $5x - 2y + 3z = 0$.

10.2 Linear Dependence

As we have seen, the span of one vector in $\mathbb{R}^n$ is usually a line, the span of two vectors in $\mathbb{R}^n$ is usually a plane, and so forth. In general, the span of k vectors in $\mathbb{R}^n$ is usually a k-dimensional subspace.

Sometimes, though, the dimension of a span is less than you would expect. For example, the span of the two vectors

$$(1, 0, 0) \qquad \text{and} \qquad (2, 0, 0)$$

is just the x-axis in $\mathbb{R}^3$. The problem is that both of these vectors lie on the x-axis, which is a 1-dimensional subspace, and therefore any linear combination of these vectors also lies on the x-axis.

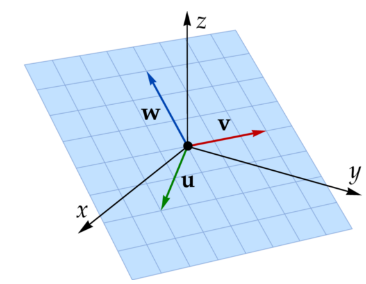
The same thing can happen with three vectors. For example, the span of the three vectors

$$(0, 2, 1), \qquad (0, 1, 3), \qquad \text{and} \qquad (0, -1, 1)$$

is just the yz-plane in $\mathbb{R}^3$. Again, the problem is that all three vectors lie on the yz-plane, so any linear combination of them also lies on this plane.

Linear Dependence

We say that k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are **linearly dependent** if they all lie on a common subspace of dimension less than k.



▲ **Figure 1:** Three linearly dependent vectors in $\mathbb{R}^3$.

Thus two vectors are linearly dependent if they lie on a common line through the origin, and three vectors are linearly dependent if they lie on a common line or plane through the origin, as shown in Figure 1.

The opposite of linear dependence is **linear independence**. We say that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if the smallest subspace that contains them has dimension k, i.e. if the span of the vectors is k-dimensional.

EXAMPLE 1

Determine whether the given vectors are linearly dependent or linearly independent.

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

SOLUTION

(a) Both of these vectors lie on the line $y = 2x$, so they are $\boxed{\text{linearly dependent}}$.

(b) These vectors lie on different lines through the origin (namely $y = -x$ and the y-axis), so they are $\boxed{\text{linearly independent}}$.

(c) All three of these vectors lie on the plane $x = y$ in $\mathbb{R}^3$ (which goes through the origin), so they are $\boxed{\text{linearly dependent}}$.

(d) The first two vectors lie on the xy-plane, but the third does not, so there is no plane through the origin that contains all three vectors. Thus these vectors are $\boxed{\text{linearly independent}}$.

By the way, since $\mathbb{R}^n$ is n-dimensional, we can have at most n linearly independent vectors in $\mathbb{R}^n$.

More than n vectors in $\mathbb{R}^n$ are always linearly dependent.

For example, the vectors $(2,1)$, $(1,3)$, and $(1,-1)$ in $\mathbb{R}^2$ are linearly dependent since they all lie in the same plane, namely $\mathbb{R}^2$ itself.

The Algebra of Linear Dependence

In most examples, it is not immediately obvious what the geometric relationship is between vectors we are given. In this case, we must use algebra to check whether the vectors are linearly independent or linearly dependent.

For two vectors this is quite easy.

Linear Dependence for Two Vectors

Two vectors are linearly dependent if and only if

1. One of them is the zero vector, or
2. The two vectors are scalar multiples of one another.

Any collection of vectors that includes the zero vector must be linearly dependent.

For example, the vectors $(4,-6,2)$ and $(-6,9,-3)$ are linearly dependent since

$$\begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}.$$

It is more difficult to check whether three vectors are linearly independent.

Linear Dependence for Three Vectors

Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent if and only if either

1. $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, or
2. $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Equivalently, three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent if and only if

1. $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and
2. $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

The idea here is that $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ must be a plane if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. If $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ does not lie on this plane, and therefore $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is three-dimensional.

EXAMPLE 2

Determine whether the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

are linearly independent or linearly dependent.

SOLUTION Clearly $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus the only question is whether $\mathbf{w}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$. The equation

$$s\begin{bmatrix}2\\8\\3\end{bmatrix} + t\begin{bmatrix}1\\5\\2\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

yields the system of equations

$$2s + t = 1, \qquad 8s + 5t = 1, \qquad 3s + 2t = 0.$$

This has $s = 2$ and $t = -3$ as a solution. We conclude that $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$, i.e.

$$\begin{bmatrix}1\\1\\0\end{bmatrix} = 2\begin{bmatrix}2\\8\\3\end{bmatrix} - 3\begin{bmatrix}1\\5\\2\end{bmatrix},$$

so these three vectors are $\boxed{\text{linearly dependent}}$.

We can generalize this test to more than three vectors as follows.

Of course, the order of the vectors is arbitrary, so $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent whenever one of them is a linear combination of the others.

Linear Dependence for k Vectors

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent if and only if

1. One of them is the zero vector, or
2. One of them is a linear combination of the previous vectors.

EXAMPLE 3

Determine whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix}1\\2\\3\\0\\0\end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix}4\\5\\2\\0\\2\end{bmatrix}$$

are linearly independent or linearly dependent.

SOLUTION None of these is the zero vector, and clearly $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$. This means that $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane. Moreover, $\mathbf{v}_3$ cannot be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, since

$$s\mathbf{v}_1 + t\mathbf{v}_2 = (s, t, t, 0, 0)$$

and $\mathbf{v}_3$ is not of this form. Therefore $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is three-dimensional.

Finally, $\mathbf{v}_4$ cannot be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ since its last coordinate is nonzero. Then $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ must be four-dimensional, so the four vectors are $\boxed{\text{linearly independent}}$.

EXERCISES

1. Are the vectors $\begin{bmatrix}1\\0\\2\end{bmatrix}$, $\begin{bmatrix}3\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ linearly dependent or linearly independent? Explain.

2. Are the vectors $\begin{bmatrix}1\\4\end{bmatrix}$, $\begin{bmatrix}3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\1\end{bmatrix}$ linearly dependent or linearly independent? Explain.

3. Which of the following pairs of vectors are linearly independent?

$$\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}3\\3\\6\end{bmatrix} \qquad \begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \begin{bmatrix}0\\4\\-8\end{bmatrix},\begin{bmatrix}0\\-3\\6\end{bmatrix} \qquad \begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}$$

4–7 ■ Determine whether the given vectors are linearly dependent or linearly independent.

4. $\begin{bmatrix}4\\5\\1\end{bmatrix}, \begin{bmatrix}3\\7\\2\end{bmatrix}, \begin{bmatrix}9\\8\\1\end{bmatrix}$

5. $\begin{bmatrix}2\\4\\3\end{bmatrix}, \begin{bmatrix}1\\3\\4\end{bmatrix}, \begin{bmatrix}1\\5\\6\end{bmatrix}$

6. $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}7\\8\\-2\\4\end{bmatrix}$

7. $\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\4\end{bmatrix}$

8–11 ■ Determine whether the span of the given vectors is a line, a plane, or a hyperplane.

8. $\begin{bmatrix}3\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}6\\-2\\-4\\2\end{bmatrix}$

9. $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix}, \begin{bmatrix}4\\4\\3\\6\end{bmatrix}$

10. $\begin{bmatrix}2\\4\\0\\2\end{bmatrix}, \begin{bmatrix}3\\6\\0\\3\end{bmatrix}, \begin{bmatrix}4\\8\\0\\4\end{bmatrix}$

11. $\begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

12. Are the following vectors linearly dependent or linearly independent? Explain.

$$\begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\\2\\1\end{bmatrix}$$