12.1 Matrices

A **matrix** is any rectangular array of numbers. For example

$$\begin{bmatrix} 3 & 0 & -2 & 5 \\ 2 & -1 & 6 & -4 \\ 8 & 13 & 3 & -2 \end{bmatrix}$$

is 3×4 matrix, i.e. a rectangular array of numbers with three rows and four columns. We usually use capital letters for matrices, e.g. A, B, and C, with lowercase letters reserved for scalars.

A vector is actually a special type of matrix, namely a matrix with only one column. In particular, a vector from $\mathbb{R}^n$ is the same thing as an $n \times 1$ matrix.

Multiplying a Matrix and a Vector

To multiply a matrix and a vector, we take the dot product of each row of the matrix with the vector. For example,

$$\begin{bmatrix} 3 & 1 & 2 & 5 \\ 1 & 4 & 3 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 21 \\ 31 \end{bmatrix}$$

Here 21 is the dot product of $(3,1,2,5)$ with $(2,3,1,2)$, and 31 is the dot product of $(1,4,3,7)$ with $(2,3,1,2)$.

Note that a matrix A can only be multiplied by a vector $\mathbf{v}$ if each row of A has the same size as $\mathbf{v}$. For example, we can only multiply a 5×8 matrix with a vector from $\mathbb{R}^8$, and the resulting product will be a vector in $\mathbb{R}^5$.

In general, the product of an $m \times n$ matrix with a vector from $\mathbb{R}^n$ is a vector in $\mathbb{R}^m$.

Multiplying Matrices

There is an operation called **matrix multiplication** that generalizes the product of a matrix and a vector. Given two matrices A and B, the product AB is the matrix obtained by taking the dot product of each row of A with each column of B. For example, if A and B are 2×2 matrices, then there are four dot products to compute:

$$\begin{bmatrix} 7 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & \\ & \end{bmatrix} \qquad \begin{bmatrix} 7 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 22 \\ & \end{bmatrix}$$

$$\begin{bmatrix} 7 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 22 \\ 10 & \end{bmatrix} \qquad \begin{bmatrix} 7 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 22 \\ 10 & 17 \end{bmatrix}$$

This product only makes sense if the rows of A and the columns of B have the same size. The result always has one row for each row of A and one column for each column of B.

EXAMPLE 1

Compute AB if $A = \begin{bmatrix} 7 & 5 & 2 & 2 \\ 3 & 1 & 1 & 0 \\ 1 & 6 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 2 & 4 \\ 1 & 0 \end{bmatrix}$.

Here A is a 3×4 matrix and B is 4×2 matrix, so AB will be a 3×2 matrix.

SOLUTION We must take the dot product of each row of A with each column of B.

$$\begin{bmatrix} \mathbf{7} & \mathbf{5} & \mathbf{2} & \mathbf{2} \\ 3 & 1 & 1 & 0 \\ 1 & 6 & 3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{0} & 3 \\ \mathbf{2} & 2 \\ \mathbf{2} & 4 \\ \mathbf{1} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{16} & \\ & \\ & \end{bmatrix} \qquad \begin{bmatrix} \mathbf{7} & \mathbf{5} & \mathbf{2} & \mathbf{2} \\ 3 & 1 & 1 & 0 \\ 1 & 6 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{3} \\ 2 & \mathbf{2} \\ 2 & \mathbf{4} \\ 1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 16 & \mathbf{39} \\ & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 & 2 & 2 \\ \mathbf{3} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 6 & 3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{0} & 3 \\ \mathbf{2} & 2 \\ \mathbf{2} & 4 \\ \mathbf{1} & 0 \end{bmatrix} = \begin{bmatrix} 16 & 39 \\ \mathbf{4} & \\ & \end{bmatrix} \qquad \begin{bmatrix} 7 & 5 & 2 & 2 \\ \mathbf{3} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 6 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{3} \\ 2 & \mathbf{2} \\ 2 & \mathbf{4} \\ 1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 16 & 39 \\ 4 & \mathbf{15} \\ & \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 & 2 & 2 \\ 3 & 1 & 1 & 0 \\ \mathbf{1} & \mathbf{6} & \mathbf{3} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & 3 \\ \mathbf{2} & 2 \\ \mathbf{2} & 4 \\ \mathbf{1} & 0 \end{bmatrix} = \begin{bmatrix} 16 & 39 \\ 4 & 15 \\ \mathbf{18} & \end{bmatrix} \qquad \begin{bmatrix} 7 & 5 & 2 & 2 \\ 3 & 1 & 1 & 0 \\ \mathbf{1} & \mathbf{6} & \mathbf{3} & \mathbf{0} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{3} \\ 2 & \mathbf{2} \\ 2 & \mathbf{4} \\ 1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 16 & 39 \\ 4 & 15 \\ 18 & \mathbf{27} \end{bmatrix}$$

Unlike multiplication of scalars, matrix multiplication is not commutative. That is, AB and BA are not necessarily the same. For example,

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & 5 \\ 16 & 6 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 7 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 23 & 9 \\ 7 & 4 \end{bmatrix}$$

However, matrix multiplication is associative. That is,

$$\boxed{A(BC) = (AB)C}$$

for any matrices A, B, and C.

Addition and Scalar Multiplication

Matrix subtraction is defined in a similar way.

There are two more basic operations involving matrices: addition and scalar multiplication. **Matrix addition** works just like vector addition, with corresponding entries of the two matrices added together:

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 1 \\ 5 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 2 \\ 7 & 3 & 7 \end{bmatrix}$$

Only two matrices of the same size can be added. Matrix multiplication distributes over addition from both the left and the right, i.e.

$$\boxed{A(B+C) = AB + AC \qquad \text{and} \qquad (A+B)C = AC + BC}$$

Scalar multiplication for matrices is also quite similar to scalar multiplication for vectors:

$$2\begin{bmatrix} 4 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 2 \\ 8 & 4 & 6 \end{bmatrix}$$

This has a variety of obvious properties, e.g.

$$k(A+B) = kA + kB \qquad \text{and} \qquad k(AB) = (kA)B = A(kB)$$

for any scalar k and matrices A and B.

Square Matrices

A matrix is called **square** if it has the same number of rows and columns. For example, 2×2 matrices are square, as are 3×3 matrices, and more generally $n \times n$ matrices.

We can take the determinant of any square matrix A, which we write as $\det(A)$. For example, if

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

then $\det(A) = 14$.

The product of two square matrices of the same size is another square matrix of that size. For example,

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 6 \\ 3 & 1 & 3 \\ 6 & 6 & 5 \end{bmatrix}$$

The determinant of a matrix product is equal to the product of the determinants:

$$\boxed{\det(AB) = \det(A)\,\det(B).}$$

A square matrix is called **diagonal** if all of its nonzero entries lie along the diagonal that goes from the upper left to the lower right. For example,

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

are diagonal matrices. The determinant of a diagonal matrix is equal to the product if the entries along the diagonal:

Note that we use square brackets for matrices and vertical lines for determinants.

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab, \quad \text{and} \quad \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Inverse Matrices

A diagonal matrix with ones along the diagonal is called an **identity matrix**:

We usually use the letter I to denote an identity matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \cdots$$

Multiplying by an identity matrix has no effect:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix}$$

Two square matrices are called **inverses** if their product is the identity matrix. For example

$$\begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$$

are inverses, since

$$\begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix can have only one inverse. If A is a square matrix, its inverse is denoted A^{-1}.

There is a simple formula for the inverse of a 2×2 matrix:

There is no simple analog of this formula for 3×3 or larger matrices.

Inverse of a 2×2 Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that $ad - bc$ is the determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

EXAMPLE 2

Find the inverse of the matrix $\begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix}$.

SOLUTION The determinant of this matrix is 2, so the inverse is

$$\frac{1}{2} \begin{bmatrix} 2 & -6 \\ -1 & 4 \end{bmatrix}$$

This simplifies to

$$\begin{bmatrix} 1 & -3 \\ -1/2 & 2 \end{bmatrix}$$

A square matrix is called **invertible** if it has an inverse. From the formula above, we see that a 2×2 matrix is invertible as long as its determinant is not zero. This rule works for matrices of any size:

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Representing Linear Systems

We can use matrices to write any linear system as a single vector equation of the form

$$A\mathbf{x} = \mathbf{b}$$

where A is the **coefficient matrix**, $\mathbf{x}$ is the vector of unknowns, and $\mathbf{b}$ is the vector of constant terms. For example, the linear system

$$\begin{aligned} 2x + 5y &= 11 \\ 3x + 4y &= 13 \end{aligned}$$

can be written in vector form as

$$\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$$

We can use inverse matrices to solve $n \times n$ linear systems. Given a linear system of the form

$$A\mathbf{x} = \mathbf{b}$$

where A is an invertible square matrix, we can multiply both sides of the equation by A^{-1} to get

$$\boxed{\mathbf{x} = A^{-1}\mathbf{b}}$$

EXAMPLE 3

Use an inverse matrix to solve the system

$$\begin{aligned} 3x + 2y &= 7 \\ x + 4y &= 5 \end{aligned}$$

SOLUTION We can write this system as

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

But

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}$$

so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 0.8 \end{bmatrix}$$

Thus $x = 1.8$ and $y = 0.8$.

EXERCISES

1–4 ■ Multiply.

1. $\begin{bmatrix} -9 & 3 & 1 \\ 2 & 9 & -3 \\ 3 & 0 & 8 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} -5 & 1 \\ 1 & 7 \\ 9 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 7 & 2 & 2 \\ 2 & 8 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 6 & 9 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 5 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

5. Find the values of x and y for which $\begin{bmatrix} x & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$.

6–9 ■ Multiply.

6. $\begin{bmatrix} 6 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 8 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 9 & 1 \\ 9 & 1 & -2 & -2 \end{bmatrix}\begin{bmatrix} -1 & -4 \\ 0 & -1 \\ 2 & 2 \\ -6 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 7 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

10–13 ■ Compute the inverse of the given matrix.

10. $\begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

14. Does the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ have an inverse? Explain.

15. Compute $5A + 6A^{-1}$ if $A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$.

16. (a) Write the linear system

$$\begin{aligned} 2x - 3y &= 5 \\ 3x + 4y &= 2 \end{aligned}$$

as an equation of the form $A\mathbf{x} = \mathbf{b}$.

(b) Use an inverse matrix to solve your equation from part (a).

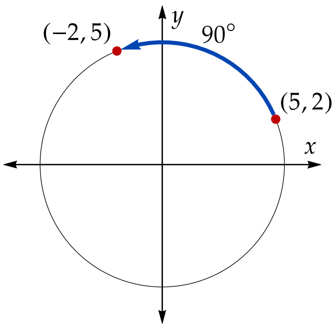
17. Given that the matrices

$$\begin{bmatrix} 2 & -5 & 3 & -2 \\ -1 & 2 & -1 & 2 \\ 1 & -3 & 2 & 1 \\ 3 & -8 & 6 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -7 & -9 & 3 & 1 \\ -2 & -2 & -1 & 1 \\ 1 & 2 & -3 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

are inverses, solve the following linear system:

$$\begin{aligned} 2x_1 - 5x_2 + 3x_3 - 2x_4 &= 3 \\ -x_1 + 2x_2 - x_3 + 2x_4 &= 2 \\ x_1 - 3x_2 + 2x_3 + x_4 &= 0 \\ 3x_1 - 8x_2 + 6x_3 + x_4 &= 2 \end{aligned}$$

12.2 Linear Transformations



▲ **Figure 1:** A 90° counterclockwise rotation.

As we have seen, we can rotate any point in the plane 90° counterclockwise around the origin by switching the two coordinates and negating the first one:

$$(5, 2) \longmapsto (-2, 5).$$

This transformation is shown in Figure 1.

This transformation is equivalent to multiplying by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

For example,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

and more generally

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

This is a simple example of a linear transformation.

Linear Transformations

A transformation of the plane is called a **linear transformation** if it corresponds to multiplying each point (x, y) by some 2×2 matrix A, i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto A \begin{bmatrix} x \\ y \end{bmatrix}.$$

It turns out that many geometric transformations of the plane are linear transformations, including:

1. Rotation of the plane by any angle around the origin.
2. Reflection of the plane across any line that goes through the origin.

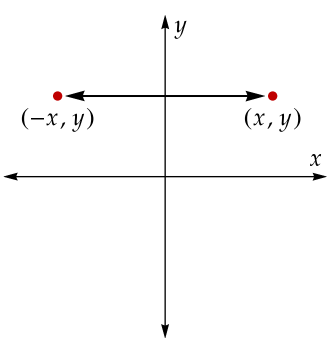
EXAMPLE 1

Describe the linear transformation of the plane corresponding to the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

SOLUTION We have

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

so this matrix negates the x-coordinate of each point of the plane. Geometrically, this corresponds to reflection across the y-axis, as shown in Figure 2.



▲ **Figure 2:** Reflection across the y-axis.

Finding the Matrix

There is a nice trick that can be used to find the matrix for a given transformation.

Column Trick

If A is a 2×2 matrix, then

$$A\begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix}$$

are the first and second columns of A, respectively.

For example,

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

The following example shows how to use this trick to find the matrix for a linear transformation.

EXAMPLE 2

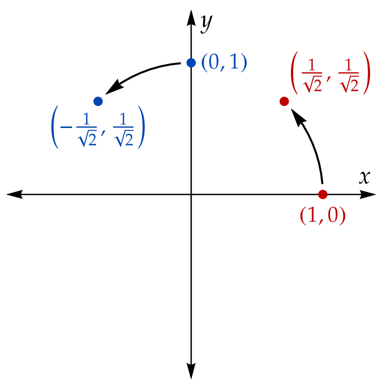
Find the matrix for a $45°$ counterclockwise rotation of the plane about the origin.

SOLUTION This transformation is shown in Figure 3. Note that $(1,0)$ maps to $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $(0,1)$ maps to $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. If A is the matrix for this transformation, it follows that

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

so these vectors are the columns of A. We conclude that

$$A=\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$$



▲ **Figure 3:** A 45° rotation of the plane.

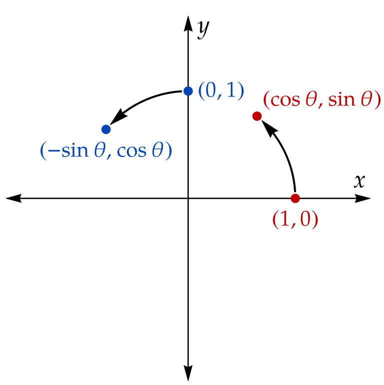
The previous example is a special case of a more general formula.

2×2 Rotation Matrices

The matrix

$$\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$$

rotates the plane counterclockwise around the origin by an angle of θ.



▲ **Figure 4:** A rotation of the plane by an angle of θ.

The justification for this formula is shown in Figure 4. If A is the matrix for this transformation, then

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}$$

so these vectors are the columns of A.

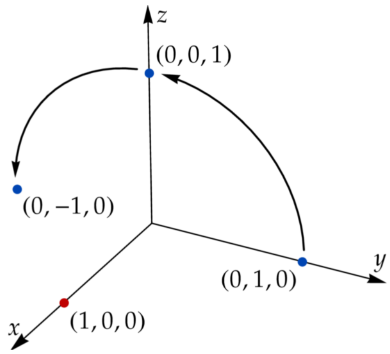


Figure 5: A 90° rotation around the x-axis.

A Closer Look — Transformations of $\mathbb{R}^3$

We can use 3×3 matrices to describe certain transformations in three dimensions, such as rotation around a line through the origin, or reflection across a plane through the origin. Such a transformation is called a **linear transformation of** $\mathbb{R}^3$.

For example, consider the 90° rotation of $\mathbb{R}^3$ about the x-axis shown in Figure 5. How can we find a 3×3 matrix A for this transformation? Well, it is obvious from the figure that

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix}.$$

Then these three vectors must be the three columns of A. We conclude that

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}$$

EXERCISES

1–4 ■ Give a geometric description of the linear transformation corresponding to the given matrix.

1. $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

2. $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

3–4 ■ Find the matrix for the reflection of $\mathbb{R}^2$ across the given line.

3. the line $y = x$

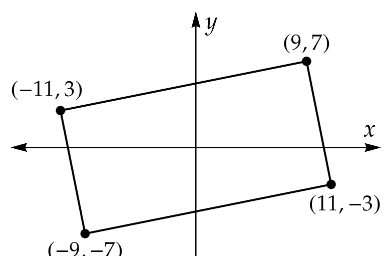
4. the x-axis

5–6 ■ Find the matrix for the given rotation of $\mathbb{R}^2$ around the origin.

5. 135° counterclockwise

6. 30° clockwise

7. The following figure shows a rectangle in the plane.



Find the new coordinates of the four vertices if this rectangle is rotated 45° counterclockwise around the origin.

12.3 Eigenvectors and Eigenvalues

The German word *eigen* can be translated into English as "characteristic", "special", or "peculiar". Hence an eigenvector is a "special vector".

Eigenvectors are certain special vectors that are associated to a square matrix. A vector $\mathbf{v}$ is called an **eigenvector** for a matrix A if the product $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. For example, if

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

then $\mathbf{v}$ is an eigenvector for A since

$$A\mathbf{v} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\mathbf{v}$$

The scalar 5 is called the **eigenvalue** associated to the eigenvector $(1, 2)$.

Here λ is the lowercase Greek letter lambda, which is traditionally used to represent eigenvalues.

Definition: Eigenvectors and Eigenvalues

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$ is called an **eigenvector** for A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ. This scalar λ is the associated **eigenvalue**

Note that an eigenvector $\mathbf{v}$ is required to be nonzero. It is true that

$$A\mathbf{0} = \lambda\mathbf{0}$$

for any matrix A and any scalar λ, but we do not count $\mathbf{0}$ as an eigenvector or λ as an eigenvalue in this case.

A 2×2 matrix typically has two different eigenvalues. More generally, an $n \times n$ matrix typically has n different eigenvalues.

EXAMPLE 1

Consider the matrix $A = \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$. Because this matrix is diagonal, each of the standard basis vectors is an eigenvector for A. In particular,

$$\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $(1, 0)$ is an eigenvector with eigenvalue 7, and $(0, 1)$ is an eigenvector with eigenvalue 4.

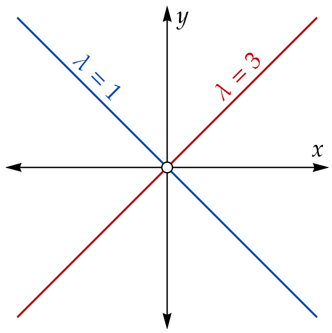
In general, the eigenvalues of a diagonal matrix are the numbers along the diagonal, and the eigenvectors are the standard basis vectors.

If A is a matrix and $\mathbf{v}$ is an eigenvector for A, then any nonzero multiple of $\mathbf{v}$ will also be an eigenvector for A. For example, since

$$\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

it is also true that

$$\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



▲ **Figure 1:** Lines of eigenvectors for the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

EXAMPLE 2

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. This matrix has $(1,1)$ as an eigenvector with eigenvalue 3:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It also has $(-1,1)$ as an eigenvector with eigenvalue 1:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

More generally, any nonzero multiple of $(1,1)$ is an eigenvector with eigenvalue 3, and any nonzero multiple of $(1,-1)$ is an eigenvector with eigenvalue 1, as shown in Figure 1.

Finding the Eigenvalues

There is a simple equation that lets you find the eigenvalues of a matrix:

The Characteristic Equation

Given an $n \times n$ matrix A, the **characteristic equation** for A is

$$\det(A - \lambda I) = 0$$

where I denotes the $n \times n$ identity matrix. The solutions to this equation are precisely the eigenvalues of A.

The reason this works is that the equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

is the same as the equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

If $A - \lambda I$ is invertible, then the only solution to this equation will be $\mathbf{v} = \mathbf{0}$. Thus there will only be nonzero solutions in the case where $A - \lambda I$ has determinant zero.

To derive this equation from the previous one, we first subtract $\lambda\mathbf{v}$ from both sides:

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

But $\lambda\mathbf{v} = \lambda I\mathbf{v}$, so we have

$$A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0}$$

and factoring out a $\mathbf{v}$ gives

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

EXAMPLE 3

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.

SOLUTION We have

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}$$

so

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - (1)(2) = \lambda^2 - 7\lambda + 10$$

Thus the characteristic equation for A is $\lambda^2 - 7\lambda + 10 = 0$. The solutions to this equation are $\boxed{\lambda = 2 \text{ and } \lambda = 5}$.

Note that $A - \lambda I$ can be obtained from A by subtracting λ from each entry on the main diagonal.

Finding the Eigenvectors

Once the eigenvalues of a matrix are known, it is fairly straightforward to find the eigenvectors. In particular, for a given scalar λ, the equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

is simply a linear system whose nonzero solutions are the eigenvectors.

EXAMPLE 4

Find an eigenvector for the matrix $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5$.

SOLUTION We wish to solve the equation

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}$$

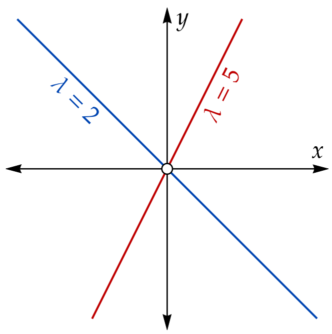
This gives us the linear system

$$\begin{aligned} 3x + y &= 5x \\ 2x + 4y &= 5y \end{aligned}$$

Both of these equations simplify to $y = 2x$, so the points on this line are eigenvectors for $\lambda = 5$. In particular, $(1, 2)$ is an eigenvector:

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Incidentally, a similar calculation reveals that the eigenvectors for $\lambda = 2$ lie on the line $y = -x$. Figure 2 shows the eigenvalues and eigenvectors for this matrix.



▲ **Figure 2:** Lines of eigenvectors for the matrix $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.

EXERCISES

1–2 ■ Find the eigenvalues of the given matrix.

1. $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} -8 & 5 \\ -4 & 1 \end{bmatrix}$

3–4 ■ Find the eigenvalues of the given matrix, and then find one eigenvector corresponding to each eigenvalue.

3. $\begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix}$

5. Find an eigenvector for the matrix $\begin{bmatrix} 1 & 1/4 \\ 3 & 2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5/2$.