11.1 Linear Systems

Recall that a **linear equation** is any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $x_1, x_2, \ldots, x_n$ are variables and $a_1, a_2, \ldots, a_n$ and b are constants. A **linear system** is a system of linear equations, i.e. a collection of linear equations involving the same variables. For example,

$$\begin{aligned} 5x + 3y - 4z &= 15 \\ 4x - 8y + 5z &= 18 \end{aligned}$$

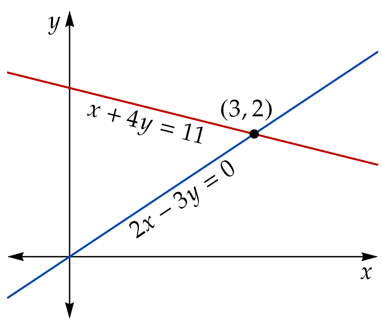
Here 2×3 is pronounced "two-by-three".

is a linear system. Specifically, this is a **2×3 system**, meaning that it involves 2 equations and 3 variables.

A **solution** to a linear system is an assignment of values to the variables that makes *all* of the equations true. For example, $x = 4$, $y = 1$, $z = 2$ is a solution to the 2×3 system above, since substituting these values in for the variables makes both equations true. We can think of this solution as an ordered triple $(4, 1, 2)$, i.e. a point in $\mathbb{R}^3$.

The set of all possible solutions to a linear system is called its **solution set**. For a linear system with n variables, the solution set is a set of points in $\mathbb{R}^n$. For example, the solution set for the 2×3 linear system above is a line in $\mathbb{R}^3$.

2×2 Systems



▲ **Figure 1:** A 2×2 system consists of two lines in $\mathbb{R}^2$.

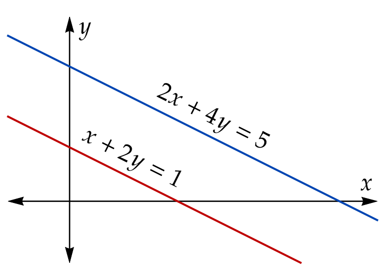
For a linear system in two variables, each equation represents a line in $\mathbb{R}^2$. Thus a 2×2 system such as

$$\begin{aligned} x + 4y &= 11 \\ 2x - 3y &= 0 \end{aligned}$$

represents two lines in $\mathbb{R}^2$, as shown in Figure 1. There is only one solution to this system, namely the point $(3, 2)$, since this is the only point that lies on both of the lines. This behavior is typical: **a 2×2 system usually has exactly one solution**.

There are exceptions to this rule. For example, the system

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 5 \end{aligned}$$



▲ **Figure 2:** A 2×2 system has no solution if the lines are parallel.

has no solutions, for multiplying the first equation through by 2 gives $2x + 4y = 2$, which contradicts the second equation. Geometrically, these two equations correspond to parallel lines in $\mathbb{R}^2$, as shown in Figure 2.

Let's see what happens if we try to solve the system in the usual way. Solving for x in the first equation gives

$$x = 1 - 2y$$

and plugging this into the second equation gives

$$2(1 - 2y) + 4y = 5.$$

The y terms on the left side cancel, leaving the contradictory equation $2 = 5$, which indicates the contradiction in the original linear system.

3×2 Systems

Now consider a 3×2 system such as

$$\begin{aligned} x + 4y &= 16 \\ 3x - 4y &= 0 \\ 6x + 2y &= 16 \end{aligned}$$

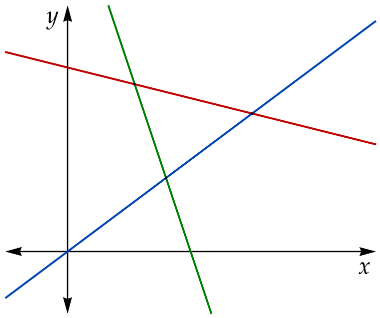
This system corresponds to three lines in $\mathbb{R}^2$, as shown in Figure 3. Because there aren't any points that lie on all three lines, this system has no solutions at all! Again, this behavior is typical: **a 3×2 system usually has no solutions**.

Of course, it is possible for a 3×2 system to have a solution, if all three lines happen to go through the same point. For example, if we change the equation $6x + 2y = 11$ to $6x + 2y = 30$, then all three lines go through the point $(4, 3)$, as shown in Figure 4. Thus the 3×2 system

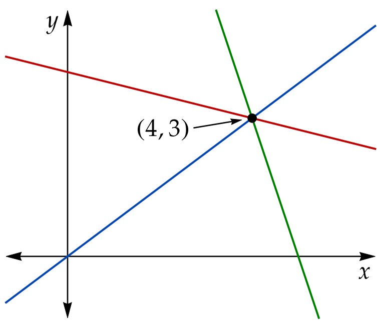
$$\begin{aligned} x + 4y &= 16 \\ 3x - 4y &= 0 \\ 6x + 2y &= 30 \end{aligned}$$

has exactly one solution, namely the point $(4, 3)$.

In general, the easiest way to check whether a 3×2 system has a solution is to find the intersection point of the first two lines, and then check whether this point also lies in the third line. If it does, then the point is a solution to the system. If it doesn't, then the system has no solutions.



▲ **Figure 3:** A 3×2 system consists of three lines in $\mathbb{R}^2$.



▲ **Figure 4:** A 3×2 system has a solution when all three lines go through a single point.

EXAMPLE 1

Determine whether the following 3×2 system has a solution:

$$\begin{aligned} 5x + 2y &= 18 \\ 3x - 4y &= 16 \\ 2x + 3y &= 9 \end{aligned}$$

SOLUTION The first two equations form a 2×2 system:

$$\begin{aligned} 5x + 2y &= 18 \\ 3x - 4y &= 16 \end{aligned}$$

Solving this system in the usual way, we find that $x = 4$ and $y = -1$. Plugging this into the third equation gives

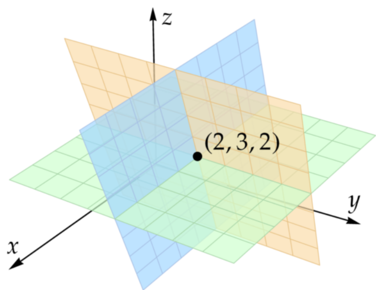
$$2(4) + 3(-1) = 9,$$

which isn't true, so this system doesn't have a solution.

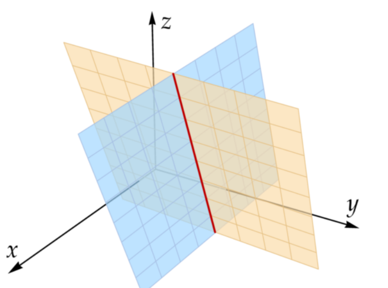
Linear Systems in $\mathbb{R}^3$

For a linear system with three variables, each linear equation represents a plane in $\mathbb{R}^3$. Thus a 3×3 system such as

$$\begin{aligned} x + 9y - z &= 27 \\ 11x - 5z &= 12 \\ 2x + y + 15z &= 37 \end{aligned}$$



▲ **Figure 5:** A 3×3 system consists of three planes in $\mathbb{R}^3$.



▲ **Figure 6:** A 2×3 system consists of two planes in $\mathbb{R}^3$.

corresponds to three planes in $\mathbb{R}^3$, as shown in Figure 5. There is only one point that lies on all three planes, namely $(2, 3, 2)$, and this is the unique solution to the system. Again, this behavior is typical: **a 3×3 system usually has exactly one solution**, since three planes in $\mathbb{R}^3$ typically intersect at a single point.

As with the two-variable case, a three-variable system with extra equations usually has no solution. For example, **a 4×3 system usually doesn't have a solution**, since four planes in $\mathbb{R}^3$ typically don't have a point in common. Of course, it is possible for a 4×3 system to have a solution, if the fourth plane happens to go through the point of intersection of the first three planes.

Finally, something interesting happens if we have *fewer* than three equations. For example, the 2×3 system

$$
\begin{aligned}
x + 9y - z &= 27 \\
11x - 5z &= 12
\end{aligned}
$$

consists of two planes in $\mathbb{R}^3$, as shown in Figure 6. These two planes intersect along a line, and every point on this line is a solution to the system. In general, **a 2×3 system usually has infinitely many solutions**, since two planes in $\mathbb{R}^3$ typically intersect along a line. Of course, this doesn't happen when the two planes are parallel, in which case the system has no solutions.

Linear Systems in $\mathbb{R}^n$

In applications of mathematics, linear systems often have a very large number of variables. For example, the PageRank algorithm, which is used by Google to order search engine results, involves a system of linear equations with tens of billions of variables. One of the primary goals of linear algebra is to understand the behavior of such large systems and to develop feasible methods for solving them.

Here is an example of a 4×6 linear system:

$$
\begin{array}{rcrcrcrcrcrcr}
-x_1 & + & 4x_2 & - & 7x_3 & + & 3x_4 & - & 9x_5 & + & 2x_6 & = & 3 \\
-4x_1 & + & 3x_2 & + & 5x_3 & + & x_4 & - & 8x_5 & - & 9x_6 & = & 12 \\
x_1 & - & 3x_2 & & & - & 9x_4 & + & 6x_5 & + & 8x_6 & = & -47 \\
7x_1 & + & 9x_2 & - & 7x_3 & - & x_4 & - & 4x_5 & + & 2x_6 & = & 11
\end{array}
$$

Here the variables are x_1, x_2, x_3, x_4, x_5, and x_6. To increase readability, we have aligned the terms of the equations into columns, with one column on the left for each variable, and a final column on the right for the constant terms.

In general, each linear equation in n variables defines a **hyperplane** in $\mathbb{R}^n$, i.e. a flat of dimension $n - 1$. For example, a linear equation in six variables defines a hyperplane in $\mathbb{R}^6$, which is a five-dimensional flat. The solution set to a linear system is the intersection of all of the corresponding hyperplanes.

Certain features of linear systems that we have observed for two and three variables carry over to the general case.

Number of Solutions to a Linear System

A linear system always has either

1. No solutions,
2. One solution, or
3. Infinitely many solutions.

The number of solutions is usually determined by the number of equations.

Types of Linear Systems

1. A linear system with fewer equations than variables is called **underdetermined**. Such a system usually has infinitely many different solutions.
2. A linear system with the same number of equations as variables is called **square**. Such a system usually has exactly one solution.
3. A linear system with more equations than variables is called **overdetermined**. Such a system usually has no solutions.

EXERCISES

1–4 ■ Determine whether the given linear system has no solutions, one solution, or infinitely many solutions.

1. $$\begin{aligned} 4x + 6y &= 5 \\ 6x + 9y &= 8 \end{aligned}$$

2. $$\begin{aligned} 2x - 3y &= 1 \\ 3x - 4y &= 3 \end{aligned}$$

3. $$\begin{aligned} 3x + 2y - 4z &= 5 \\ 2x - 3y + 5z &= 8 \end{aligned}$$

4. $$\begin{aligned} x + 2y - 3z &= 4 \\ 2x + 4y - 6z &= 5 \end{aligned}$$

5. For what value of a does the following linear system have a solution?

$$\begin{aligned} 2x + 3y &= 11 \\ x + 4y &= 8 \\ 3x - 2y &= a \end{aligned}$$

6. For what value of a does the following linear system have *no* solutions?

$$\begin{aligned} 3x + 2y + 4z &= 7 \\ 6x + 4y + az &= 3 \end{aligned}$$

11.2 Row Reduction

In this section, we will learn a method called **row reduction** that can be used to solve linear systems.

Augmented Matrices

Before discussing row reduction, we need to introduce a new way of representing linear systems. When we write a linear system such as

$$\begin{aligned} -x_1 + 4x_2 - 7x_3 + 3x_4 - 9x_5 + 2x_6 &= 3 \\ -4x_1 + 3x_2 + 5x_3 + x_4 - 8x_5 - 9x_6 &= 12 \\ x_1 - 3x_2 \phantom{{}- 7x_3} - 9x_4 + 6x_5 + 8x_6 &= -47 \\ 7x_1 + 9x_2 - 7x_3 - x_4 - 4x_5 + 2x_6 &= 11 \end{aligned}$$

it's only the numbers that are really important for specifying the system. As long as we know what variables are being used, it isn't really necessary to write them in each equation. For this reason, it is common to specify a linear system by putting the coefficients and constant terms into a **augmented matrix**, i.e.

$$\left[\begin{array}{rrrrrr|r} -1 & 4 & -7 & 3 & -9 & 2 & 3 \\ -4 & 3 & 5 & 1 & -8 & -9 & 12 \\ 1 & -3 & 0 & -9 & 6 & 8 & -47 \\ 7 & 9 & -7 & -1 & -4 & 2 & 11 \end{array}\right]$$

Here "augmented" means that this matrix has two parts, separated by a vertical line. The coefficients of the linear system appear to the left of the line, and the constant terms are on the right.

From now on, we will almost always use augmented matrices for writing linear systems. In particular, row reduction involves manipulating a system of equations using certain operations, and we will be keeping our equations in matrix form during these manipulations.

Elementary Row Operations

The method of row reduction involves simplifying a linear system using three "moves" known as **elementary row operations**.

Elementary Row Operations

1. Switch two rows of a matrix.
2. Multiply a row of a matrix by a nonzero scalar.
3. Add a scalar multiple of one row of a matrix to another row.

The important thing about these three operations is that none of them changes the solution set to a linear system. In general, two linear systems are said to be **equivalent** if they have the same solution sets. Given a linear system, the idea of row reduction is to use these three row operations to find an equivalent linear system that is simpler than the original.

We now discuss each of the three operations individually.

1. **Switch two rows of a matrix.**

   In the following example, we switch the second and third rows of a matrix:

$$\left[\begin{array}{ccc|c} 3 & 4 & 7 & 2 \\ 5 & 8 & 6 & 9 \\ 2 & 4 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 4 & 7 & 2 \\ 2 & 4 & 3 & 1 \\ 5 & 8 & 6 & 9 \end{array}\right]$$

   Switching two rows has very little effect on the corresponding linear system. Indeed, all we have done is switch the order in which the second and third equations are listed:

$$\begin{aligned} 3x + 4y + 7z &= 2 \\ 5x + 8y + 6z &= 9 \\ 2x + 4y + 3z &= 1 \end{aligned} \quad \to \quad \begin{aligned} 3x + 4y + 7z &= 2 \\ 2x + 4y + 3z &= 1 \\ 5x + 8y + 6z &= 9 \end{aligned}$$

   Clearly we can use this operation whenever we like without affecting the solutions to a linear system.

2. **Multiply a row of a matrix by a nonzero scalar.**

   In the following example, we multiply the third row of a matrix by 2:

$$\left[\begin{array}{ccc|c} 3 & 4 & 7 & 2 \\ 5 & 8 & 6 & 9 \\ 2 & 4 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 4 & 7 & 2 \\ 5 & 8 & 6 & 9 \\ 4 & 8 & 6 & 2 \end{array}\right]$$

   This corresponds to multiplying both sides of the third equation by 2:

$$2x + 4y + 3z = 1 \quad \to \quad 4x + 8y + 6z = 2$$

   Since these two equations are equivalent, this operation does not affect the solution set of the linear system.

   Incidentally, when performing row reduction, it is much more common to *divide* a row by a nonzero scalar. For example, we might divide the first row of a matrix by 3:

$$\left[\begin{array}{cc|c} 3 & 9 & 6 \\ 2 & 7 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 7 & 4 \end{array}\right]$$

   This is a valid row operation, since it is the same as multiplying the first row by $1/3$.

3. **Add a multiple of one row of a matrix to another.**

   In the following example, we add 2 times the third row of a matrix to the first row:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 4 \\ 9 & 5 & 7 & 6 \\ 2 & 4 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 7 & 9 & 8 & 6 \\ 9 & 5 & 7 & 6 \\ 2 & 4 & 3 & 1 \end{array}\right]$$

   Note that the new first equation follows from the original equations, since

$$7x + 9y + 8z = (3x + y + 2z) + 2(2x + 4y + 3z) = 4 + 2(1) = 6.$$

   Note also that this operation is reversible, since we can change the new matrix back to the original matrix by adding -2 times the third row to the first row. That is, the original first equation follows from the new first equation and the third equation:

$$3x + y + 2z = (7x + 9y + 8z) - 2(2x + 4y + 3z) = 6 - 2(1) = 4.$$

Since each equation in the new matrix follows from the equations of the old matrix and vice-versa, the two linear systems must have the same solution set.

Gaussian Elimination

The idea of row reduction is to use elementary row operations to simplify a linear system until its solutions become apparent. Here is a simple example of this method:

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 7 & 11 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Starting with the matrix on the left, we first add −2 times the first row to the second row, resulting in the matrix in the middle. We then add −3 times the second row to the first row, which yields the matrix on the right. The rightmost matrix corresponds to the system of equations

$$\begin{aligned} 1x + 0y &= 2 \\ 0x + 1y &= 1 \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} x &= 2 \\ y &= 1 \end{aligned}$$

Thus we have solved the given linear system by performing two row operations.

In general, the goal of row reduction is to put a matrix into a **reduced echelon form** such as

The word *echelon* is a military term that refers to certain diagonal formations of troops, ships, or aircraft. Here the 1's form an "echelon" across the diagonal of the matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Though any row operations are allowed, there is a certain method called **Gaussian elimination** that will reliably put a matrix into reduced form. The following example illustrates Gaussian elimination for a 2×2 system.

EXAMPLE 1

Use row reduction to solve the following linear system:

$$\begin{aligned} 3x - 3y &= 9 \\ 2x - 3y &= 2 \end{aligned}$$

SOLUTION The matrix form for this system is

$$\left[\begin{array}{cc|c} 3 & -3 & 9 \\ 2 & -3 & 2 \end{array}\right]$$

The first step in Gaussian elimination is to obtain a 1 in the upper-left position. In this case, we can make a 1 by multiplying the first row through by 1/3

$$\left[\begin{array}{cc|c} 3 & -3 & 9 \\ 2 & -3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} \mathbf{1} & -1 & 3 \\ 2 & -3 & 2 \end{array}\right]$$

This 1 is called a **pivot**. Now that we have a pivot, we can use it to make the other numbers in the same column into 0's. In this case, we can make the 2 below the pivot into a 0 by adding -2 times the first row to the second row:

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & -3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & -1 & -4 \end{array}\right]$$

Next we want a pivot for the second column, which we can obtain by multiplying the second row by -1:

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & -1 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

Again, we can use the pivot to clear the other numbers in the same column. Specifically, we can add the second row to the first row to make a 0 above the new pivot:

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

This matrix is now in reduced form, and the solution is $(x, y) = (7, 4)$.

The main steps of Gaussian elimination are:

1. Make a pivot. This should be in the upper-left position for the first pivot, and otherwise should be one step down and to the right of the previous pivot.
2. Use the pivot to make the rest of the numbers in the same column into 0's.

These steps are repeated until the matrix is in reduced form.

EXAMPLE 2

Use row reduction to solve the following linear system:

$$\begin{aligned} 2x - 8y + 6z &= 2 \\ -3x + 16y - 5z &= -7 \\ -3x + 15y - 9z &= -12 \end{aligned}$$

SOLUTION Here are the nine row operations necessary to solve this linear system. The explanations for the steps are on the left.

Step 1. We multiply the first row by 1/2 to create a pivot in the upper left.

Steps 2–3. We clear the other numbers in the first column by adding 3 times the first row to both the second and third rows.

Step 4. We multiply the second row by 1/4 to create a second pivot.

Steps 5–6. We clear the other numbers in the second column. This involves adding 4 times the second row to the first row, and adding -3 times the second row to the third row.

Step 7. We multiply the third row by $-1/3$ to create a third pivot.

Steps 8–9. We clear the other numbers in the third column, adding -7 times the third row to the first row and -1 times the third row to the second row.

$$\left[\begin{array}{ccc|c} 2 & -8 & 6 & 2 \\ -3 & 16 & -5 & -7 \\ -3 & 15 & -9 & -12 \end{array}\right]$$

$$\xrightarrow{1} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 1 \\ -3 & 16 & -5 & -7 \\ -3 & 15 & -9 & -12 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 1 \\ 0 & 4 & 4 & -4 \\ -3 & 15 & -9 & -12 \end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 1 \\ 0 & 4 & 4 & -4 \\ 0 & 3 & 0 & -9 \end{array}\right]$$

$$\xrightarrow{4} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 0 & -9 \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 0 & -9 \end{array}\right] \xrightarrow{6} \left[\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -6 \end{array}\right]$$

$$\xrightarrow{7} \left[\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{8} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -17 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{9} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The final matrix corresponds to the linear system $x = -17$, $y = -3$, $z = 2$, so the solution is $(-17, -3, 2)$.

Finally, one complication that sometimes arises during Gaussian elimination is that a matrix has a 0 in a position that we would like to place a pivot. For example, the matrix

$$\left[\begin{array}{cc|c} 0 & -3 & 3 \\ -2 & -8 & 4 \end{array}\right]$$

has a 0 in the upper-left, which interferes with putting a pivot in this position. When this happens, the solution is to **switch the problematic row with a later row**. For the matrix above, we would switch the first row with the second row and then continue the row reduction:

$$\left[\begin{array}{cc|c} 0 & -3 & 3 \\ -2 & -8 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} -2 & -8 & 4 \\ 0 & -3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 4 & -2 \\ 0 & -3 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} \mathbf{1} & 4 & -2 \\ 0 & \mathbf{1} & -1 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & \mathbf{0} & 2 \\ 0 & \mathbf{1} & -1 \end{array}\right]$$

This complication can also arise in the middle of larger row reductions.

EXAMPLE 3

Use row reduction to solve the following linear system:

$$\begin{aligned} 2x - 6y - 6z &= -8 \\ x - 3y - 6z &= 8 \\ -x + 2y + 2z &= 6 \end{aligned}$$

SOLUTION Here are the first few steps of the row reduction:

$$\left[\begin{array}{ccc|c} 2 & -6 & -6 & -8 \\ 1 & -3 & -6 & 8 \\ -1 & 2 & 2 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} \mathbf{1} & -3 & -3 & -4 \\ 1 & -3 & -6 & 8 \\ -1 & 2 & 2 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \mathbf{1} & -3 & -3 & -4 \\ \mathbf{0} & 0 & -3 & 12 \\ -1 & 2 & 2 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} \mathbf{1} & -3 & -3 & -4 \\ \mathbf{0} & 0 & -3 & 12 \\ \mathbf{0} & -1 & -1 & 2 \end{array}\right]$$

As you can see, there is a 0 in the spot where we would like our next pivot. We can fix this by switching the second and third row, and then continuing the row reduction as usual:

$$\left[\begin{array}{ccc|c} \mathbf{1} & -3 & -3 & -4 \\ 0 & 0 & -3 & 12 \\ 0 & -1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} \mathbf{1} & -3 & -3 & -4 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -3 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} \mathbf{1} & -3 & -3 & -4 \\ 0 & \mathbf{1} & 1 & -2 \\ 0 & 0 & -3 & 12 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & 0 & -10 \\ 0 & \mathbf{1} & 1 & -2 \\ 0 & 0 & -3 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & -10 \\ 0 & \mathbf{1} & 1 & -2 \\ 0 & 0 & \mathbf{1} & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & -10 \\ 0 & \mathbf{1} & \mathbf{0} & 2 \\ 0 & 0 & \mathbf{1} & -4 \end{array}\right]$$

Thus the solution is $(-10, 2, -4)$.

EXERCISES

1–2 ■ Write the matrix that corresponds to the given linear system.

1.
$$\begin{aligned} y + z &= 5 \\ 2z - x &= 3 \\ 3y - 2x &= 1 \end{aligned}$$

2.
$$\begin{aligned} x_3 + x_5 &= 1 \\ x_1 - x_2 + x_3 &= 2 \\ x_4 &= 3 \\ -x_1 + 3x_4 &= 0 \end{aligned}$$

3–5 ■ Use row reduction to solve the given linear system.

3.
$$\begin{aligned} -2x - 6y &= -2 \\ 4x + 15y &= -2 \end{aligned}$$

4.
$$\begin{aligned} -3x - 6y + 6z &= 3 \\ x + 4y - 4z &= 3 \\ 3x + 8y - 7z &= 4 \end{aligned}$$

5.
$$\begin{aligned} x_1 - 4x_2 - 3x_3 + 3x_4 &= 1 \\ -2x_1 + 7x_2 + 3x_3 - 6x_4 &= 0 \\ -x_1 + 6x_2 + 6x_3 + 3x_4 &= 1 \\ -2x_1 + 7x_2 + 5x_3 - 11x_4 &= -5 \end{aligned}$$

6–9 ■ Row reduce the given matrix.

6. $\left[\begin{array}{cc|c} 2 & 6 & 8 \\ 2 & 4 & 7 \end{array}\right]$

7. $\left[\begin{array}{cc|c} 0 & 4 & 8 \\ 3 & 6 & -3 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 2 & 8 & 2 & 0 \\ 0 & 2 & 8 & -8 \\ -2 & -10 & -11 & 8 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 2 & 8 & 5 & 8 \\ 1 & 5 & 0 & 4 \end{array}\right]$

11.3 No Solutions

Linear systems sometimes have no solutions at all. For example, the system

$$\begin{aligned} 2x + 4y &= 10 \\ 3x + 6y &= 17 \end{aligned}$$

has no solution, since the corresponding lines are parallel in $\mathbb{R}^2$. Algebraically, if $2x + 4y = 10$, then $3x + 6y$ must be 15, not 17.

A system of equations with no solutions is sometimes said to be **inconsistent**.

Here are the first few steps of the row reduction for the above system:

$$\left[\begin{array}{cc|c} 2 & 4 & 10 \\ 3 & 6 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ 3 & 6 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ \mathbf{0} & 0 & 2 \end{array}\right]$$

At this point, the equation corresponding to the second row is

$$0x + 0y = 2$$

or more succinctly

$$0 = 2$$

which is a contradiction.

In general, **a row whose coefficients are all zero but whose constant term is nonzero indicates a contradiction**. If such a row arises during a row reduction, it means that the original linear system had no solutions.

3×3 Systems with No Solutions

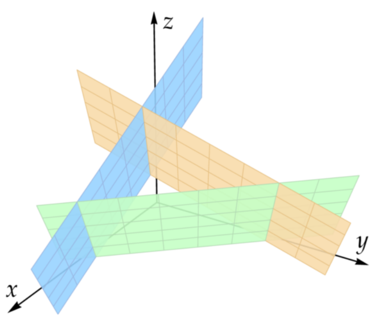


Figure 1: Three planes that do not intersect at a common point.

It is easy to see when a 2×2 system has no solutions, since the two lines must be parallel. For a 3×3 system, though, a contradiction can be much less obvious, and can involve all three of the equations. Geometrically, this corresponds to the situation shown in Figure 1. The three planes in this figure have no point in common, even though no two of the planes are parallel.

An example of this phenomenon is the system

$$\begin{aligned} 2x + 4y + 4z &= 2 \\ 3x + 4y + 2z &= 5 \\ 5x + 8y + 6z &= 4 \end{aligned}$$

Even though no two of these planes are parallel, this 3×3 system has no solutions. The reason is that the *sum* of the first two equations is

$$5x + 8y + 6z = 7$$

which contradicts the third equation.

More generally, a 3×3 system will have no solutions if the third equation contradicts *any* linear combination of the first two. For example, the linear system

$$\begin{aligned} 2x + 6y - 4z &= 2 \\ x - 5y + 5z &= 5 \\ 7x - 3y + 7z &= 6 \end{aligned}$$

has no solutions, and the reason is that *two* times the first equation plus *three* times the second equation is

$$7x - 3y + 7z = 19$$

which contradicts the third equation.

The following example illustrates how to use row reduction to detect a contradiction in a 3×3 system.

EXAMPLE 1

Solve the following linear system.

$$\begin{aligned} 2x + 4y + 4z &= 2 \\ 3x + 4y + 2z &= 5 \\ 5x + 8y + 6z &= 4 \end{aligned}$$

SOLUTION We row reduce the matrix in the usual way:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 3 & 4 & 2 & 5 \\ 5 & 8 & 6 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 3 & 4 & 2 & 5 \\ 5 & 8 & 6 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -2 & -4 & 2 \\ 5 & 8 & 6 & 4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Arguably the contradiction was clear after the third row operation, since we had obtained the equations $-2y - 4z = 2$ and $-2y - 4z = -1$.

We can stop the row reduction at this point, since the last row is a contradiction ($0 = -3$). This means that the original linear system had no solutions.

Overdetermined Systems

As we have seen, a linear system with more equations than unknowns usually has no solutions. Again, the reason is always a contradiction in the original equations. For example, the system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 3y &= 1 \\ 5x + 9y &= 3 \end{aligned}$$

has no solution, and the reason is that the first equation plus twice the second equation is

$$5x + 9y = 4$$

which contradicts the third equation. This contradiction can easily be detected using row reduction:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 5 & 9 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -6 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

The first step of this row reduction is actually two row operations. Specifically, we add -2 times the first row to the second row, and we add -5 times the first row to the third row.

The third row is now the equation $0 = -1$, which is a contradiction.

Of course, it's possible for an overdetermined system to have a solution. For example, the linear system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 3y &= 1 \\ 5x + 9y &= 4 \end{aligned}$$

has $(-1, 1)$ as a solution. In this case, the third equation is a *consequence* of the first two equations. Specifically, the third equation is equal to the first equation plus twice the second equation.

Here is the corresponding row reduction:

Again, the first and third steps of this row reduction each consist of two row operations. From now on, we will always skip steps like this during row reduction.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 5 & 9 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -6 & -6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

This time there is no contradiction in the third row, since the third equation is $0 = 0$. In general, **a row of zeroes obtained during row reduction indicates that one of the original equations was a consequence of the others**. If such a row arises during a row reduction, the proper procedure is to move it to the bottom of the matrix and ignore it for the rest of the reduction.

EXERCISES

1–2 ■ Use row reduction to solve the given linear system.

1. $$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y + 9z &= 8 \\ x + 4y + z &= 10 \end{aligned}$$

2. $$\begin{aligned} 3x + 12y &= 6 \\ 5x + 11y &= 1 \\ 7x + 10y &= -4 \end{aligned}$$

3–8 ■ Solve the linear system corresponding to the given matrix.

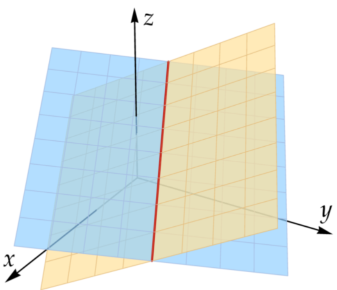
3. $$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 4 & 8 & 8 \\ 1 & 7 & 9 \end{array}\right]$$

4. $$\left[\begin{array}{cc|c} 2 & -4 & 10 \\ -3 & 7 & -9 \\ 1 & -2 & 5 \\ 2 & -5 & 4 \end{array}\right]$$

5. $$\left[\begin{array}{cc|c} 2 & -6 & 4 \\ -3 & 9 & -6 \\ 2 & -4 & -2 \\ -5 & 11 & 2 \\ 4 & -9 & -1 \end{array}\right]$$

6. $$\left[\begin{array}{ccc|c} -2 & -6 & 8 & 4 \\ -2 & -8 & 8 & 8 \\ -1 & -7 & 7 & 13 \\ -2 & -7 & 4 & 6 \end{array}\right]$$

11.4 Infinitely Many Solutions



▲ **Figure 1:** This 2×3 system has infinitely many solutions.

Linear systems sometimes have infinitely many different solutions. For example, a 2×3 system such as

$$\begin{aligned} 2x + 2y + 6z &= 14 \\ 2x - y + 3z &= 5 \end{aligned}$$

represents two planes in $\mathbb{R}^3$. Two planes usually intersect along a line, as shown in Figure 1, and each point on this line is a solution to the linear system.

When a linear system has infinitely many solutions, it is possible to solve for some of the variables in terms of the others. For example, in the 2×3 system above, it is possible to solve for x and y in terms of z:

$$x = 4 - 2z \qquad \text{and} \qquad y = 3 - z.$$

Of course it would also be possible to solve for x and z in terms of y, or for y and z in terms of x. Thus, it is our choice which of the three variables serves as a free variable.

In this case, we say that z is a **free variable**, meaning that it is free to take any value at all in a solution. Once the value of z is chosen, the two formulas above determine the values of x and y. For example, if $z = 0$, then $x = 4$ and $y = 3$, which gives the solution $(4, 3, 0)$. Similarly, if $z = 1$, then $x = 2$ and $y = 2$, which gives the solution $(2, 2, 1)$.

We can use the free variable z to give a parametric equation for the solution set:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2t \\ 3 - t \\ t \end{bmatrix}$$

Since z is a free variable, we can set it equal to the parameter t and then give the corresponding formulas for x and y. The result is a parametric equation for the line of intersection of the two planes.

All of this depends on being able to solve for some of the variables in terms of others. Fortunately, this is exactly what row reduction does for a system with infinitely many solutions.

EXAMPLE 1

Find a parametric description of the solutions to the following linear system.

$$\begin{aligned} 2x + 2y + 6z &= 14 \\ 2x - y + 3z &= 5 \end{aligned}$$

SOLUTION Here are the steps for row reducing the corresponding matrix:

$$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 14 \\ 2 & -1 & 3 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 2 & -1 & 3 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & -3 & -3 & -9 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Note that there isn't space for a third pivot, so this is as far as this matrix can be reduced. The system of equations is now

$$x + 2z = 4 \qquad \text{and} \qquad y + z = 3$$

which we can write as

$$x = 4 - 2z \qquad \text{and} \qquad y = 3 - z.$$

Thus the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4-2t \\ 3-t \\ t \end{bmatrix}.$$

Multiple Free Variables

It is possible for a linear system to have more than one free variable. For example, consider the 2×4 system

$$\begin{aligned} x_1 + 3x_2 - 4x_3 + 4x_4 &= 4 \\ x_1 + 4x_2 - 7x_3 + 6x_4 &= 3 \end{aligned}$$

We row reduce the corresponding matrix:

$$\left[\begin{array}{cccc|c} \mathbf{1} & 3 & -4 & 4 & 4 \\ 1 & 4 & -7 & 6 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} \mathbf{1} & 3 & -4 & 4 & 4 \\ \mathbf{0} & 1 & -3 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{0} & 5 & -2 & 7 \\ 0 & \mathbf{1} & -3 & 2 & -1 \end{array}\right]$$

This gives the equations

$$x_1 + 5x_3 - 2x_4 = 7, \qquad x_2 - 3x_3 + 2x_4 = -1.$$

Each column without a pivot in the reduced matrix corresponds to a free variable.

Essentially we have solved for x_1 and x_2 in terms of x_3 and x_4. Indeed, we can rewrite these equations as

$$x_1 = 7 - 5x_3 + 2x_4, \qquad x_2 = -1 + 3x_3 - 2x_4.$$

The result is that both x_3 and x_4 are free variables. If we want to parameterize the solution set, we need two parameters, with one for x_3 and one for x_4:

We always need one parameter for each free variable.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7-5s+2t \\ -1+3s-2t \\ s \\ t \end{bmatrix}.$$

Geometrically, this solution set is a plane in $\mathbb{R}^4$.

In general, **the number of free variables in a linear system is usually equal to the number of variables minus the number of equations**. In this case, four variables and two equations led to two free variables.

EXAMPLE 2

Find a parametric description of the solutions to the following linear system.

$$\begin{aligned} -2x_1 + 2x_2 - 6x_3 + 8x_4 - 8x_5 &= -2 \\ -4x_1 + x_2 - 15x_3 + 13x_4 - 13x_5 &= 2 \end{aligned}$$

SOLUTION This system has five variables and two equations, so we are expecting three free variables. We row reduce the corresponding matrix:

$$\left[\begin{array}{rrrrr|r} -2 & 2 & -6 & 8 & -8 & -2 \\ -4 & 1 & -15 & 13 & -13 & 2 \end{array}\right] \to \left[\begin{array}{rrrrr|r} \mathbf{1} & -1 & 3 & -4 & 4 & 1 \\ -4 & 1 & -15 & 13 & -13 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrr|r} \mathbf{1} & -1 & 3 & -4 & 2 & 1 \\ \mathbf{0} & -3 & -3 & -3 & 3 & 6 \end{array}\right] \to \left[\begin{array}{rrrrr|r} \mathbf{1} & -1 & 3 & -4 & 2 & 1 \\ 0 & \mathbf{1} & 1 & 1 & -1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrr|r} \mathbf{1} & \mathbf{0} & 4 & -3 & 1 & -1 \\ 0 & \mathbf{1} & 1 & 1 & -1 & -2 \end{array}\right]$$

This gives us the equations

$$x_1 + 4x_3 - 3x_4 + x_5 = -1, \qquad x_2 + x_3 + x_4 - x_5 = -2.$$

Here x_3, x_4, and x_5 are free variables.

As you can see, we have solved for x_1 and x_2 in terms of x_3, x_4, and x_5. Thus the general solution is

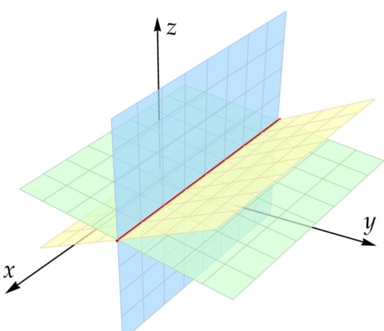
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 - 4s + 3t - u \\ -2 - s - t + u \\ s \\ t \\ u \end{bmatrix}.$$

This solution set is a three-dimensional flat in $\mathbb{R}^5$.

Redundant Equations

A linear system can have more free variables than expected if one of the equations is a consequence of the others. For example, consider the 3×3 system

$$\begin{aligned} x + 9y - z &= 27 \\ x - 8y + 16z &= 10 \\ 2x + y + 15z &= 37 \end{aligned}$$



▲ **Figure 2:** It is possible for three planes to intersect along a line.

Though a 3×3 system usually has a unique solution, in this system the third equation is a consequence of the first two. Specifically, the third equation here is simply the sum of the first two equations. As a result, any solution to the first two equations is also a solution to the third equation, so there is a whole line of solutions, as shown in Figure 2.

Redundant equations lead to rows of zeroes during row reduction. For example, here is what happens if we row reduce the matrix for the 3×3 system above:

$$\left[\begin{array}{rrr|r} \mathbf{1} & 9 & -1 & 27 \\ 1 & -8 & 16 & 10 \\ 2 & 1 & 15 & 37 \end{array}\right] \to \left[\begin{array}{rrr|r} \mathbf{1} & 9 & -1 & 27 \\ \mathbf{0} & -17 & 17 & -17 \\ \mathbf{0} & -17 & 17 & -17 \end{array}\right]$$

The first and third steps here each consist of two row operations.

$$\to \left[\begin{array}{rrr|r} \mathbf{1} & 9 & -1 & 27 \\ 0 & \mathbf{1} & -1 & 1 \\ 0 & -17 & 17 & -17 \end{array}\right] \to \left[\begin{array}{rrr|r} \mathbf{1} & \mathbf{0} & 8 & 18 \\ 0 & \mathbf{1} & -1 & 1 \\ 0 & \mathbf{0} & 0 & 0 \end{array}\right]$$

Because of the row of zeroes, only the first two columns have pivots, and therefore z is a free variable. In fact, we have the equations

$$x + 8z = 18, \qquad y - z = 1$$

and thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 - 8t \\ 1 + t \\ t \end{bmatrix}.$$

In general, a **redundant equation** in a linear system is an equation that is a consequence of the previous equations. A linear system with redundant equations behaves as though the extra equations weren't there. For example, the 3×3 system above has one redundant equation, so it behaves more like a 2×3 system, with one free variable and a line of solutions.

Columns Without Pivots

When row reducing a matrix, it is sometimes not possible to create a pivot in a certain column. For example, consider the following linear system:

$$\begin{aligned} x + 3y + 2z &= 5 \\ x + 3y + 3z &= 7 \end{aligned}$$

This system should have one free variable, so we are expecting to be able to solve for x and y in terms of z. However, we quickly run into trouble if we try to row reduce:

$$\left[\begin{array}{ccc|c} \mathbf{1} & 3 & 2 & 5 \\ 1 & 3 & 3 & 7 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|c} \mathbf{1} & 3 & 2 & 5 \\ \mathbf{0} & 0 & 1 & 2 \end{array}\right]$$

With a 0 in the desired position and no later rows to switch with, there is no way to obtain a pivot immediately down and to the right of the first pivot.

The problem is that there is no way to solve these equations for x and y in terms of z. Indeed, it follows from the original equations that $z = 2$, so z can't play the role of a free variable for this system.

The standard solution to this problem is to treat the 1 in the third column as a pivot:

$$\left[\begin{array}{ccc|c} \mathbf{1} & 3 & 2 & 5 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|c} \mathbf{1} & 3 & \mathbf{0} & 1 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right]$$

This matrix is now considered reduced, and the corresponding equations are

$$x + 3y = 1, \qquad z = 2.$$

Now y is the free variable, with $x = 1 - 3y$, so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - 3t \\ t \\ 2 \end{bmatrix}.$$

As a general rule, **if it is not possible to obtain a pivot in a certain column, simply move on to the next column**. After the row reduction is complete, whichever columns don't have pivots can serve as free variables for the resulting parametrization.

EXAMPLE 3

Find a parametric description of the solutions to the following linear system.

$$\begin{aligned} -2x_1 + 4x_2 + 2x_3 - \ \ 8x_4 + 4x_5 &= -8 \\ 3x_1 - 6x_2 - 2x_3 + 11x_4 - 7x_5 &= 13 \\ x_1 - 2x_2 - 5x_3 + \ \ 8x_4 + \ \ x_5 &= -3 \end{aligned}$$

SOLUTION Here are the step in row reducing the associated matrix. Both the second and fourth columns present problems during the reduction, so we end up with pivots in the first, third, and fifth columns:

$$\left[\begin{array}{ccccc|c} -2 & 4 & 2 & -8 & 4 & -8 \\ 3 & -6 & -1 & 10 & -8 & 14 \\ 1 & -2 & -5 & 8 & 1 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & -2 & -1 & 4 & -2 & 4 \\ 3 & -6 & -1 & 10 & -8 & 14 \\ 1 & -2 & -5 & 8 & 1 & -3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & -2 & -1 & 4 & -2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 2 \\ 0 & 0 & -4 & 4 & 3 & -7 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & -2 & -1 & 4 & -2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -4 & 4 & 3 & -7 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & -3 & 5 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & -3 & 5 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 14 \\ 0 & 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

The free variables are x_2 and x_4, since these are the columns without pivots, and we have the equations

$$x_1 - 2x_2 + 3x_4 = 14, \qquad x_3 - x_4 = 4, \qquad x_5 = 3.$$

Thus the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 14 + 2s - 3t \\ s \\ 4 + t \\ t \\ 3 \end{bmatrix}.$$

In this case, the solution set is a plane in $\mathbb{R}^5$.

EXERCISES

1–6 ■ For each of the following reduced matrices, state which variables are free, and find a parametric equation for the solution set to the corresponding linear system.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$

2. $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 & 2 \end{array}\right]$

3. $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 5 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & -3 & 0 \end{array}\right]$

5. $\left[\begin{array}{cccc|c} 1 & 3 & 0 & -2 & 8 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right]$

6. $\left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$

7. Find a 2×4 linear system whose solution set is the plane

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - s + 3t \\ s \\ 1 - 4t \\ t \end{bmatrix}$$

8. Find a parametric equation for the solution set to the following linear system:

$$\begin{aligned} 2x + 6y - 2z &= 6 \\ -2x - 3y + 8z &= -15 \end{aligned}$$

9. The planes $x + 3y + 6z = 5$ and $3x + 2y + 4z = 8$ intersect along a line L. Find a parametric equation for L.

10. Describe the solution set to the following linear system:

$$\begin{aligned} -3x + 3y - 6z &= -6 \\ -x + 3y + 2z &= 4 \\ -3x + 7y + 2z &= 6 \end{aligned}$$

11. The hyperplanes

$$x_1 + x_2 + 2x_3 - 3x_4 + x_5 = 4 \qquad \text{and} \qquad x_1 + 2x_2 + 2x_3 - 6x_4 + 3x_5 = 8$$

intersect along a three-dimensional flat in $\mathbb{R}^5$. Find a parametric equation for this flat.

12–13 ■ Row reduce the given matrix, skipping over columns without pivots, and find a parametric equation for the solution set to the corresponding linear system.

12. $\left[\begin{array}{cccc|c} -1 & 4 & 2 & -1 & -2 \\ 2 & -8 & -7 & -4 & 7 \end{array}\right]$

13. $\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 2 & 4 & 4 & 6 & 4 \\ 3 & 6 & 6 & 7 & 4 \end{array}\right]$