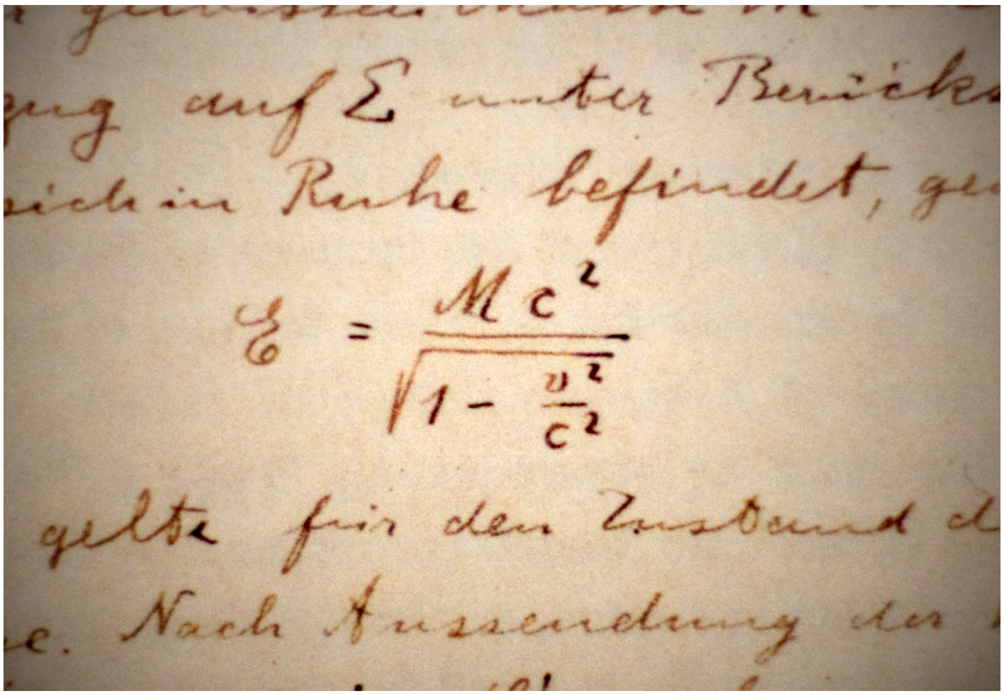
3 Algebraic Methods



◀ The first appearance of the equation $E = Mc^2$ in Einstein's handwritten notes.[1]

So far, the only general class of differential equations that we know how to solve are directly integrable equations, i.e. equations of the form

$$y' = f(x).$$

In this chapter, we will learn two new algebraic methods for solving differential equations, known as **integrating factors** and **separation of variables**. These methods will allow us to solve a wide variety of first-order equations.

Both of these methods are based on the idea of implicit differentiation. Given any equation involving x, y and a constant C, such as

$$x^2 + y^2 = C,$$

we can differentiate both sides implicitly to obtain a differential equation involving x and y:

$$2x + 2yy' = 0.$$

[1] Photo by Peat Bakke, licensed under CC BY 2.0.

By reversing this process, we can find the general solution to certain differential equations. For example, the equation

$$3y^2y' + e^{2x} = 0$$

integrable, since it is the derivative of the equation

$$y^3 + \frac{1}{2}e^{2x} = C.$$

Solving for y now yields the general solution. Implicit differentiation and the solution of integrable equations are discussed in Section 3.1.

Most differential equations are not integrable, but it is often possible to put a differential equation into integrable form using algebraic manipulations. In some cases, we can use algebra to **separate the variables** of a differential equation, putting it into the form

$$f(y)\,y' = g(x).$$

In this case, we can find a solution by integrating $f(y)$ and $g(x)$ separately. In other cases, the right approach is to multiply through by a certain function of x known as an **integrating factor**. This method can be used to solve any **linear differential equation** of the form

$$f(x)\,y' + g(x)\,y = h(x).$$

We will discuss separation of variables in Section 3.2, and integrating factors in Section 3.3.

Finally, in Section 3.4 we discuss several applications of integrating factors and separation of variables to science, including Newton's law of cooling and rate equations for chemical reactions.

3.1 Integrating Both Sides

Implicit differentiation is a technique from calculus for taking the derivative of expressions involving an unknown function $y(x)$. The idea is to use the normal algorithm for differentiation, and to simply write y' whenever we would usually need to take the derivative of y.

For example, to take the derivative of something like $(\sin x)^3$, we would normally use the power rule and the chain rule:

$$\frac{d}{dx}\left[(\sin x)^3\right] = 3(\sin x)^2 \cos x.$$

Here the $\cos x$ comes from the chain rule, since $\cos x$ is the derivative of $\sin x$.

We use exactly the same procedure to take the derivative of something like y^3, except that we write y' whenever we need the derivative of y:

$$\frac{d}{dx}\left[y^3\right] = 3y^2 y'.$$

Here the $3y^2$ comes from the product rule, and we multiply by y' because of the chain rule.

EXAMPLE 1

Take the derivative of both sides of the following equation:

$$\sin(y) + x^2 y = x^3.$$

SOLUTION We can use the chain rule to take the derivative of $\sin(y)$:

$$\frac{d}{dx}\left[\sin(y)\right] = \cos(y)\, y'.$$

The derivative of $x^2 y$ requires the product rule:

$$\frac{d}{dx}\left[x^2 y\right] = 2xy + x^2 y'.$$

In general, the product rule states that

$$\frac{d}{dx}\left[uv\right] = u'v + uv'.$$

In this case, $u = x^2$ and $v = y$, so $u' = 2x$ and $v' = y'$.

Thus the derivative is

$$y' \cos(y) + 2xy + x^2 y' = 3x^2.$$

We can use implicit differentiation to find a differential equation that has a given general solution. For example, suppose we want to find a differential equation whose general solution is

$$y = Cx^2.$$

We start by solving for the constant C:

$$yx^{-2} = C.$$

If we now take the derivative of both sides, the C disappears:

Since C is a constant, its derivative is zero.

$$y'x^{-2} - 2yx^{-3} = 0.$$

This is a differential equation whose general solution is $y = Cx^2$. We can simplify the equation a bit by multiplying through by x^3:

$$xy' - 2y = 0.$$

EXAMPLE 2

Find a differential equation whose general solution is $y = \ln(x + C)$.

SOLUTION To solve this equation for C, we first take the exponential of both sides:

$$e^y = x + C.$$

Thus

$$e^y - x = C.$$

Taking the derivative of both sides gives

$$e^y y' - 1 = 0.$$

This is the differential equation we wanted. We can simplify it by solving for y':

$$y' = e^{-y}.$$

Integrating Differential Equations

Now that we know how take derivatives implicitly, we can reverse the process to solve differential equations. For example, given a differential equation like

$$2yy' = 4x^3 + 2,$$

we can *integrate both sides* to get an equation involving x and y:

$$y^2 = x^4 + 2x + C.$$

Technically, there might be a constant on both sides of the antiderivative, i.e.

$$y^2 + C_1 = x^4 + 2x + C_2$$

However, we can combine the two constants by subtracting C_1 from both sides of this equation.

We can now solve for y to get the general solution to the given differential equation:

$$y = \pm\sqrt{x^4 + 2x + C}.$$

EXAMPLE 3

Find the general solution to the equation $y^2 y' = e^{3x}$.

SOLUTION We can integrate both sides to get

$$\frac{1}{3}y^3 = \frac{1}{3}e^{3x} + C.$$

Solving for y gives

$$y = \sqrt[3]{e^{3x} + 3C}.$$

Since C is an arbitrary constant, $3C$ is actually itself an arbitrary constant. If we let $A = 3C$, we can rewrite our general solution as

$$y = \sqrt[3]{e^{3x} + A},$$

where A is an arbitrary constant.

EXAMPLE 4

Solve the following initial value problem:

$$e^{4y}\, y' = x, \qquad y(0) = 1.$$

SOLUTION Integrating both sides yields

$$\frac{1}{4}e^{4y} = \frac{1}{2}x^2 + C.$$

Though we could solve for y at this point, it is easier to start by substituting in the initial condition:

$$\frac{1}{4}e^4 = 0 + C.$$

Then $C = \frac{1}{4}e^4$, so the solution becomes

$$\frac{1}{4}e^{4y} = \frac{1}{2}x^2 + \frac{1}{4}e^4.$$

Solving for y gives

$$y = \frac{1}{4}\ln\left(2x^2 + e^4\right).$$

In general, the antiderivative of $f(y)\,y'$ is

$$\int f(y)\,dy.$$

In this case, the antiderivative of $e^{4y}\,y'$ is

$$\int e^{4y}\,dy = \frac{1}{4}e^{4y} + C.$$

Sometimes it is possible to recognize one side of an equation as the result of the product rule.

EXAMPLE 5

Find the general solution to the differential equation $x^3 y' + 3xx^2 y = 1$.

SOLUTION The left side of this equation is the result of a product rule, with the two factors being x^3 and y. Integrating both sides gives

$$x^3 y = x + C.$$

Solving for y yields the general solution:

$$y = x^{-2} + Cx^{-3}.$$

A Closer Look Implicit Solutions

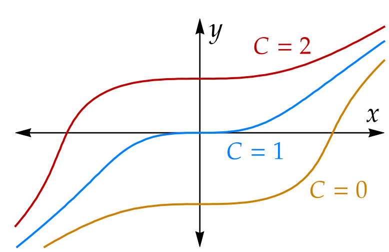
Sometimes after you integrate both sides of a differential equation it is not possible to algebraically solve for y. For example, consider the equation

$$(3y^2 + e^y)\,y' = x^2.$$

Integrating both sides gives

$$y^3 + e^y = \frac{1}{3}x^3 + C.$$

Unfortunately, there is no way to solve this equation algebraically for y. This **implicit solution** still describes the solutions of the equation, in the sense that the corresponding curves in the xy-plane are graphs of the solution functions (see Figure 1), but there is no way to write explicit algebraic formulas for these solutions.



▲ **Figure 1:** Three curves of the form $y^3 + e^y = \frac{1}{3}x^3 + C$.

A Closer Look **Integrating Second-Order Equations**

The method of integrating both sides can also be used on second-order equations, although the antiderivatives are often much more difficult. For example, consider the equation

$$yy'' + (y')^2 = 6x.$$

Though it may not be obvious, the left side is a result of the product rule: it is the derivative of the product yy'. Integrating both sides gives

$$yy' = 3x^2 + C.$$

We have now reduced to a first-order equation, and we can integrate both sides again to obtain the general solution. Specifically, we get

$$\frac{1}{2}y^2 = x^3 + Cx + D,$$

where D is an arbitrary constant. Multiplying by two and taking the square root gives

$$y = \pm\sqrt{2x^3 + Ax + B},$$

where $A = 2C$ and $B = 2D$ are arbitrary constants.

EXERCISES

1–2 ■ Take the derivative of both sides of the given equation with respect to x.

1. $xy + \ln y = \sin(2x)$

2. $e^{2y} - xy^2 = 1$

3–6 ■ Find a differential equation whose general solution is the given formula.

3. $y = C\sqrt{x}$

4. $y = \pm\sqrt{x + C}$

5. $y = \tan(x + C)$

6. $y = e^{2x} + Ce^{-x}$

7–12 ■ Find the general solution to the given differential equation by integrating both sides.

7. $yy' = e^x$

8. $y'e^{2y} = x^3$

9. $y'\sqrt{y} = \cos x$

10. $x^4y' + 4x^3y = x^4$

11. $e^{3x}y' + 3e^{3x}y = e^{5x}$

12. $y' \tan x + y \sec^2 x = \cos x$

13–14 ■ Solve the given initial value problem.

13. $y' \cos y = e^{-x}, \;\; y(0) = 0$

14. $-y^{-2}y' = 2x, \;\; y(0) = 3$

3.2 Separation of Variables

Sometimes we must manipulate an equation algebraically before we can integrate both sides. For example, consider the equation

$$y' = \frac{\cos x}{3y^2}.$$

The right side cannot be integrated, since it involves y but not y'. However, if we multiply through by $3y^2$, we get

$$3y^2 y' = \cos x.$$

This equation can be integrated, yielding

$$y^3 = \sin x + C.$$

Solving for y gives $y = (\sin x + C)^{1/3}$.

In general, if we can get a differential equation into the form

$$f(y)\, y' = g(x)$$

for some functions f and g, then we can try to solve it by integrating both sides. This technique is called **separation of variables**, since it involves moving all of the y's to one side of the equation and all of the x's to the other side.

Separation of Variables

A differential equation is called **separable** if it can be put in the form

$$f(y)\, y' = g(x),$$

for some functions f and g. In this case, the solutions are given by

$$\int f(y)\, dy = \int g(x)\, dx.$$

EXAMPLE 6

Find the general solution to the differential equation $y' = xe^{2y}$.

SOLUTION We can divide through by e^{2y} to separate the variables:

$$e^{-2y} y' = x.$$

The solutions are given by

$$\int e^{-2y}\, dy = \int x\, dx.$$

Integrating yields

$$-\frac{1}{2}e^{-2y} = \frac{1}{2}x^2 + C.$$

We must now solve for y. We first multiply through by -2 to get

$$e^{-2y} = -x^2 + A,$$

where $A = -2C$. Taking the logarithm of both sides and dividing through by -2 yields

$$y = -\frac{1}{2}\ln(A - x^2).$$

A Closer Look **Missing Solutions**

Sometimes separation of variables misses certain constant solutions. For example, consider the equation

$$y' = 2xy^2.$$

We can divide through by y^2 to separate variables:

$$y^{-2}y' = 2x$$

Integrating both sides yields

$$-y^{-1} = x^2 + C$$

so

$$y = -\frac{1}{x^2 + C}.$$

It may seem that we have found the general solution, but we are actually missing the constant solution $y = 0$. The culprit is the first step where we divided through by y^2, which is only possible if $y \neq 0$.

EXAMPLE 7

Solve the following initial value problem:

$$x^2y' = y^2, \qquad y(1) = 3.$$

SOLUTION To separate the variables, we must divide through by both x^2 and y^2:

$$y^{-2}y' = x^{-2}.$$

Then the solutions are given by

$$\int y^{-2}\,dy = \int x^{-2}\,dx.$$

Integrating yields

$$-y^{-1} = -x^{-1} + C.$$

Instead of solving for y, we plug in the initial condition $y(1) = 3$ immediately, which tells us that $C = 2/3$. Thus we have the equation

$$-\frac{1}{y} = -\frac{1}{x} + \frac{2}{3}.$$

Solving for y and simplifying yields

$$y = \frac{3x}{3 - 2x}.$$

For an initial value problem, it is usually easiest to solve for C immediately after integrating.

Autonomous Equations

One common type of differential equation are those that involve y and y' but not x, i.e. differential equations of the form

$$y' = f(y).$$

Such equations are known as **autonomous equations**. For an autonomous equation, we can separate variables by dividing through by the entire right side $f(y)$.

In the case where y depends on t, autonomous equations are also known as **time-invariant equations**.

EXAMPLE 8

Find the general solution to the differential equation $y' = y^2 + 1$

SOLUTION Dividing through by $y^2 + 1$ separates the variables:

$$\frac{1}{y^2+1}\, y' = 1.$$

The solutions are given by

$$\int \frac{1}{y^2+1}\, dy = \int 1\, dx.$$

Integrating gives

$$\arctan y = x + C,$$

so

$$y = \tan(x + C).$$

Recall that

$$\int \frac{1}{x^2+1}\, dx = \arctan(x) + C.$$

Note that the result of integrating an autonomous equation is always just $x + C$ on the right, and thus the general solution to an autonomous equation is always a formula for y as a function of $x + C$.

A Closer Look Separation of Variables using Differentials

There is a nice way of viewing separation of variables using differentials, which looks a little bit different from our method. Given an equation like

$$-y' = 3x^2 e^y,$$

we can write it as

$$-\frac{dy}{dx} = 3x^2 e^y,$$

To separate variables, we now divide through by e^y and *multiply through by* dx:

$$-e^{-y}\, dy = 3x^2\, dx.$$

Here, the dx and dy by themselves are *differentials*, which represent small changes in the values of x and y as we travel along a solution curve. We can integrate both sides of this equation to get the solution:

$$\int -e^{-y}\, dy = \int 3x^2\, dx.$$

This method is presented in many textbooks, and you should feel free to use it if you prefer it. It always yields the same results as our reverse implicit differentiation method.

Really the differentials dx and dy represent *infinitesimal* changes in the values of x and y. Infinitesimals are not often used in mathematics, since they are not part of the real number system, but reasoning using infinitesimals is common in the sciences.

EXERCISES

1–8 ■ Use separation of variables to find the general solution to the given equation. (Do not worry about missing solutions.)

1. $e^{-2x} y' = 2y^2$

2. $y' = e^{y} \ln x$

3. $y' = x y^2$

4. $y' = y^2$

5. $y' = e^{2y}$

6. $y' = 3y^{2/3}$

7. $y' = e^{x+y}$

8. $y' - x y^2 = x$

9–12 ■ Solve the given initial value problem.

9. $e^x y' = x y^2, \;\; y(0) = -1$

10. $y' = 2x \sec y, \;\; y(0) = \dfrac{\pi}{6}$

11. $e^x y y' = e^{3x}, \;\; y(0) = -2$

12. $y' - 1 = y^2, \;\; y(0) = 1$

3.3 Integrating Factors

Consider the following differential equation:

$$xy' + 3y = x^4.$$

This equation cannot be integrated directly, and it is not possible to separate the variables. Is there any good way to solve it?

There is actually a very clever trick we can use to make this equation integrable. Consider what happens if we multiply through by x^2:

$$x^3y' + 3x^2y = x^6.$$

As you can see, the left side is now the result of a product rule (being the derivative of x^3y), which makes it possible to integrate both sides. This yields

$$x^3y = \frac{1}{7}x^7 + C$$

and therefore

$$y = \frac{1}{7}x^4 + Cx^{-3}.$$

The factor of x^2 that we used here is called a **integrating factor**, since multiplying by this factor makes the equation integrable.

First-Order Linear Equations

A differential equation of the form

$$f(x)\,y' + g(x)\,y = h(x)$$

is called a **first-order linear equation**. Such an equation is integrable if

$$f'(x) = g(x).$$

Every first-order linear equation can be made integrable by multiplying through by an appropriately chosen integrating factor.

If $f'(x) = g(x)$, then the differential equation is just

$$f(x)\,y' + f'(x)\,y = h(x),$$

and the left side is the derivative of $f(x)\,y$.

When searching for an integrating factor, keep in mind that the goal is to make the derivative of $f(x)$ equal to $g(x)$.

EXAMPLE 9

Find the general solution to the equation $x^2y' + 6xy = x + 1$.

SOLUTION This equation isn't integrable, since $6x$ isn't the derivative of x^2. However, we can make it integrable if we multiply through by x^4:

$$x^6y' + 6x^5y = x^5 + x^4.$$

The left side is now the derivative of x^6y. Integrating both sides gives

$$x^6y = \frac{1}{6}x^6 + \frac{1}{5}x^5 + C$$

and solving for y yields

$$y = \frac{1}{6} + \frac{1}{5}x^{-1} + Cx^{-6}.$$

EXAMPLE 10

Find the general solution to the equation $y' + 2y = x$.

SOLUTION Multiplying by a power of x won't work here, since multiplying by x^n gives

$$x^n y' + 2x^n y = x^{n+1}.$$

and $2x^n$ is never the derivative of x^n. Instead, we need to multiply through by e^{2x}:

$$e^{2x} y' + 2e^{2x} y = xe^{2x}.$$

The left side is now the derivative of $e^{2x} y$. This lets us integrate both sides, using integration by parts on the right side:

$$e^{2x} y = \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} + C.$$

Solving for y gives

$$y = \frac{1}{2} x - \frac{1}{4} + Ce^{-2x}.$$

EXAMPLE 11

Find the general solution to the equation $y' = 1 + y \tan x$.

SOLUTION We start by putting this equation into the standard form for a linear equation:

$$y' - y \tan x = 1.$$

We can now look for an integrating factor, although a little bit of guessing and checking is required. The right strategy is to multiply through by $\cos x$:

$$y' \cos x - y \sin x = \cos x.$$

The left side is the derivative of $y \cos x$. Integrating both sides gives

$$y \cos x = \sin x + C$$

so

$$y = \tan x + C \sec x.$$

An equation must be put into the form

$$f(x) y' + g(x) y = h(x)$$

before you can search for integrating factors.

A Closer Look Existence of Integrating Factors

It is not hard to show that every first-order linear equation has an integrating factor. First, by dividing through by the coefficient of y', we can put any first-order linear equation into the form

$$y' + g(x) y = h(x).$$

Once an equation is in this form, the desired integrating factor is $e^{G(x)}$, where $G(x)$ is any antiderivative of $g(x)$. For example, the equation

$$y' + (\cos x) y = \cos^3 x$$

has integrating factor $e^{\sin x}$. Although this procedure works for any first-order linear equation, it is usually easier to simply guess the integrating factor than to divide through by the coefficient of y' and then use the formula $e^{G(x)}$.

EXERCISES

1–8 ■ Use an integrating factor to find the general solution to the given differential equation.

1. $x^2y' + 3xy = 1$

2. $xy' = x^5 + 2y$

3. $y' + 5y = e^x$

4. $y' = x + y$

5. $y' + y\cot x = \cos x$

6. $\dfrac{xy'}{1+y} = 1$

7. $xy' \ln x + y = x^2$

8. $y' + 2xy = x$

9–10 ■ Solve the given initial value problem.

9. $xy' + 2y = x^2$, $y(1) = 1$

10. $y' = e^x + 3y$, $y(0) = 4$

3.4 Applications

In this section we consider a few applications of separation of variables and integrating factors to the natural sciences. Many more such applications are described in the exercises.

Newton's Law of Cooling

Newton's law of cooling is a differential equation that predicts the cooling of a warm body placed in a cold environment. According to the law, the rate at which the temperature of the body decreases is proportional to the difference in temperatures between the body and its environment. In symbols,

This law was first formulated by Isaac Newton in 1701 in an anonymous article entitled *Scala Graduum Caloris* ("Scale of the Degrees of Heat").

$$\boxed{\frac{dT}{dt} = -r(T - T_e)}$$

where T is the temperature of the object, T_e is the (constant) temperature of the environment, and r is a constant of proportionality.

This equation is linear, so we ought to be able to solve it using integrating factors. We start by adding rT to both sides:

▲ Newton illustrated his law by describing the cooling of hot iron.[1]

$$\frac{dT}{dt} + rT = rT_e.$$

This is now in the standard form for a linear equation. The integrating factor is e^{rt}:

$$e^{rt}\frac{dT}{dt} + re^{rt}T = re^{rt}T_e.$$

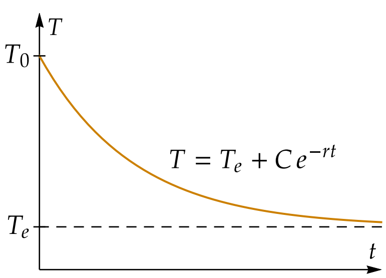
Integrating both sides gives

$$e^{rt}T = e^{rt}T_e + C$$

and hence

$$\boxed{T = T_e + Ce^{-rt}}$$

where C is an arbitrary constant.



▲ **Figure 1:** Cooling of a warm body.

Figure 1 shows the cooling of a warm body over time as predicted by Newton's law of cooling. The behavior is very similar to exponential decay, except that the temperature T approaches T_e instead of 0 as $t \to \infty$. Indeed, Newton's law of cooling can be interpreted as saying that the temperature *difference* $T - T_e$ decays exponentially over time.

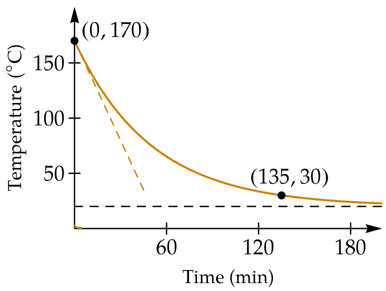
EXAMPLE 1

An apple pie with an initial temperature of 170 °C is removed from the oven and left to cool in a room with an air temperature of 20 °C. Given that the temperature of the pie initially decreases at a rate of 3.0 °C/min, how long will it take for the pie to cool to a temperature of 30 °C?

SOLUTION The pie should obey Newton's law of cooling with $T_e = 20\,°\mathrm{C}$. Thus

$$T(t) = 20 + Ce^{-rt} \qquad \text{and} \qquad T'(t) = -rCe^{-rt}$$

[1] Photo by Jeff Kubina, licensed under CC BY-SA 2.0, cropped from original.



▲ **Figure 2:** The temperature function $T(t)$ in Example 1. The diagonal dashed line shows the initial $3.0\,°\mathrm{C/min}$ rate of temperature decrease.

for some constants C and r. We also have the initial conditions

$$T(0) = 170\,°\mathrm{C} \qquad \text{and} \qquad T'(0) = -3.0\,°\mathrm{C/min}.$$

Plugging these in gives the equations

$$170 = 20 + C \qquad \text{and} \qquad -3 = -rC,$$

so $C = 150\,°\mathrm{C}$ and $r = 0.02$. Thus

$$T = 20 + 150e^{-0.02t}.$$

To find how long it will take for the temperature to reach $30\,°\mathrm{C}$, we plug in 30 for T and solve for t. The result is that $t = 135$ min, as shown in Figure 2.

Reaction Rates

In chemistry, the rate at which a chemical reaction occurs is determined by a differential equation called a **rate equation**. For a reaction with a single reactant, the concentration C of the reactant obeys a rate equation of the form

$$\boxed{\frac{dC}{dt} = -rC^n,}$$

where r is a constant called the **rate constant**, and n is a constant called the **order** of the reaction.

The study of the rates at which chemical reactions occur is a branch of chemistry known as **chemical kinetics**. For more complicated reactions, the chemical kinetics can involve a system of differential equations, with one equation for each reactant.

A Closer Look The Logistic Equation

Another important differential equation for the sciences is the **logistic equation**, which is often used as a model for population growth in an environment with limited resources. The equation is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{P_{\max}}\right)$$

where $P_{\max}$ is the maximum population that the given environment can support, sometimes called the **carrying capacity**. This equation is autonomous (time-independent), and can therefore be solved using separation of variables; however, the integral that arises is difficult, and requires integration by partial fractions. The resulting solution is

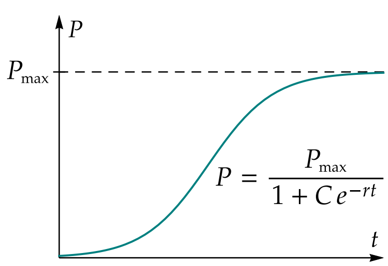
$$P = \frac{P_{\max}}{1 + Ce^{-rt}}$$

where C is an arbitrary constant.

Figure 3 shows the graph of a typical solution to the logistic equation. Note that the population grows quickly at first (as with exponential growth), but the rate of increase slows as the population approaches the maximum. As $t \to \infty$, the population asymptotically approaches the carrying capacity, i.e.

$$\lim_{t\to\infty} P(t) = P_{\max}.$$

The solutions to the logistic equation are known as **logistic functions**, and can be used to model any situation where a variable is growing but the growth is bounded above.



▲ **Figure 3:** Logistic population growth.

Logistic functions are sometimes called **sigmoid functions** because of the S-like shape of their graphs, though this term is also used more broadly to refer to any function whose graph has an S-like shape.

▲ A sample of nitrogen dioxide gas.[2]

Typically the order of the reaction is the same as its **molecularity**, i.e. the number of molecules of the reactant that must come together to react. For example, the decomposition of sulfuryl chloride

$$SO_2Cl_2 \longrightarrow SO_2 + Cl_2.$$

is a first-order reaction ($n = 1$), since it involves the decomposition of a single molecule of SO_2Cl_2. On the other hand, the decomposition of nitrogen dioxide

$$2\,NO_2 \longrightarrow 2\,NO + O_2.$$

is a second order reaction ($n = 2$), since it only occurs when two molecules of NO_2 come together.

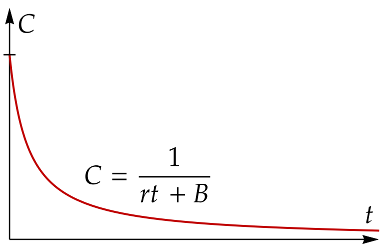
First and second order reactions behave somewhat differently. The rate equation for a first-order reaction is

$$\frac{dC}{dt} = -rC,$$

which is an instance of the exponential decay equation. Thus the concentration of the reactant decays exponentially. A second-order reaction, on the other hand, is governed by the equation

$$\frac{dC}{dt} = -rC^2,$$

We can solve this equation using separation of variables. Dividing through by C^2 gives



▲ **Figure 4:** Concentration of a second-order reactant.

$$C^{-2}\frac{dC}{dt} = -r,$$

so the solutions are given by

$$\int C^{-2}\,dC = \int -r\,dt.$$

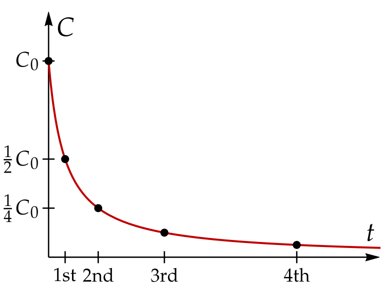
Integrating gives

$$-C^{-1} = -rt + A$$

for some constant A, so

$$\boxed{C = \frac{1}{rt + B}}$$

where $B = -A$.



▲ **Figure 5:** Halving times for the reactant in a second-order reaction. Note the increasing half-life.

Figure 4 shows the decrease in the concentration of a second-order reactant according to this equation. This decrease behaves somewhat differently than exponential decay, with the concentration decreasing very quickly at first but having a very long tail. In particular, the half-life of the reactant is very small at first but increases over the course of the reaction, as shown in Figure 5.

EXAMPLE 2

The decomposition of nitrogen dioxide (NO_2) is a second-order reaction. During a chemistry experiment, nitrogen dioxide with an initial concentration of 0.20 M is decomposed at a high temperature. If 90% of the NO_2 decomposes during the first ten seconds, what is the rate constant for the reaction?

[2]Photo by W. Oelen, licensed under CC BY-SA 3.0, via Wikimedia Commons

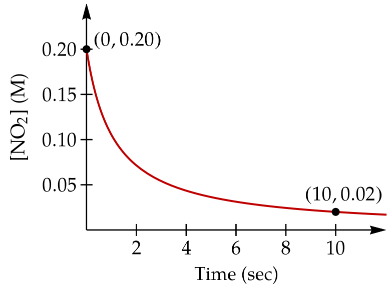


Figure 6: The concentration of NO_2 in Example 2.

SOLUTION Figure 6 shows the concentration $[NO_2]$ of nitrogen dioxide in this example. We know that

$$[NO_2] = \frac{1}{rt + B},$$

and we are given that $[NO_2] = 0.20$ M at $t = 0$, and $[NO_2] = 0.020$ M (10% of 0.20 M) at $t = 10$ sec. This information yields two equations:

$$0.20\text{ M} = \frac{1}{B} \qquad \text{and} \qquad 0.020\text{ M} = \frac{1}{r(10\text{ sec}) + B}.$$

Then $B = 5.0/\text{M}$, so the rate constant r is $4.5/(\text{M}\cdot\text{sec})$.

EXERCISES

1. A bottle of water with an initial temperature of 25 °C is placed in a refrigerator with an internal temperature of 5 °C. Given that the temperature of the water is 21 °C ten minutes after it is placed in the refrigerator, what will the temperature of the water be after one hour?

2. In 1974, Stephen Hawking discovered that black holes emit a small amount of radiation, causing them to slowly evaporate over time. According to Hawking, the mass M of a black hole obeys the differential equation

$$\frac{dM}{dt} = -\frac{k}{M^2}$$

where $k = 1.26 \times 10^{23}\ \text{kg}^3/\text{year}$.

(a) Use separation of variables to find the general solution to this equation

(b) After a supernova, the remnant of a star collapses into a black hole with an initial mass of 6.00×10^{31} kg. How long will it take for this black hole to evaporate completely?

3. The decomposition of hydrogen iodide is a second-order reaction:

$$2\,\text{HI} \longrightarrow \text{H}_2 + \text{I}_2$$

Initially the concentration of a sample of hydrogen iodide is 0.250 M, and the concentration is decreasing at a rate of 1.00×10^{-4} M/sec.

(a) How long will it take for half of the hydrogen iodide to be consumed?

(b) How long will it take for three quarters of the hydrogen iodide to be consumed?

4. When a hospital patient is administered morphine intravenously, the volume V of morphine in the bloodstream obeys the equation

$$\frac{dV}{dt} = r - kV$$

where r is the flow rate of the morphine, and $k = 0.35/\text{hour}$.

(a) Use integrating factors to find the general solution to the above equation.

(b) A nurse connects a patient to a morphine drip that administers morphine at a rate of 1.5 mg/hour. How much morphine will there be in the patient's bloodstream one hour later?

5. The velocity of an object moving through a fluid can be modeled by the **drag equation**

$$\frac{dv}{dt} = -kv^2$$

where k is a constant.

(a) Find the general solution to this equation.

(b) An object moving through the water has an initial velocity of 16 m/sec. After 2.0 seconds, the velocity has decreased to 12 m/sec. What will the velocity be after ten seconds?

6. Water is being drained from a spout in the bottom of a cylindrical tank. According to **Torricelli's law**, the volume V of water left in the tank obeys the differential equation

$$\frac{dV}{dt} = -k\sqrt{V}$$

where k is a constant.

(a) Use separation of variables to find the general solution to this equation.

(b) Suppose the tank initially holds 30.0 L of water, which initially drains at a rate of 1.80 L/min. How long will it take for tank to drain completely?