2 Growth, Decay, and Oscillation

The city of Suzhou in Jiangsu Province, China. Suzhou is the fastest growing city in the world, with an annual population growth of 6.5% between the years 2000 and 2014.[1]

As we have discussed, differential equations are commonly used in science to model the behavior of dynamical systems. In this chapter, we consider some of the simplest possible behaviors for a dynamical system: growth, decay, and oscillation.

Of course, the simplest model for growth is **linear growth**, for which a variable y that increases at a constant rate. This corresponds to the differential equation

$$\frac{dy}{dt} = r,$$

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[1] Photo by Mudaxiong, cropped from original, licensed under CC BY-SA 3.0, via Wikimedia Commons

where r is the (constant) growth rate. The solutions to this equation are linear functions

$$y = y_0 + rt.$$

Note that the growth rate r is the slope of the linear function, and the constant term $y_0 = y(0)$ is the initial value of y. We assume that the reader is quite familiar with linear growth, and we will not discuss it further.

The second simplest model for growth is **exponential growth**, which corresponds to the differential equation

Since we are interested in applications, we use t as the independent variable throughout this chapter. If we were using x, this equation could be written

$$y' = ky.$$

.

$$\frac{dy}{dt} = ky.$$

The solutions to this equation are exponential functions of the form

$$y = y_0 e^{kt}.$$

Such a function only grows over time when $k > 0$; when $k < 0$, the function decreases asymptotically to zero, which is known as **exponential decay**. Both exponential growth and exponential decay are quite common in science, and we will discuss several applications of these in Sections 2.1 and 2.2.

Excluding growth and decay, the next simplest type of behavior for a dynamical system is oscillation. The simplest model for oscillation is **harmonic oscillation**, which corresponds to the second-order differential equation

Again, this is the same as the equation

$$y'' = -ry$$

except that we are using t instead of x.

$$\frac{d^2y}{dt^2} = -ry \qquad (r > 0).$$

The solutions to this equation are sinusoidal functions of the form

$$y = C\cos(\omega t) + D\sin(\omega t)$$

where C and D are constants and $\omega = \sqrt{r}$. This solution can also be written as

$$y = A\cos(\omega t + \phi)$$

where A and ϕ are constants. We will discuss several examples of harmonic oscillation in Sections 2.3 and 2.4

2.1 Exponential Growth

In this section we discuss the differential equation

$$\frac{dy}{dt} = ky,$$

where k is a positive constant. In words, this equation says that the rate at which the variable y changes is proportional to the value of y; thus the larger y becomes, the more quickly it increases. The result is that y grows, slowly at first and then very quickly, a phenomenon known as **exponential growth**.

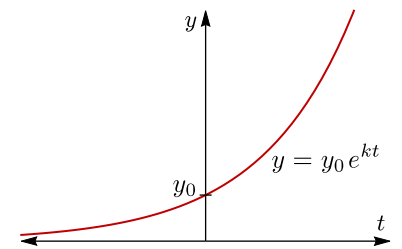**The Exponential Growth Equation**

The **exponential growth equation** is the differential equation

$$\frac{dy}{dt} = ky \qquad (k > 0).$$

Its solutions are **exponential functions** of the form

$$y = y_0 e^{kt}$$

where $y_0 = y(0)$ is the initial value of y.

▲ **Figure 1:** Exponential growth.

Figure 1 shows the graph of a typical exponential function, assuming $y_0 > 0$ and $k > 0$. Because of the factor of e^t, an exponential function increases quite quickly as t increases, as illustrated in Figure 2.

t	e^t
1	2.72
2	7.39
5	148.41
10	22,026.47
20	485,165,195.41

▲ **Figure 2:** The exponential function e^t grows very quickly as t increases.

EXAMPLE 1

Solve the following initial value problem.

$$\frac{dy}{dt} = 3y, \qquad y(0) = 4.$$

SOLUTION   According to the formulas above, the solution is $y(t) = 4e^{3t}$.

EXAMPLE 2

Solve the following initial value problem.

$$\frac{dy}{dt} = 3y, \qquad y(2) = 8.$$

SOLUTION   This time the initial value is at $t = 2$ instead of $t = 0$, so we have some work to do. We know that y has the form

$$y(t) = y_0 e^{3t}$$

for some constant y_0. Plugging in $y(2) = 8$ gives

$$8 = y_0 e^6,$$

so $y_0 = 8e^{-6}$. Then

$$y(t) = (8e^{-6}) e^{3t} = 8e^{3t-6}.$$

The Growth Constant

The constant k in the equations

$$\frac{dy}{dt} = ky \qquad \text{and} \qquad y = y_0 e^{kt}$$

is called the **growth constant** or **exponential growth rate**. It controls how rapidly the exponential function grows—higher values of k correspond to faster growth, while lower values of k correspond to more gradual growth.

In general, e^x only makes sense if x is a pure number. If x is a quantity with units, then e^x is meaningless.

No matter what the units are for y, the units for k are always inverse time. For example, k could be something like 0.27/sec, which is the same as 16.2/hour. Note then that the product kt is always a dimensionless number, which is why it makes sense to compute e^{kt}.

EXAMPLE 3

A variable y is growing exponentially. Initially y has a value of 200. Three hours later, it has grown to 500. What is the growth constant for y?

SOLUTION   Since $y(0) = 200$, we know that

$$y(t) = 200 e^{kt}$$

for some constant k. Substituting in $y(3) = 500$ gives the equation

$$500 = 200 e^{3k}.$$

To solve for k here, we first divide by 200 to get
$$e^{3k} = 2.5.$$
Then $3k = \ln(2.5)$, so $k = \ln(2.5)/3$.

Solving for k yields

$$k = \frac{\ln(2.5)}{3} \approx \boxed{0.305/\text{hour}}.$$

Note that we needed *two* pieces of information about y in the last example to determine the value of k. Even though the exponential growth equation is a first-order equation, it is common in applications to not know the value of k beforehand. The result is that the solution

$$y = y_0 e^{kt}$$

has *two* unknown constants, so we need two pieces of information about y to determine y_0 and k.

EXAMPLE 4

A variable y is growing exponentially. Given that $y(0) = 5$ and $y'(0) = 8$, compute $y(2)$.

SOLUTION   Since $y(0) = 5$, we know that

$$y(t) = 5 e^{kt}$$

for some constant k. To substitute in $y'(0) = 8$, we take the derivative of this equation:

$$y'(t) = 5k e^{kt}.$$

A different way to find k in this example is to substitute both $y(0) = 5$ and $y'(0) = 8$ directly into the differential equation
$$\frac{dy}{dt} = ky.$$
This gives $8 = 5k$, so $k = 1.6$.

Substituting 0 for t and 8 for $y'(t)$ yields $8 = 5k$, so $k = 8/5 = 1.6$. Then

$$y(2) = 5 e^{(1.6)(2)} \approx \boxed{122.66}$$

Other Measures of Growth

In many applications, it makes sense to consider the *reciprocal* of the growth constant k:

$$\tau = \frac{1}{k}$$

Using τ instead of k, the exponential growth equation and its solution can be written as

$$\frac{dy}{dt} = \frac{y}{\tau} \qquad \text{and} \qquad y = y_0 e^{t/\tau}.$$

▲ **Figure 3:** The e-folding time τ is the amount of time it takes for y to grow by a factor of e.

The main advantage of τ over k is that it has units of time, which makes it a much more intuitive measure of the growth rate. Indeed, since $y(\tau) = e$, we can interpret τ as the amount of time that it takes for the function y to grow by a factor of e, as shown in Figure 3. For this reason, τ is known as the **e-folding time** for the exponential function.

Another related measure of the exponential growth rate is the **doubling time** T_d. This is the amount of time that it takes for the exponential function to double in value, as shown in Figure 4. We can find a formula for the doubling time by solving the equation

$$e^{kT_d} = 2$$

for T_d. The result is

$$T_d = \frac{\ln 2}{k} = \tau \ln 2$$

Note that $\ln 2 \approx 0.6931$, so the doubling time is approximately 69% of the e-folding time.

▲ **Figure 4:** The doubling time T_d is the amount of time it takes for y to double in value.

Population Growth

Exponential growth is often used to model the growth of populations of organisms in a resource-rich environment. Here "resource-rich" means that there is plenty of food and other resources necessary for the population to grow. For example, the initial growth of a bacteria colony in a petri dish is often modeled as exponential.

The justification for this model is that the rate at which a population of organisms grows should be proportional to the number of organisms, assuming that the organisms reproduce at a constant rate. For example, if you double the size of a population, then this should precisely double the rate at which the population bears offspring, and should therefore double the rate at which the size of the population increases.
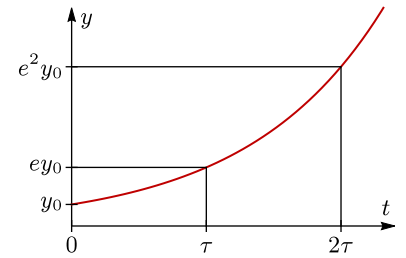
What this means is that the population P of a given organism in a resource-rich environment should satisfy the differential equation
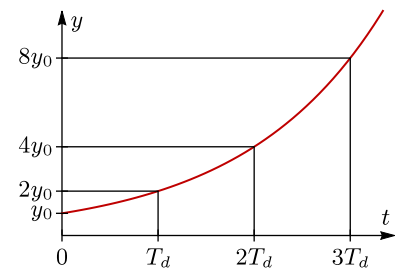
$$\frac{dP}{dt} = kP,$$

where k is some constant that depends on the rate of reproduction. Thus the population grows exponentially:

$$P = P_0 e^{kt}.$$

Of course, this model predicts that the population P will grow indefinitely, which cannot be true in any real situation. Eventually any population will run out of resources such as food or space to grow. However, the exponential model often gives fairly accurate results in cases where the short-term growth of a population is not inhibited by limited resources.

▲ A colony of *Paenibacillus dendritiformis* bacteria.[2]

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[2]Photo from the laboratory of Dr. Eshel Ben Jacob, licensed under CC BY-SA 3.0, via Wikimedia Commons

EXAMPLE 5

During a biology experiment, a certain culture of cells grows exponentially with a growth constant of 0.04/minute. If there are 5,000 cells at the beginning of the experiment, how large will the culture be one hour later?

SOLUTION   The population $P(t)$ is given by the equation

$$P(t) = 5{,}000e^{0.04t}.$$

We are measuring t in minutes, so one hour later corresponds to $t = 60$. Since

$$P(60) = 5{,}000e^{0.04(60)} \approx 55{,}115.9,$$

the population after one hour will be approximately 55,000 cells.

EXAMPLE 6

A population of bacteria initially consists of 20,000 cells. Twenty minutes later, the population has grown to 50,000 cells. How quickly is the population increasing at that time?

SOLUTION   Assuming constant relative growth rate, the population $P(t)$ is given by the equation

$$P(t) = 20{,}000e^{rt}. \tag{1}$$

for some constant r. Plugging in $P(20) = 50{,}000$ gives the equation

$$50{,}000 = 20{,}000e^{k(20)}.$$

and solving for k yields

$$k = \frac{\ln 2.5}{20} \approx 0.045815.$$

To find the rate of increase at $t = 20$ min, we must compute $P'(20)$. Taking the derivative of equation (1) gives

$$P'(t) = 20{,}000ke^{kt}$$

so

$$P'(20) = 20{,}000(0.045815)e^{(0.045815)(20)} \approx 2290.73.$$

Thus the population is increasing at a rate of roughly 2,300 cells/min.

To solve the equation

$$20{,}000e^{20k} = 50{,}000$$

for k, we first divide by 20,000 to get

$$e^{20k} = 2.5,$$

and then we take the natural logarithm of both sides, which yields

$$20k = \ln(2.5).$$

We can now divide through by 20 to obtain the value of k.

EXERCISES

1. Solve the following initial value problem:

$$\frac{dy}{dt} = 3y, \qquad y(\ln 2) = 40.$$

2. At the beginning of an experiment, a culture of *E. coli* contains 15,000 cells. Two hours later, the population has grown to a size of 80,000 cells. Assuming exponential growth, what is the *e*-folding time τ?

3. During a nuclear chain reaction, the number N of free neutrons in a sample of fissile material obeys the differential equation

$$\frac{dN}{dt} = kN,$$

where k a constant known as the **neutron multiplication factor**.

(a) Suppose the sample initially contains 100 free neutrons. If the neutron multiplication factor is $7.0/\mu s$, how many free neutrons will there be $2.0\ \mu s$ later?

(b) How quickly will the number of free neutrons be increasing at that time?

4. The population of the United States is currently 320 million, and is increasing at a rate of 2.4 million/year. Assuming exponential growth, what is the doubling time of the population?

5. A cell culture is growing exponentially with a doubling time of 3.00 hours. If there are 11,500 cells initially, how long will it take for the cell culture to grow to 30,000 cells?

2.2   Exponential Decay

When the growth constant k is negative, the function

$$y = y_0 e^{kt}$$

does not actually grow. Instead, the value of y approaches zero as $t \to \infty$. This is known as **exponential decay**.

   When discussing exponential decay, it is common in applications to write $k = -r$, where r is a positive constant known as the **decay constant**. In this case, the differential equation takes the following form.

This is just the exponential growth equation with $-r$ substituted in for k.

The Exponential Decay Equation

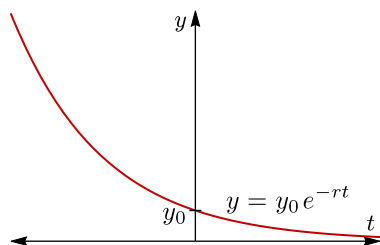The **exponential decay equation** is the differential equation

$$\frac{dy}{dt} = -ry$$

where r is a positive constant. Its solutions have the form

$$y = y_0 e^{-rt}$$

where $y_0 = y(0)$ is the initial value of y.

▲ **Figure 5:** Exponential decay.

   Figure 5 shows the graph of a typical exponential decay. As with exponential growth, there are two alternatives to using the decay constant r when describing exponential decay:

1. The **time constant** (or *e*-**folding time**) of y is the quantity $\tau = 1/r$, and represents the amount of time that it takes for the value of y to be divided by e.

2. The **half-life** of y is the amount of time that it takes for the value of y to be cut in half. It can be found by solving the equation $e^{-rt} = 1/2$ for t.

As we shall see, exponential decay can be used to model such diverse phenomena as radioactive decay, electric circuits, and chemical reactions.

Radioactive Decay

See the EPA website for a list of radioisotopes commonly used in industry.

A substance whose atoms are inherently unstable is called **radioactive**. For such a substance, a certain fixed proportion of the atoms decay during each time interval. If N is the number of atoms in a sample of the substance, then N will satisfy the differential equation

$$\frac{dN}{dt} = -rN.$$

Here r is the decay constant, which represents the rate at which individual atoms tend to decay. For example, if $r = 0.003$/hour, it means that about 0.3% of the atoms will decay each hour.

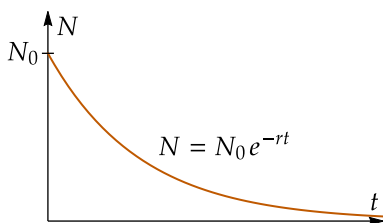   The number of atoms in a radioactive sample decays exponentially with time:

$$N = N_0 e^{-rt},$$

▲ **Figure 6:** Exponential decay of a radioactive sample.

where N_0 is the number of atoms int the sample at time $t = 0$ (see Figure 6). Equivalently, the total mass M of atoms in a sample will decay exponentially:

$$M = M_0 e^{-rt}.$$

EXAMPLE 7

Cesium-137 has a half-life of approximately 30.17 years. If a 0.300-mole sample of ^{137}Cs is left in a storage closet, how much ^{137}Cs will be left after four years?

SOLUTION   The amount $N(t)$ of ^{137}Cs will obey an equation of the form

$$N(t) = 0.30e^{-rt},$$

where r is a constant. Since the half-life is 30.17 years, we know that

$$e^{-r(30.17)} = \frac{1}{2}.$$

Solving for r gives $r \approx 0.022975$/year. Then

$$N(4) = 0.300e^{-(0.022975)(4)} = 0.273659,$$

so there should be 0.274 moles left after four years.

EXAMPLE 8

A radiochemist prepares a cobalt sample containing 0.100 moles of ^{58}Co. According to readings from a Geiger counter, the atoms of ^{58}Co in the sample appear to be decaying at a rate of 6.79×10^{-7} moles/min. Based on this information, what is the half-life of ^{58}Co?

SOLUTION   The amount $N(t)$ of ^{58}Co will obey an equation of the form

$$N(t) = 0.100e^{-rt},$$

where r is a constant. Then

$$N'(t) = -0.100re^{-rt}.$$

We are given that $N'(0) = -6.79 \times 10^{-7}$ moles/min, which gives us the equation

$$-6.79 \times 10^{-7} = -0.100r.$$

Solving for r gives $r = 6.79 \times 10^{-6}$/min. The half-life is the value of t for which

$$e^{-rt} = \frac{1}{2}.$$

Plugging in r and solving for t yields a half life of 102,084 minutes, which is about 70.9 days.

▲ **Figure 7:** A sample of cesium-137

$N'(0)$ is negative since N is decreasing.

RC Circuits

Figure 8 shows a simple kind of electric circuit known as an **RC circuit**. This circuit has two components:

- A **resistor** is any circuit component—such as a light bulb—that resists the flow of electric charge. Applying voltage to a resistor will force current through it, with the amount of current given by **Ohm's law**

$$I = \frac{V}{R}. \tag{1}$$

Here V is the applied voltage, I is the resulting current, and R is a constant called the **resistance** of the resistor.
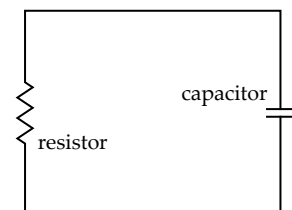
▲ **Figure 8:** An RC circuit.

This application assumes you are familiar with the basics of electric circuits, as well as the SI units corresponding to various circuit-related quantities. Here is a summary:

- Electric charge is a basic property of matter, inherent to charged particles such as electrons and protons. It is measured in **coulombs (C)**.

- Voltage is a difference in electric potential that causes charge to move through a circuit. It is measured in **volts (V)**.

- Electric current is the rate at which electric charge flows through a circuit. It is measured in **amps (A)**, where 1 A = 1 C/sec.

- Resistance is a property of resistors. It is measured in **ohms (Ω)**, where 1 Ω = 1 V/A.

- Capacitance is a property of capacitors. It is measured in **farads (F)**, where 1 F = 1 C/V.

- A **capacitor** is a circuit component that stores electric charge. A charged capacitor can supply voltage to a circuit, with the amount of voltage given by the equation

$$V = \frac{Q}{C}. \tag{2}$$

Here Q is the charge stored in the capacitor and C is a constant called the **capacitance** of the capacitor.

In an RC circuit, the voltage produced by a capacitor is applied directly across a resistor. Substituting equation (2) into equation (1) yields a formula for the resulting current:

$$I = \frac{Q}{RC}. \tag{3}$$

This current represents the flow of charge out of the capacitor, with

$$\boxed{\frac{dQ}{dt} = -I.}$$

Substituting equation (3) into this equation yields the differential equation

$$\frac{dQ}{dt} = -\frac{Q}{RC}.$$

Thus the charge Q will decay exponentially, with decay constant

Equivalently, the time constant for the RC circuit is
$$\tau = RC.$$

$$\boxed{r = \frac{1}{RC}}$$

In the case where the resistor is a light bulb, the result is that the bulb lights up at first, but becomes dimmer and dimmer over time as the capacitor discharges.

EXAMPLE 9

A 0.25 F capacitor holding a charge of 2.0 C is attached to a 1.6 Ω resistor. How long will it take for the capacitor to expend 1.5 C of its initial charge?

SOLUTION   The charge on the capacitor will decay exponentially according to the formula

$$Q = Q_0 e^{-rt},$$

where $Q_0 = 2.0$ C and

Since seconds are the SI unit of time, the units for r must be number per second.

$$r = \frac{1}{RC} = \frac{1}{(1.6\ \Omega)(0.25\ \text{F})} = 2.5/\text{sec}.$$

If the capacitor expends 1.5 C of its charge, it will have 0.5 C left. Plugging this into the formula for Q gives
$$0.5 = 2.0 e^{-(2.5)t},$$
and solving for t yields $t = 0.5545$. Thus the capacitor will expend 1.5 C of charge in approximately 0.55 seconds.

EXERCISES

1. A sample of an unknown radioactive isotope initially weighs 5.00 g. One year later the mass has decreased to 4.27 g.

   (a) How quickly is the mass of the isotope decreasing at that time?

   (b) What is the half life of the isotope?

2. During a certain chemical reaction, the concentration $[C_4H_9Cl]$ of butyl chloride obeys the rate equation

$$\frac{d[C_4H_9Cl]}{dt} = -r[C_4H_9Cl],$$

where $r = 0.1223/\text{sec}$ is the **rate constant** for the reaction. How long will it take for this reaction to consume 90% of the initial butyl chloride?

3. A capacitor with a capacitance of 5.0 F holds an initial charge of 350 C. The capacitor is attached to a resistor with a resistance of 9.0 Ω.

   (a) How quickly will the charge held by the capacitor initially decrease?

   (b) How quickly will the charge be decreasing after 20 seconds?

4. An **LR circuit** consists of a resistor attached to an electrical component called an **inductor**, which supplies voltage to the resistor according to the formula

$$V = -L\frac{dI}{dt}.$$

Here L is a constant called the **inductance** of the inductor.

   (a) Combine the equation above with Ohm's law to obtain a differential equation for the current $I(t)$ that involves the constants L and R.

   (b) The current $I(t)$ in an LR circuit decays exponentially. Find a formula for the decay constant in terms of L and R.

5. For a planetary atmosphere of ideal gas of uniform temperature T, the atmospheric pressure $P(h)$ and density $\rho(h)$ at a height h above the ground are related by the equations

$$P = \rho RT \qquad \text{and} \qquad \frac{dP}{dh} = -\rho g,$$

where R is the specific gas constant and g is the acceleration due to gravity.

The first of these equations is the molar form of the **ideal gas law**, while the second is the equation for hydrostatic pressure.

   (a) Combine the given equations to obtain a single differential equation for P involving the constants g, R, and T.

   (b) The atmospheric pressure in such an atmosphere varies with height according to the formula $P(h) = P_0 e^{-rh}$, where P_0 is the pressure at ground level. Find a formula for the decay constant r in terms of g, R, and T.

2.3   Oscillation

So far, we have used differential equations to describe functions that grow or decay over time. The next most common behavior for a function is to **oscillate**, meaning that it increases and decreases in a repeating pattern. There is a simple differential equation that leads to this behavior.

The Harmonic Oscillator Equation

The **harmonic oscillator equation** is the differential equation

$$\frac{d^2 y}{dt^2} = -ry \qquad (r > 0).$$

Its solutions have the form

$$y = C\cos(\omega t) + D\sin(\omega t),$$

where C and D are constants, and $\omega = \sqrt{r}$.

This is the **Cartesian form** of the solution. There is also a **polar form** that we will describe in a short while.

As we will see shortly, the formula $y = C\cos(\omega t) + D\sin(\omega t)$ actually describes simple **sinusoidal oscillation**, also known as **harmonic oscillation**. The constant $\omega = \sqrt{r}$ is called the **angular frequency** of the oscillation. The square root comes from taking the second derivative; for if

The angular frequency ω has units of radians per unit time, e.g. rad/sec.

$$y = C\cos(\omega t) + D\sin(\omega t)$$

then taking the second derivative gives

$$\frac{d^2 y}{dt^2} = -\omega^2 C\cos(\omega t) - \omega^2 D\sin(\omega t)$$

and thus y satisfies the equation

$$\frac{d^2 y}{dt^2} = -\omega^2 y.$$

It would also work to let $\omega = -\sqrt{r}$, but this actually yields the same solutions as $\omega = \sqrt{r}$, so it is simpler to use a positive value for ω.

We conclude that $r = \omega^2$, and hence $\omega = \sqrt{r}$.

EXAMPLE 10

Solve the following initial value problem.

$$\frac{d^2 y}{dt^2} = -25y, \qquad y(0) = 3, \qquad y'(0) = 10.$$

SOLUTION   We know that
$$y(t) = C\cos(5t) + D\sin(5t)$$
for some constants C and D, which means that

$$y'(t) = -5C\sin(5t) + 5D\cos(5t).$$

Remember that $\cos(0) = 1$ and $\sin(0) = 0$.

Plugging $y(0) = 3$ into the first equation and $y'(0) = 10$ into the second equation yields $C = 3$ and $D = 2$, so the solution is

$$y(t) = 3\cos(5t) + 2\sin(5t).$$

EXAMPLE 11

Solve the following boundary value problem.

$$\frac{d^2y}{dt^2} = -4y, \qquad y(0) = 1, \qquad y\left(\frac{\pi}{8}\right) = 3\sqrt{2}$$

SOLUTION   We know that

$$y(t) = C\cos(2t) + D\sin(2t)$$

for some constants C and D. The two boundary values yield the equations

$$1 = C \qquad \text{and} \qquad 3\sqrt{2} = C\cos\left(\frac{\pi}{4}\right) + D\sin\left(\frac{\pi}{4}\right).$$

Plugging $C = 1$ and $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ into the second equation and solving for D gives $D = 5$, and hence

$$y(t) = \cos(2t) + 5\sin(2t).$$

Cartesian and Polar Forms

Our general solution

$$y = C\cos(\omega t) + D\sin(\omega t)$$

to the harmonic oscillator equation is called the **Cartesian form** of the solution. For many applications, it is more convenient to write the solution in the following **polar form**.

Polar Form for the Solution

The solutions to the harmonic oscillator equation

$$\frac{d^2y}{dt^2} = -ry \qquad (r > 0).$$

can also be written as

$$y = A\cos(\omega t + \phi)$$

where A and ϕ are constants.

The constant A is called the **amplitude**, and ϕ is the **phase angle**.

Functions of the form

$$y = A\cos(\omega t + \phi)$$

are sometimes called **sinusoidal functions**. The graph of such a function is a simple sine wave, as shown in Figure 9. The solutions to the harmonic oscillator equation are precisely the sinusoidal functions, and any variable y that obeys the harmonic oscillator equation undergoes sinusoidal oscillation.

It is not obvious that the Cartesian and polar forms of a sinusoidal function are equivalent. To see this, consider a sinusoidal function in polar form:
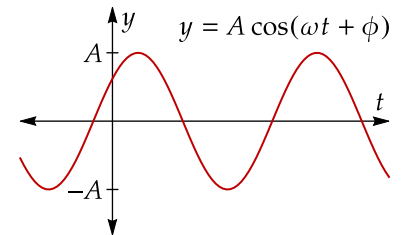
$$y = A\cos(\omega t + \phi)$$

Using the sum of angle formula for cosine, we can expand the right side to get

$$y = A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t).$$

This has the form $y = C\cos(\omega t) + D\sin(\omega t)$, where

▲ **Figure 9:** The graph of a sinusoidal function.

Here we used the identity

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

$$C = A\cos(\phi) \qquad \text{and} \qquad D = -A\sin(\phi)$$

The algebra here is similar to the conversion between rectangular and polar coordinates, where $(x, y) = (C, -D)$ and $(r, \theta) = (A, \phi)$.

Conversely, for any values of C and D it is always possible to find values for A and ϕ that satisfy the above equations. In particular,

$$A = \sqrt{C^2 + D^2}, \qquad \cos(\phi) = \frac{C}{A}, \qquad \text{and} \qquad \sin(\phi) = -\frac{D}{A}$$

These formulas let us convert sinusoidal functions between Cartesian and polar forms.

EXAMPLE 12

Find the Cartesian form of the sinusoidal function $y(t) = 4\cos\left(3t + \dfrac{\pi}{3}\right)$.

SOLUTION   We have $A = 4$ and $\phi = \pi/3$, so

$$C = 4\cos\left(\frac{\pi}{3}\right) = 2 \qquad \text{and} \qquad D = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3}$$

and therefore

$$y(t) = 2\cos(3t) - 2\sqrt{3}\sin(3t).$$

EXAMPLE 13

Find the polar form of the sinusoidal function $y(t) = 6\cos(4t) + 6\sin(4t)$.

SOLUTION   We have $C = 6$ and $D = 6$, so

$$A = \sqrt{C^2 + D^2} = \sqrt{6^2 + 6^2} = 6\sqrt{2}.$$

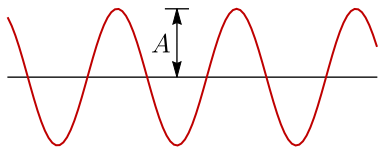Then

$$\cos(\phi) = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sin(\phi) = -\frac{6}{6\sqrt{2}} = -\frac{1}{\sqrt{2}},$$
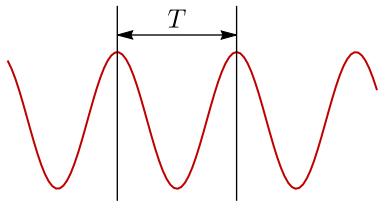
Here we were able to just recognize the angle ϕ from its cosine and sine, but in general we can find ϕ using inverse trigonometric functions.

and hence $\phi = -\pi/4$. We conclude that

$$y(t) = 6\sqrt{2}\cos\left(4t - \frac{\pi}{4}\right).$$

▲ **Figure 10:** The amplitude of an oscillation.

▲ **Figure 11:** One period of an oscillation.

Properties of Oscillation

The two most important properties of any oscillation are its amplitude and its period. The **amplitude** of an oscillation is simply its maximum value A, as shown in Figure 10. For a sinusoidal oscillation, this is the coefficient of the cosine when the function is expressed in polar form. The **period** of an oscillation is the amount of time T that it takes for the oscillation to go through one complete cycle, as shown in Figure 11. For a sinusoidal oscillation, the period is given by the formula

$$T = \frac{2\pi}{\omega}$$

where ω is the angular frequency.

Closely related to the period of an oscillation is its **frequency**, which is defined by the formula

$$f = \frac{1}{T}$$

The frequency is measured in units of inverse time (i.e. number per unit time), and can be interpreted as the rate at which oscillations occur. For example, a sinusoidal function with a frequency of 3/sec undergoes three full oscillations each second, while a sinusoidal function with a frequency of 0.5/sec undergoes half of an oscillation each second, or one full oscillation every two seconds.

The SI unit for frequency is the **hertz** (Hz), where 1 Hz = 1/sec.

Note that the frequency is not the same as the angular frequency. These are related by the formula

$$\omega = 2\pi f$$

Thus the angular frequency also measures the rate of oscillation, but it is much less natural than either the period or the frequency. Indeed, angular frequency is only really important because it appears in the formulas for sinusoidal oscillation.


EXAMPLE 14

A harmonic oscillator satisfies the differential equation

$$\frac{d^2 y}{dt^2} = -0.34\,y.$$

What is the period of oscillation?

SOLUTION   The angular frequency is $\omega = \sqrt{0.34} \approx 0.5831$, so $T = \dfrac{2\pi}{\omega} \approx 10.78$.

Finally, every sinusoidal oscillation has a **phase angle** ϕ, which describes the state of the oscillation when $t = 0$. A phase angle of $\phi = 0$ corresponds to an oscillation that starts at its maximum value at $t = 0$. Phase angles less than 0 correspond to starting earlier in the cycle, and phase angles greater than 0 correspond to starting later in the cycle, as shown in Figure 12.
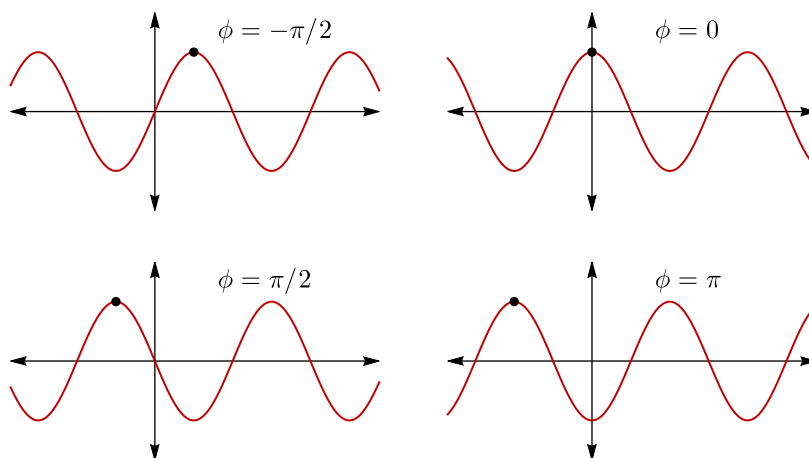
◀ **Figure 12:** Sinusoidal oscillations with four different phase angles ϕ.

A Closer Look | Solving the Harmonic Oscillator Equation

For $\omega > 0$, we have stated without proof that the solutions to the equation

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

are the functions $y = C\cos(\omega t) + D\sin(\omega t)$, where C and D can be any constants. It is easy to see that these functions are indeed solutions, but how can we be sure that every solution has this form?

We can prove this as follows. Suppose that $y(t)$ is a solution to the above equation, and define functions $C(t)$ and $D(t)$ by the formulas

$$C(t) = y(t)\cos(\omega t) - \frac{y'(t)}{\omega}\sin(\omega t), \qquad D(t) = y(t)\sin(\omega t) + \frac{y'(t)}{\omega}\cos(\omega t).$$

Observe that

$$y(t) = C(t)\cos(\omega t) + D(t)\sin(\omega t).$$

We wish to show that $C(t)$ and $D(t)$ are constant functions. To prove this, we take the derivative of each using the product rule. The derivative of $C(t)$ is

$$C'(t) = y'(t)\cos(\omega t) - \omega y(t)\sin(\omega t) - \frac{y''(t)}{\omega}\sin(\omega t) - y'(t)\cos(\omega t).$$

The first and last terms cancel, leaving

$$C'(t) = -\omega y(t)\sin(\omega t) - \frac{y''(t)}{\omega}\sin(\omega t).$$

Substituting in $y''(t) = -\omega^2 y(t)$ causes the two remaining terms to cancel, giving us

$$C'(t) = 0.$$

Thus $C(t)$ is a constant function. A similar computation shows that $D(t)$ is a constant function, and therefore $y(t)$ has the desired form.

These formulas for $C(t)$ and $D(t)$ were obtained by solving the equations

$$y = C\cos(\omega t) + D\sin(\omega t)$$

$$y' = -\omega C\sin(\omega t) + \omega D\cos(\omega t)$$

for C and D.

EXERCISES

1. Solve the following initial value problem:

$$\frac{d^2y}{dt^2} = -5y, \qquad y(0) = 3, \qquad y'(0) = 10.$$

2. Solve the following boundary value problem:

$$\frac{d^2y}{dt^2} = -y, \qquad y(0) = 4, \qquad y\left(\frac{2\pi}{3}\right) = 1.$$

3. Solve the following boundary value problem:

$$\frac{d^2y}{dt^2} = -4y, \qquad y\left(-\frac{\pi}{6}\right) = 4, \qquad y\left(\frac{\pi}{6}\right) = 16.$$

4. Express the sinusoidal function $y(t) = 10\cos\left(6t + \frac{\pi}{4}\right)$ in Cartesian form.

5. Express each of the following sinusoidal functions in polar form.
   (a) $y(t) = \cos(3t) + \sin(3t)$
   (b) $y(t) = \sin(5t)$
   (c) $y(t) = -\sqrt{2}\cos t + \sqrt{6}\sin t$

6. Find the amplitude of the following sinusoidal function:

$$y(t) = 12\cos(4t) - 5\sin(4t).$$

7. Find the phase angle of the following sinusoidal function:

$$y(t) = 3\cos(2t) + 4\sin(2t).$$

Express your answer in degrees.

8. Find the period of the following sinusoidal function:

$$y(t) = 7.2\cos(5.4t + 1.2)$$

9. A sinusoidal function $y(t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} = -5000y.$$

What is the frequency of the oscillation?

10. A sinusoidal function $y(t)$ has phase angle $\pi/3$. Given that $y(0) = 5$, what is the amplitude?

2.4   Models of Oscillation

In this section we give three examples of oscillating physical systems that can be modeled by the harmonic oscillator equation. Such models are ubiquitous in physics, but are also used in chemistry, biology, and social science to model oscillatory behavior.

Mass-Spring Systems

spring

block

frictionless table $x = 0$

▲ **Figure 13:** A simple mass-spring system.

Consider the simple mass-spring system shown in Figure 13, which consists of a block with mass m attached to spring whose other end is fixed. According to **Hooke's Law**, the force that the spring exerts on the block is given by the equation

$$F = -kx.$$

Here k is the **spring constant**, and x is the horizontal position of the block, with $x = 0$ being the rest position.

Newton's second law ($F = ma$) can be written as

$$F = m\frac{d^2x}{dt^2},$$

We are neglecting friction in this computation. Adding a frictional force to the model would make this an example of **damped harmonic oscillations**.

where $x(t)$ is the horizontal position of the block. Assuming the spring is the only horizontal force affecting the block, this gives us the differential equation
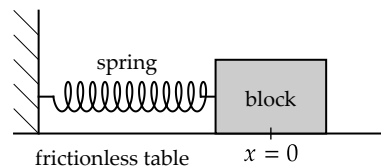
$$-kx = m\frac{d^2x}{dt^2},$$

which we can rewrite as

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x.$$

This is a form of the harmonic oscillator equation, with angular frequency

An object whose position oscillates sinusoidally is said to undergo **harmonic motion**.

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

EXAMPLE 15

A 3.0 kg mass is attached to a spring with a spring constant of 4.0 kg/sec^2. The spring is stretched 0.80 m from its rest position and then the mass is released. What is the speed of the mass 1.0 sec later?

SOLUTION   The mass will undergo harmonic motion with an angular frequency of

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.0}{3.0}} = 1.1547 \text{ rad/sec}.$$

The position $x(t)$ of the mass obeys a formula of the form

$$x(t) = C\cos(\omega t) + D\sin(\omega t),$$

for some constants C and D. Taking the derivative gives

$$x'(t) = -\omega C\sin(\omega t) + \omega D\cos(\omega t)$$

We are given that $x(0) = 0.80$ and $x'(0) = 0$, and plugging these in gives $C = 0.80$ and $D = 0$, so the speed of the mass at $t = 1.0$ sec is

$$x'(1.0) = -(1.1547)(0.80)\sin(1.1547) = -0.84 \text{ m/sec}.$$

LC Circuits

Figure 14 shows a simple kind of electric circuit known as an **LC circuit**. This circuit consists of a capacitor connected to a circuit component **inductor**, which is essentially just a coil of wire. Unlike a resistor, which always resists the flow of current, an inductor tends to oppose *changes* to the flow of electric current. That is, it's difficult to start pushing current through an inductor, but once the current gets going, it's difficult to make it stop.
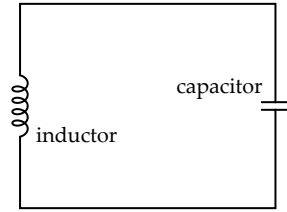
▲ **Figure 14:** An LC circuit.

The voltage drop V across an inductor is given by the formula

$$V = L\frac{dI}{dt}$$

Here L is a constant called the **inductance** of the inductor. Note that the inductor has positive voltage drop (like a resistor) when the current is increasing, but when the current is decreasing the voltage drop is negative, meaning that the inductor actually pulls current through it.

The SI unit of inductance is the **henry** (**H**), where 1 H = 1 V · sec/A.

Combining the formula above with the equation $V = Q/C$ for the capacitor yields

$$L\frac{dI}{dt} = \frac{Q}{C}.$$

As in an RC circuit, the electric current I is that same as the rate at which the capacitor is discharging, so

$$\boxed{I = -\frac{dQ}{dt}.}$$

Substituting this into the previous equation yields the differential equation

$$-L\frac{d^2Q}{dt^2} = \frac{Q}{C}.$$

which we can rewrite as

$$\frac{d^2Q}{dt^2} = -\left(\frac{1}{LC}\right)Q.$$

This equation describes sinusoidal oscillation with angular frequency

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

▲ An inductor is just a coil of wire. Its electrical properties derive from the magnetic field it creates when current flows through it.

Thus the charge held in the capacitor oscillates according to the formula

$$Q = A\cos(\omega t + \phi),$$

where A is the amplitude of the oscillations (measured in coulombs, the SI unit of charge), and ϕ is the initial phase of the circuit.

Roughly speaking, if we assume that the capacitor begins charged, then the capacitor begins by discharging through the inductor, slowly at first but picking up speed as the inductor lets more current through. Once the capacitor is fully discharged, the inductor continues pushing current through the circuit, which drains even more charge from the capacitor, leaving it with a negative total charge. The capacitor then reverses the flow of current to regain the lost charge, but the same thing happens again, with the inductor continuing to push current through in the reverse direction until the capacitor is back to its initial charged state. The cycle thus continues indefinitely.
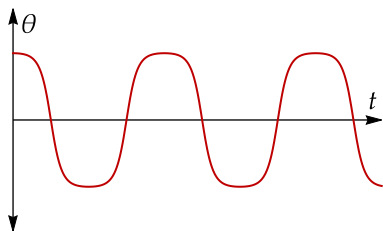
The capacitor is like the "spring" in an LC circuit (with spring constant $1/C$), while the inductor is the "mass".
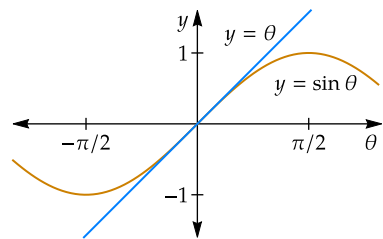
Here we use D and E for the constants, since the letter C represents capacitance.
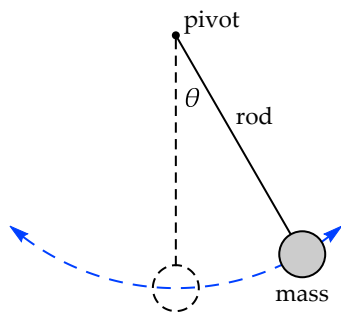
▲ **Figure 15:** A pendulum.

▲ **Figure 16:** Anharmonic motion of a pendulum with initial conditions $\theta(0) = 0.99\pi$ and $\theta'(0) = 0$. Note that the graph is not actually a sine wave.

▲ **Figure 17:** The graphs of $y = \sin\theta$ and $y = \theta$ nearly coincide for θ close to zero.

Pendulums

A **pendulum** consists of a mass suspended from a rod that swings from a fixed pivot point, as shown in Figure 15. If $\theta(t)$ denotes the angle of the string from the vertical, then θ obeys the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g\sin\theta}{L}, \tag{1}$$

where g is the acceleration due to gravity (usually 9.8 m/sec^2), and L is the length of the string.

Unfortunately, equation (1) is not an instance of the harmonic oscillator equation, because the right side involves $\sin\theta$ instead of θ. This means that a pendulum is actually an **anharmonic** oscillator, meaning that the oscillation is not actually sinusoidal. For example, Figure 16 shows the noticeably anharmonic motion of a pendulum that starts from rest at $\theta(0) = 0.99\pi$.

Though the motion of a pendulum is anharmonic, we can make a **harmonic approximation** for the motion in the case where θ isn't too large. This depends on the linear approximation

$$\sin\theta \approx \theta,$$

which is quite accurate when θ is close to zero, as shown in Figure 17. Indeed, as long as θ stays between $-14°$ and $14°$, this approximation is accurate to within 1%. Replacing $\sin\theta$ with θ in equation (1) gives us the approximate differential equation

$$\frac{d^2\theta}{dt^2} \approx -\left(\frac{g}{L}\right)\theta.$$

This equation describes approximate harmonic motion, with angular frequency

$$\boxed{\omega \approx \sqrt{\frac{g}{L}}}$$

EXAMPLE 17

A swinging pendulum with a length of 2.0 m has an initial position of $\theta(0) = 0.10$ rad and an initial angular velocity of $\theta'(0) = -0.12$ rad/sec. What will the position of the pendulum be 0.80 sec later?

SOLUTION   The pendulum will move approximately harmonically with angular velocity

$$\omega \approx \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{2.0}} = 2.2136 \text{ rad/sec}^2.$$

Then

$$\theta(t) \approx C\cos(\omega t) + D\sin(\omega t) \qquad \text{and} \qquad \theta'(t) \approx -\omega C\sin(\omega t) + \omega D\cos(\omega t)$$

for some constants C and D. Plugging in the initial conditions gives $C = 0.10$ rad and $D = -0.05421$ rad. Then

$$\theta(0.50) \approx (0.10)\cos(1.77088) - (0.05421)\sin(1.77088) = -0.0730036,$$

so the pendulum will be at an angle of approximately -0.073 rad.

EXERCISES

1. A pendulum with a length of 0.30 m starts from rest at an angle of 0.18 rad. How quickly will the angle of the pendulum be changing 0.20 sec later?

2. A 2.0 μF capacitor is connected to an inductor. If the resulting system oscillates with a frequency of 3.0 kHz (i.e. 3000/sec), what is the inductance of the inductor?

3. A 3.5 kg mass has been attached to a spring with a spring constant of 24 kg/sec^2. If the mass is oscillating with an amplitude of 1.6 m, what is the maximum speed of the mass during the oscillation?

4. A charged capacitor with a capacitance of 3.0 F is attached to an inductor with an inductance of 0.20 H. The initial current in the circuit is zero, but after 1.0 sec the current has increased to 8.0 A. What was the initial charge on the capacitor?

5. When two masses m_1 and m_2 are connected by a spring, the length L of the spring obeys the differential equation

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 L}{dt^2} = -kL,$$

where k is the spring constant.
   (a) Suppose a 3.0 kg mass is attached to a 5.0 kg mass by a spring with a spring constant of 12 kg/sec^2. What is the period of the resulting oscillations?
   (b) In a carbon monoxide (CO) molecule, the bond between the carbon and oxygen atoms can be modeled as a spring with spring constant 1.13×10^{39} u/sec^2. Given that the carbon atom has a mass of 12.0 u and the oxygen atom has a mass of 16.0 u, at what frequency will the molecule tend to vibrate?

6. A pendulum with a length of 20.0 cm released from rest at an angle of 45.0°.
  - (a) What period does the harmonic approximation predict for this pendulum? Assume that the acceleration due to gravity is 9.81 m/sec^2.
  - (b) The pendulum is measured to have an actual period of 0.933 seconds. What was the percentage error in the period predicted by the harmonic approximation?